Algebraic geometry of the ring of continuous functions

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Abstract

Maximal ideals of the ring of continuous functions on a compact space correspond to points of the space. For an algebraic geometer, this is exciting. But if we try to go further than this, it doesn't really lead anywhere.

1. Let X be a compact Hausdorff space. We work throughout with the ring $C(X)$ of continuous, real-valued functions on X. If $A \subseteq X$, let

$$I(A) = \{f \in C(X) : f(x) = 0 \ \forall x \in A\}$$

be the ideal of functions that vanish on A. If $\mathfrak{a} \subseteq C(X)$ is an ideal, let

$$V(\mathfrak{a}) = \{x \in X : f(x) = 0 \ \forall f \in \mathfrak{a}\}$$

be the common vanishing locus of the functions in $\mathfrak{a}$. As usual,

(1) (a) If $\mathfrak{a} \subseteq \mathfrak{b}$ then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.
(b) $V(0) = X$ and $V(1) = \varnothing$.
(c) $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
(d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(2) (a) If $A \subseteq B$ then $I(B) \subseteq I(A)$.
(b) $I(X) = 0$ and $I(\varnothing) = (1)$.
(c) $I(\bigcup A_i) = \bigcap I(A_i)$.

(3) $A \subseteq V(I(A))$ and $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$.

(4) $V(I(V(\mathfrak{a}))) = V(\mathfrak{a})$ and $I(V(I(A))) = I(A)$.

2. If $x \in X$ is a point then $I(x)$ is a maximal ideal, since it is the kernel of the surjective homomorphism $C(X) \to \mathbb{R}$ sending f to $f(x)$. Conversely,

Proposition. *Every maximal ideal is the ideal of a point.**

Proof. Let $\mathfrak{m}$ be maximal, and suppose that $V(\mathfrak{m})$ is empty. For each $x \in X$, choose $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$ and let U_x be the set where f_x does not vanish. Since $x \in U_x$, the collection of all U_x cover X, so we can extract a finite subcover $U_{x_1}, \ldots, U_{x_k}$. Now $f_{x_1}^2 + \cdots + f_{x_k}^2$ never vanishes, hence is a unit, but this is impossible since $\mathfrak{m}$ is a proper ideal.

Thus there is a point $x \in V(\mathfrak{m})$, so $\mathfrak{m} \subseteq I(V(\mathfrak{m})) \subseteq I(x)$, and since $\mathfrak{m}$ is maximal, $\mathfrak{m} = I(x)$. □

3. Since maximal ideals of $C(X)$ correspond to points just as in algebraic geometry, we would like prime ideals to correspond to subvarieties in some sense: maybe subspaces, or closed subspaces, or submanifolds if X is a manifold—really, we would like the prime ideals to tell us what we should mean by "subvariety." We are sorely disappointed:

Proposition. *If $\mathfrak{p}$ is prime then $V(\mathfrak{p})$ consists of a single point.*

Proof. Since $\mathfrak{p}$ is proper, $V(\mathfrak{p})$ consists of at least one point as we saw above. Suppose that $x, y \in V(\mathfrak{p})$ and $x \neq y$. Let U and V be neighborhoods of x and y, respectively, with $U \cap V = \varnothing$. Since X is compact and Hausdorff, it is completely regular, so there are bump functions f and g supported in U and V, respectively, with $f(x) = g(y) = 1$. Now $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, but $fg = 0 \in \mathfrak{p}$, which is a contradiction. □

Observe in particular that every prime ideal is contained in a unique maximal ideal.

4. Thus disappointed, we might hope that every prime ideal is maximal, that is, that there are no strange prime ideals properly contained in $I(x)$. But we are disappointed further: let $X = [-1, 1]$ and

$$\mathfrak{r} = \left\{ f \in C(X) : \lim_{x \to 0} \frac{f(x)}{x^k} = 0 \ \forall k \geq 0 \right\}.$$

This $\mathfrak{r}$ is not prime, for if $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$ then $f \notin \mathfrak{r}$ and $g \notin \mathfrak{r}$, but $fg = 0 \in \mathfrak{r}$. But $\mathfrak{r}$ is radical, for if $f^n \in \mathfrak{r}$ then

$$\lim_{x \to 0} \left| \frac{f(x)}{x^k} \right| = \lim_{x \to 0} \sqrt[n]{\left| \frac{f(x)^n}{x^{kn}} \right|} = 0$$

