1.3.2. \( \sigma = (1 \ 13 \ 5 \ 10)(3 \ 15 \ 8)(4 \ 14 \ 11 \ 7 \ 12 \ 9) \).
\( \tau = (1 \ 14)(2 \ 9 \ 15 \ 13 \ 4)(3 \ 10)(5 \ 12 \ 7)(8 \ 11) \). 

1.3.15. First we show that an \( m \)-cycle has order \( m \). If \( \sigma = (a_1 \ a_2 \ \ldots \ a_m) \) is a \( m \)-cycle, then \( \sigma(a_1) = a_2 \), \( \sigma^2(a_1) = a_3 \), \( \sigma^3(a_1) = a_4 \), and similarly \( \sigma^k(a_1) \) is different from \( a_1 \) for all \( k < m \), but \( \sigma^m \) is the identity.

Now let \( \sigma_1, \ldots, \sigma_k \) be disjoint cycles of lengths \( m_1, \ldots, m_k \), let \( \sigma = \sigma_1 \cdots \sigma_m \), and let \( m \) be the least common multiple of \( m_1, \ldots, m_k \). Since disjoint cycles commute, \( \sigma^m = \sigma_1^m \cdots \sigma_k^m = 1 \), so \( |\sigma| \leq m \).

1.4.8. Every field contains a 0 and a 1 which are not equal, so let

\[
A = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \quad B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.
\]

Then

\[
AB = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \quad BA = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}
\]

where \( 2 = 1 + 1 \neq 1 \). Thus \( AB \neq BA \), so \( GL_n(F) \) is not abelian.

1.6.1. (a) \( \varphi(x^n) = \varphi(x \cdot x \cdots x) = \underbrace{\varphi(x) \varphi(x) \cdots \varphi(x)}_{n \text{ times}} = \varphi(x)^n \).

(b) First, \( \varphi(1) = \varphi(1 \cdot 1) = \varphi(1) \varphi(1) \), so \( 1 = \varphi(1) \), so the claim is true for \( n = 0 \). Next, \( \varphi(x \varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1) = 1 \), so \( \varphi(x^{-1}) = \varphi(x)^{-1} \). If \( n > 0 \) then

\[
\varphi(x^{-n}) = \underbrace{\varphi(x^{-1}x^{-1} \cdots x^{-1})}_{n \text{ times}} = \underbrace{\varphi(x^{-1}) \varphi(x^{-1}) \cdots \varphi(x^{-1})}_{n \text{ times}} = \underbrace{\varphi(x^{-1}) \varphi(x^{-1}) \cdots \varphi(x^{-1})}_{n \text{ times}} = \varphi(x)^{-n}.
\]

1.6.9. An element of \( S_4 \) is one of the following: the identity, a transposition, a product of disjoint transpositions, a 3-cycle, or a 4-cycle. Thus an element of \( S_4 \) has order at most 4. But \( D_{24} \) has an element of order 24, namely \( r \).