SPINOR SHEAVES AND COMPLETE INTERSECTIONS OF QUADRICS

by

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SPINOR SHEAVES AND COMPLETE INTERSECTIONS OF QUADRICS

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We show that the bounded derived category of coherent sheaves on a general complete intersection of four quadrics in $\mathbb{P}^{2n-1}$, $n \geq 4$, has a semi-orthogonal decomposition

$$\langle \mathcal{O}(-2n+9), \ldots, \mathcal{O}(-1), \mathcal{O}, \mathcal{D} \rangle,$$

where $\mathcal{D}$ is the derived category of twisted sheaves on a certain non-Kähler complex 3-fold. To do this, we develop a theory of "spinor sheaves" on singular quadrics, generalizing the spinor bundles on smooth quadrics.

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Contents

Chapter 1

Overview

In §1.1 we review earlier results on intersections of quadrics and state the main theorem. In §1.2 we discuss it in the context of string theory. In §1.3 we discuss it in relation to Orlov's result on matrix factorizations and derived categories of hypersurfaces and indulge in a little speculation.

In Chapter 2 we review the main techniques used in this dissertation: linear spaces on quadrics, spinor sheaves, stable sheaves, Fourier–Mukai transforms, semi-orthogonal decompositions, matrix factorizations, twisted sheaves, and ordinary double points. In Chapter 3 we introduce spinor sheaves on singular quadrics and develop the theory we need to work with them. In Chapter 4 we prove the main theorem.

1.1 Intersections of Quadrics

In 1954 Weil was looking for evidence of his famous conjectures, and thus wanted to count points on varieties over $\mathbb{F}_q$. To a general complete intersection X of two quadrics in $\mathbb{P}^{2n-1}$ he associated the following variety [51]: let L be the line those quadrics span in the space $\mathbb{P}^{\binom{2n+1}{2}-1}$ of quadrics in $\mathbb{P}^{2n-1}$ and $\mathcal{M}$ the double cover of L branched over the $2n$ points of L corresponding to singular quadrics. Then $\mathcal{M}$ is a hyperelliptic curve of genus $n-1$, on which he knew how to count points, and he was able to relate the number of points on X to the number on $\mathcal{M}$, and thus compute the zeta function of X. Hirzebruch [22] had just computed the Hodge numbers of complete intersections in projective space; the Hodge diamond of X is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \vdots & & & & \\ & & & & 1 & & & & \\ 0 & \cdots & 0 & n-1 & & n-1 & 0 & \cdots & 0 \\ & & & & 1 & & & & \\ & & & & \vdots & & & & \\ & & & & 1, & & & & \end{array}$$

which contains that of $\mathcal{M}$

$$\begin{array}{ccc} & 1 & \\ n-1 & & n-1 \\ & 1 & \end{array}$$

in the middle, and Weil was able to verify his conjectures in this example.

In his 1972 thesis, Reid [48] showed that this is not merely a coincidence of Hodge numbers, but that the weight 1 Hodge structure of $\mathcal{M}$ is isomorphic to the weight $2n-3$ Hodge structure of X, or equivalently that the Jacobian of $\mathcal{M}$ is isomorphic to the intermediate Jacobian of X.* He also showed that the Jacobian of $\mathcal{M}$ is isomorphic to the Fano variety of $\mathbb{P}^{n-2}$s on X. Donagi [17] clarified this, showing that the Abel-Jacobi map from this Fano variety to the intermediate Jacobian is an isomorphism. Donagi also observed that X can be recovered from $\mathcal{M}$, which gives a Torelli theorem for X.

Thus the moduli space of degree 0 line bundles on $\mathcal{M}$ is isomorphic to the variety of

*Even better, $\mathcal{M}$ is a direct summand of X in the category of motives.

