Worksheet 10

March 5, 2008

1. Consider the surfaces $z^2 = 65 - 16x^2 - 9y^2$ and $x = 2$. Both contain the point $P = (2, 0, 1)$.
   (a) Write down a vector that is perpendicular to the first surface at P.
   (b) Write down a vector that is perpendicular to the second surface at P.
   (c) The two surfaces intersect in a curve. Find a parametric description of the tangent line to this curve at P. (Hint: parts (a) and (b) are relevant.)
   (d) Say something about tangent planes.

2. The U.S. post office will accept a box for shipment only if the sum of the length and the girth (distance around) is at most 108 inches.
   (a) Find the dimensions of the largest acceptable box with square front and back, as you might have done in Math 221.
   (b) Now do not assume that the front and back are square, and find the dimensions of the largest acceptable box.

3. You are going to make a 1-foot by 12-foot piece of metal into a gutter by folding x inches from each side up to an angle θ.
   (a) Draw a picture.
   (b) Express the volume of the gutter as a function of x and θ.
   (c) Which values of x and θ maximize the volume of the gutter?

4. Least Squares Approximation
   (a) Sketch some points $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane whose x-coordinates are not all equal, and a line $y = mx + b$. Draw dotted lines from each point straight down (or up) to the main line.
   (b) Write an expression for vertical distance from a given point (x_i, y_i) to the line.
   (c) Think of the points as fixed and m and b as variable. Let $f(m, b)$ be the sum of the squares of the vertical distances from each point to the line. Write an expression for $f(m, b)$.
   (d) Find the values of m and b that minimize $f(m, b)$. Hint: you may find the abberviations

   $$\bar{x} = \frac{x_1 + \cdots + x_n}{n} \qquad \bar{y} = \frac{x_1 + \cdots + x_n}{n} \qquad \overline{x^2} = \frac{x_1^2 + \cdots + x_n^2}{n} \qquad \overline{xy} = \frac{x_1 y_1 + \cdots + x_n y_n}{n}$$

   useful. Find m first.

5. Recall that the *determinant* of a 2×2 matrix

   $$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   is $\det M = ad - bc$. Recall also that the *identity matrix* is

   $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   We define the *eigenvalues* of M to be the roots of the polynomial $\det(M - tI)$.

(a) Find the eigenvalues of $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

(b) Consider a quadratic polynomial in t with real roots. Show that the constant term is positive if and only if the roots have the same sign.

(c) Let $f(x, y)$ be a function of two variables. Consider the matrix

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Observe that the determinant of this matrix shows up in the second deritative test.

(d) Show that the second deritive test is equivalent to the following:

  i. If the eigenvalues are both positive then f has a local minimum.
  ii. If the eigenvalues are both negative then f has a local maximum.
  iii. If the eigenvalues have opposite signs then f has a saddle point.