*If X is not compact, there are more maximal ideals: the ideal $\mathfrak{a} = C_c(X)$ of functions with compact support is proper, hence is contained in a maximal ideal, but $V(\mathfrak{a}) = \varnothing$. In fact, the maximal ideals of $C(X)$ are in bijection with the points of the Stone-Čech compactification of X.

so $f \in \mathfrak{r}$. Now $\mathfrak{r}$ is a radical ideal properly contained in $I(0)$, and since a radical ideal is the intersection of all primes that contain it, this implies that there are prime ideals properly contained in $I(0)$.

We remark that while $\mathfrak{r}$ is not prime, its contraction to $C^\infty(X) \subseteq C(X)$ is prime, as follows. If $f \in \mathfrak{r}$ is C^∞ then by l'Hôpital's rule,

$$0 = \lim_{x \to 0} \frac{f(x)}{x^{k+1}} = \lim_{x \to 0} \frac{f'(x)}{(k+1)x^k}$$

for all $k \geq 0$, so $f' \in \mathfrak{r}$. Thus $f^{(k)}(0) = 0$ for all $k \geq 0$, so $\mathfrak{r} \cap C^\infty(X)$ is the kernel of the map $C^\infty(X) \to \mathbb{R}[[x]]$ sending a function to its Taylor series

$$f \mapsto f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and $\mathbb{R}[[x]]$ is an integral domain.

The pathological example just described comes in an infinite family: given any function g that vanishes only at 0, let

$$\mathfrak{r}_g = \left\{ f \in C(X) : \lim_{x \to 0} \frac{f(x)}{g(x)^k} = 0 \ \forall k \geq 0 \right\}.$$

As before, $\mathfrak{r}_g$ as radical. It is not zero since $e^{-1/g^2} \in \mathfrak{r}_g$. If $h \in \mathfrak{r}_g$ then $\mathfrak{r}_h$ is properly contained in $\mathfrak{r}_g$ (since $h \notin \mathfrak{r}_h$). Thus there is a huge, messy lattice of radical ideals, and hence of primes, living under $I(x)$.

5. A discussion of the Nullstellensatz may shed light the preceding example.

Proposition.

(1) $V(\mathfrak{a})$ is closed.

(2) $V(I(A)) = \bar{A}$.

(3) $I(A)$ is radical.

Proof.

(1) Each $f \in S$ is continuous and $V(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} f^{-1}(0)$.

(2) $V(I(A))$ is a closed set containing A, so $\bar{A} \subseteq V(I(A))$. Since X is compact and Hausdorff, it is completely regular, so if $x \notin \bar{A}$ there is a function f with $f(x) = 1$ and $f|_{\bar{A}} = 0$. This $f \in I(A)$, so $x \notin V(I(A))$.

(3) If $f(x)^n = 0$ then $f(x) = 0$. □

Parts 1 and 2 of the proposition say that the vanishing locus of an ideal is closed and every closed set is the vanishing locus of some ideal. (Thus the Zariski topology on X is the same as the original topology.) Part 3 says that the ideal of a subset of X is radical. If there were a fourth part, it would be the Nullstellensatz: $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, or equivalently, every radical ideal is the ideal of some subset of X. But the radical ideals are nice if and only if the prime ideals are nice.

Proposition. *The following are equivalent:*

(1) Every radical ideal is $I(A)$ for some $A \subseteq X$.

(2) Every prime ideal is $I(A)$ for some $A \subseteq X$.

Proof. Every prime ideal is radical, so (1) $\Rightarrow$ (2). Every radical ideal is an intersection of primes, so (2) $\Rightarrow$ (1). □

And we have seen that (2) fails, even for very nice spaces like $[-1, 1]$.

The reader may object that we should not expect a Nullstellensatz since we are working with real-valued functions and $\mathbb{R}$ is not algebraically closed. But this is not the problem: everything we have said goes through for complex-valued functions with only slight modifications.

6. As wild as the primes of $C(X)$ are, at least we can study them locally. If $x \in X$, let

$$J(x) = \{f \in C(X) : f|_U = 0 \text{ for some open } U \ni x\}$$

be the ideal of functions that vanish in a neighborhood of x, so $C(X)/J(x)$ is the ring of germs of continuous functions at x.

