0.3.12. Since \( a \) and \( n \) are not relatively prime, there is an integer \( d > 1 \) such that \( d \mid a \) and \( d \mid n \). Observe that \( d \leq n \). Take \( b = n/d \), so \( 1 \leq b < n \). Then \( ab = a(n/d) = n(a/d) \equiv 0 \pmod{n} \).

Now suppose there is an \( c \) such that \( ac \equiv 1 \pmod{n} \). Then mod \( n \), we have

\[
b \equiv 1 \cdot b \equiv (ac)b \equiv (ab)c \equiv 0 \cdot c \equiv 0,
\]

but this is not true since \( 1 \leq b < n \).

0.3.15. (a) Since 13 is prime and 20 = \( 2^2 \cdot 5 \), they have no common prime factors, hence are relatively prime. 17 is an inverse since \( 13 \cdot 17 = 221 \equiv 1 \pmod{20} \).

1.1.1. (a) No. \( 5 - 3 - 1 = 1 \), but \( 5 - (3 - 1) = 3 \).

(b) Yes.

\[
(a \ast b) \ast c = (a + b + ab) \ast c = a + b + ab + c + ac + bc + abc
\]

\[
a \ast (b \ast c) = a \ast (b + c + bc) = a + b + c + bc + ab + ac + abc
\]

which are equal.

(c) No. \( (1 \ast 1) \ast 2 = 2/5 \ast 2 = 12/25 \), but \( 1 \ast (1 \ast 2) = 1 \ast 3/5 = 8/25 \).

(d) Yes. This is just addition of fractions. For a proof,

\[
[(a, b) \ast (c, d)] \ast (e, f) = (ad + bc, bd) \ast (e, f) = ((ad + bc)f + bde, bdf)
\]

\[
(a, b) \ast [(c, d) \ast (e, f)] = (a, b) \ast (cf + de, df) = (adf + b(cf + de), bdf)
\]

which are equal.

(e) No. \( (8/4)/2 = 1 \), but \( 8/(4/2) = 4 \).

1.1.6. Most of these sets are not closed under addition.

(a) Yes. Suppose that \( b \) and \( d \) are odd. Then when we write \( a/b + c/d = (ad + bc)/(bd) \) in lowest terms, the denominator will divide \( bd \), hence will be odd. Thus the set is closed under addition. Addition in \( \mathbb{Q} \) is associative. The additive identity 0 is in the set by hypothesis. If \( a/b \) is in lowest terms with \( b \) odd then its additive inverse \( -(a)/b \) is still in lowest terms.

(b) No. \( 1/6 + 1/6 = 2/6 = 1/3 \).

(c) No. \( 1/2 + 1/2 = 1 \).

(d) No. \( 3/2 + (-1) = 1/2 \).

(e) Yes. Every element of this set can be written uniquely as \( n/2 \), where \( n \in \mathbb{Z} \). The set is closed under addition because \( \mathbb{Z} \) is: \( m/2 + n/2 = (m + n)/2 \). Addition in \( \mathbb{Q} \) is associative. The additive identity \( 0/2 = 0 \) is in the set. If \( n/2 \) is in the set then so is its additive inverse \( -(n)/2 \).

(f) No. \( 1/2 + 1/3 = 5/6 \).

1.2.2. An arbitrary element \( x \in D_{2n} \) can be written in terms of the generators as \( x = s^{k_1}r^{i_1}s^{k_2}r^{i_2} \ldots s^{k_m}r^{i_m} \) where the powers \( k_1, k_2, \ldots, k_m, i_1, i_2, \ldots, i_m \in \mathbb{Z} \). Since \( rs = sr^{-1} \), we can move all the \( s \)'s to the left and \( r \)'s to the right, so \( x = s^{k_i}r^i \) with \( k, i \in \mathbb{Z} \). Since \( s^2 = 1 \), we have \( x = r^i \) or \( x = sr^i \). If \( x \) is not a power of \( r \) then \( rx = rsr^i = sr^{-1}r^i = sr^i(r^{-1}r) = xx^{-1} \).