Worksheet 7

February 13, 2008

If $f(x, y)$ is a real-valued function of two variables and I is a set of real numbers, we the $f^{-1}(I)$ to be the set of points (x, y) such that $f(x, y)$ is in S. We call it the *inverse image* or *pullback* of I. This is just notation—we are not suggesting that f^{-1} is a legitimate function, and we are certainly not raising anything to the power -1.

1. Let $f(x, y) = 2xy$.
   (a) Sketch $f^{-1}(-1)$, $f^{-1}(0)$, $f^{-1}(1/2)$, $f^{-1}(1)$, and $f^{-1}(2)$. We have another name for the inverse image of a single number—what is it?
   (b) Sketch $f^{-1}([1, 2])$, $f^{-1}([0, 1/2])$, and $f^{-1}([-1, 1])$. What do you notice about f^{-1} of a closed interval?
   (c) Sketch $f^{-1}((1, 2))$, $f^{-1}((0, 1/2))$, and $f^{-1}((-1, 1))$. What do you notice about f^{-1} of an open interval?

2. Let

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \dfrac{2xy}{x^2 + y^2} & \text{otherwise.} \end{cases}$$

   (a) Sketch $f^{-1}(1)$, $f^{-1}(\sqrt{3}/2)$, $f^{-1}(1/2)$, $f^{-1}(0)$, $f^{-1}(-1/2)$, and $f^{-1}(2)$. Only one of these should contain the origin. You may find it convenient to work in polar coordinates.
   (b) Sketch $f^{-1}([1/2, \sqrt{3}/2])$. Is it open? Is it closed? What about $f^{-1}((1/2, \sqrt{3}/2))$?
   (c) Sketch $f^{-1}([-1/2, 1/2])$. Is it open? Is it closed? What about $f^{-1}((-1/2, 1/2))$?
   (d) How is this function different from the one in the previous problem?
   (e) Sketch $f^{-1}([1/2, 1])$ and $f^{-1}([1/2, 2])$.

3. Let U be a set of points in the plane. Argue that the following are equivalent:
   (a) U is open.
   (b) For each $\mathbf{u}$ in U there is a $\delta > 0$ such that if $|\mathbf{u} - \mathbf{v}| < \delta$ then $\mathbf{v}$ is also in U.

4. Write down the δ-ϵ definition of continuity.

5. Argue that the following are equivalent:
   (a) f is continuous.
   (b) If I is an open interval then $f^{-1}(I)$ is open.