Proposition. *The prime ideals contained in $I(x)$ are in bijection with the prime ideals of $C(X)/J(x)$.*

Proof. It suffices to show that $J(x) \subseteq \mathfrak{p}$ for all primes $\mathfrak{p} \subseteq I(x)$. If $f \in J(x)$ then f vanishes in a neighborhood U of x. Let g be a function supported in U with $g(x) = 1$. Then $g \notin I(x)$, so $g \notin \mathfrak{p}$, but $fg = 0 \in \mathfrak{p}$, so $f \in \mathfrak{p}$. □

We give a second proof, based on the following fact:

Proposition. *The ring of germs $C(X)/J(x)$ is isomorphic to the localization $C(X)_{I(x)}$.*

Proof. First we show that any $f \notin I(x)$ becomes a unit in $C(X)/J(x)$. It suffices to show that f^2 becomes a unit. Let $y = f(x)^2$ and $g = \max\{f^2, y/2\}$. Then g never vanishes, hence is a unit, and $g - f^2 \in J(x)$.

Second we show that the natural map $C(X) \to C(X)_{I(x)}$ sends any $f \in J(x)$ to 0. Since $f \in J(x)$, f vanishes on a neighborhood U of x. Let g be a function supported in U with $g(x) = 1$. Then $g \notin I(x)$ and $fg = 0$, so $f/1 = 0/g$. □

Now for any ring R and prime $\mathfrak{p}$, the primes of $R_{\mathfrak{p}}$ are in bijection with the primes contained in $\mathfrak{p}$.

7. We conclude with some remarks about finite generation.

Proposition. *If a radical ideal* $\mathfrak{r}$ *is finitely generated then* $V(\mathfrak{r})$ *is open.*

Proof. Suppose that $\mathfrak{r} = (f_1, \ldots, f_k)$. For each i, $|f_i|^2 = f_i^2 \in \mathfrak{r}$, so $|f_i| \in \mathfrak{r}$. Let $f = \sqrt{|f_1| + \cdots + |f_k|}$. Then $f \in \mathfrak{r}$, so there are $g_1, \ldots, g_k \in C(X)$ such that $f = g_1 f_1 + \cdots + g_k f_k$, so

$$\begin{aligned} f &= g_1 f_1 + \cdots + g_k f_k \\ &\le |g_1||f_1| + \cdots + |g_k||f_k| \\ &\le (|g_1| + \cdots + |g_k|)(|f_1| + \cdots + |f_k|) \\ &= (|g_1| + \cdots + |g_k|) f^2 \end{aligned}$$

Outside of $V(\mathfrak{r}) = V(f)$, we can divide by f to get

$$1 \le (|g_1| + \cdots + |g_k|) f,$$

but if $f(x) = 0$ this inequality fails. Thus the complement of $V(\mathfrak{r})$ is the pullback of the closed set $[1, \infty)$ via the continuous function $(|g_1| + \cdots + |g_k|) f$. □

Since $V(\mathfrak{r})$ is also closed, when X is connected the only finitely generated radical ideals are 0 and the whole ring. In particular, if $X = [-1, 1]$ then $I(0)$ is not finitely generated. We contrast this with the C^∞ situation:

Proposition. *In* $C^\infty(\mathbb{R}^n)$ *we have* $I(0) = (x_1, \ldots, x_n)$.

Proof. We follow Lemma 2.1 of Milnor's *Morse Theory.* Let $f \in I(0)$. Then

$$\begin{aligned} f(x_1, \ldots, x_n) &= \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt \\ &= \int_0^1 \left(\frac{\partial f}{\partial x_1}(tx_1, \ldots, tx_n)\, x_1 + \cdots + \frac{\partial f}{\partial x_n}(tx_1, \ldots, tx_n)\, x_n \right) dt. \end{aligned}$$

Taking $g_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt$ we get $f = g_1 x_1 + \ldots + g_n x_n$. □

Neither $C(X)$, nor $C^\infty(X)$ if X is a smooth manifold, is typically Noetherian: to produce an ascending chain of ideals, we need only produce a descending chain of closed sets, which is easy.

8. In conclusion, while it is well-known that algebraic geometry produces some pretty terrible topological spaces, here we have seen that topology produces some pretty terrible rings.

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