Note on Spin

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From their Dynkin diagrams we know that the groups $\mathrm{Spin}(n)$ are isomorphic to other classical Lie groups for small n, but if one wants to compute one needs the explicit isomorphisms, which are hard to find written down. The most economical way to describe the isomorphisms is to describe the actions on $\mathbb{R}^n$.

- $\mathrm{Spin}(3) = \mathrm{Sp}(1)$. Identify $\mathbb{R}^3$ with the imaginary quaternions and let $q \in \mathrm{Sp}(1)$ act by $x \mapsto qx\bar{q}$.

- $\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Identify $\mathbb{R}^4$ with $\mathbb{H}$ and let $(p,q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ act by $x \mapsto px\bar{q}$.

- $\mathrm{Spin}(5) = \mathrm{Sp}(2)$. Let $\mathrm{Sp}(2)$ act on the 16-dimensional vector space $\mathfrak{gl}_2(\mathbb{H})$ by conjugation. If we endow $\mathfrak{gl}_2(\mathbb{H})$ with the (real) inner product $\langle X, Y\rangle = \mathrm{Re}(\operatorname{tr} XY^*)$ then $\mathrm{Sp}(2)$ acts by isometries. The 10-dimensional subspace $\mathfrak{sp}(2)$ of skew-adjoint matrices is invariant—this is the adjoint representation. The 1-dimensional subspace of real scalar matrices is also invariant—this is the trivial representation. The remaining 5-dimensional subspace of traceless self-adjoint matrices is the one we want.

- $\mathrm{Spin}(6) = \mathrm{SU}(4)$. Let $\mathbb{C}^4$ have basis $e_1, \ldots, e_4$ and let $V = \bigwedge^2 \mathbb{C}^4$. The usual Hermitian inner product on $\mathbb{C}^4$ induces one on V. If $v \in V$, define $*v \in V$ by the formula

  $$u \wedge *v = \langle u, v\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

  for all $u \in V$. Then the natural action of $\mathrm{SU}(4)$ on V fixes the the subspace $\{v \in V : v = *v\}$ of self-dual forms, which has real dimension 6.