$\mathbb{P}^{n-2}$s on X. In a series of papers starting in 1976 [16], Desale and Ramanan described various moduli spaces of bundles on $\mathcal{M}$ in terms of varieties of linear spaces on X. In 1995 Bondal and Orlov [6] gave a categorical explanation of this: viewing $\mathcal{M}$ as the fine moduli space of spinor bundles on X, they used the universal bundle as the kernel of a Fourier–Mukai transform to embed $D(\mathcal{M})$ in $D(X)$ and showed that

$$D(X) = \langle \mathcal{O}_X(-2n+5), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, D(\mathcal{M}) \rangle. \tag{1.1.1}$$

Thus any moduli space of objects of $D(\mathcal{M})$ is isomorphic to one of objects of $D(X)$. This can also be seen as a refinement of the Hodge-theoretic results mentioned above. Note that if we set $n = 2$, so X and $\mathcal{M}$ are dual elliptic curves, we recover one of Mukai's original examples [41] of a derived equivalence.

Next we consider a general complete intersection X of three quadrics in $\mathbb{P}^{2n-1}$, the associated plane L in the space of quadrics, and the double cover $\mathcal{M}$ of L branched over the locus of singular quadrics, which is a smooth curve of degree $2n$. Mukai [42] initiated this study in the case $n = 3$, so X and $\mathcal{M}$ are K3 surfaces, describing $\mathcal{M}$ as the moduli space of spinor bundles on X. It need not be a fine moduli space, so we only get a twisted pseudo-universal bundle, twisted by some Brauer class $\alpha \in H^2(\mathcal{M}, \mathcal{O}^*_{\mathcal{M}})$. In his thesis on twisted sheaves, Căldăraru [11] showed that the Fourier–Mukai transform with this twisted bundle as kernel is an equivalence $D(X) \cong D(\mathcal{M}, \alpha^{-1})$.

Following Mukai, many authors studied three quadrics when $n > 3$; to name a few, O'Grady [43] studied the Hodge structure, Desale [5] studied varieties of linear spaces on X as moduli spaces of bundles on $\mathcal{M}$, and Laszlo [34] proved a Torelli theorem. Again

$\mathcal{M}$ is a (not necessarily fine) moduli space of spinor bundles on X, and

$$D(X) = \langle \mathcal{O}_X(-2n+7), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, D(\mathcal{M}, \alpha^{-1})\rangle. \tag{1.1.2}$$

Before considering more than three quadrics, let us mention the situation with fewer than two. For a single quadric $Q \subset \mathbb{P}^{2n-1}$, Kapranov [27] showed that

$$D(Q) = \langle \mathcal{O}_Q(-2n+3), \ldots, \mathcal{O}_Q(-1), \mathcal{O}_Q, S_+, S_-\rangle \tag{1.1.3}$$

where S_+ and S_- are the two spinor bundles on Q. This is analogous to (1.1.1) and (1.1.2): the subcategory $\langle S_+, S_-\rangle$ is the derived category of two points, which we can view as the double cover $\mathcal{M}$ of a point L in the space of quadrics. Even Beĭlinson's description [3] of the derived category of $\mathbb{P}^{2n-1}$

$$D(\mathbb{P}^{2n-1}) = \langle \mathcal{O}_{\mathbb{P}^{2n-1}}(-2n+1), \ldots, \mathcal{O}_{\mathbb{P}^{2n-1}}(-1), \mathcal{O}_{\mathbb{P}^{2n-1}}\rangle$$

fits into the sequence, viewing $\mathbb{P}^{2n-1}$ as the complete intersection of zero quadrics.

For more than three quadrics there is a problem: the hypersurface $\Delta \subset \mathbb{P}^{\binom{2n+1}{2}-1}$ of singular quadrics is singular in codimension 2, so the linear space L must now meet its singular locus, so $L \cap \Delta$ and $\mathcal{M}$ are singular. In particular the derived category of $\mathcal{M}$ is unpleasant to work with, so to describe $D(X)$ one can either ignore $\mathcal{M}$ or resolve its singularities.

Kapranov [28] described $D(X)$ as a quotient of the derived category of modules over a generalized Clifford algebra, analogous to Berštein–Gelfan'd–Gelfan'd's description of $D(\mathbb{P}^{2n-1})$ as a quotient of the derived category of modules over an exterior algebra.

Bondal and Orlov [7] equipped $\mathcal{M}$ with a related sheaf of algebas $\mathcal{B}$, viewed $(\mathcal{M}, \mathcal{B})$ as a non-commutative resolution of singularities of $\mathcal{M}$, and stated that

$$D(X) = \langle \mathcal{O}_X(-2n+2m+1), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, D(\mathcal{B}\text{-mod})\rangle$$

when $n \geq m$, where m is the number of quadrics. Kuznetsov [31] proved this and more using his homological projective duality. For three quadrics, $\mathcal{B}$ is just an Azumaya algebra, so this is equivalent to (1.1.2), but in general $\mathcal{B}$ is less tame.

We will take a more geometric approach to the complete intersection of four quadrics: rather than taking a non-commutative resolution of $\mathcal{M}$, we take a non-Kähler one $\hat{\mathcal{M}} \to \mathcal{M}$. Our resolution is modular; that is, the points of $\hat{\mathcal{M}}$ parametrize sheaves on X. The smooth points of $\mathcal{M}$ parametrize stable sheaves on X, and its singular points, which are ordinary double points, parametrize S-equivalence classes of properly semi-stable sheaves. The points of $\hat{\mathcal{M}}$ over smooth points of $\mathcal{M}$ will parametrize the same stable sheaves, and an exceptional line over an ODP of $\mathcal{M}$ will parametrize semi-stable sheaves in the corresponding S-equivalence class. Again there is a twisted pseudo-universal bundle, and

Theorem. $D(X) = \langle \mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, D(\hat{\mathcal{M}}, \alpha^{-1})\rangle$.

Our construction will be guided by the geometry of linear spaces on the quadrics $Q \in L$; indeed, this result can be viewed as making explicit the geometry underlying Kuznetsov's result. But our approach does not work for more than four quadrics.

1.2 String Theory

Taking $n = 4$ in the theorem above, we get a twisted derived equivalence between Calabi–Yau 3-folds.* Such derived equivalences are related to string theory.

In string theory, spacetime is modeled not as $\mathbb{R}^4$, as it is in earlier physical theories, but as $\mathbb{R}^4 \times X$ for some compact complex 3-fold X equipped with a Ricci-flat Kähler metric. The number and types of particles and the strengths of their interactions depend on the topology of X and the geometry of its complex and symplectic structures.

There are several competing models of string theory: type I, type IIA, type IIB, and two "heterotic" string theories. *Mirror symmetry* is a relationship, somewhat mysterious to mathematicians, between Ricci-flat complex 3-folds X and $\check{X}$ (or more properly between families of these) in which type IIA string theory on X isomorphic to type IIB string theory on $\check{X}$ and vice versa. Each of the five string theories mentioned above has two simplifications, confusingly called the A-twist and the B-twist, which are also interchanged by mirror symmetry. The A-twist depends only on the symplectic structure of X and the B-twist only on the complex structure. *Homological mirror symmetry* is a conjecture of Kontsevich [30] that attempts to make this mathematically precise: the B-twist is encoded in the derived category $D(X)$ and the A-twist in the *Fukaya category* $\mathrm{Fuk}(\check{X})$, and the conjecture is that for mirror pairs X and $\check{X}$, $D(X) \cong \mathrm{Fuk}(\check{X})$ and $D(\check{X}) \cong \mathrm{Fuk}(X)$. It may be that two Calabi–Yaus X and Y have the same mirror, and so $D(X) \cong D(Y)$. All derived equivalences are expected to arise from such "double mirror" relationships.

*A complex geometer might object to calling $\hat{\mathcal{M}}$ Calabi–Yau, since Yau's proof of the Calabi conjecture applies only to Kähler manifolds. What we mean is that it is simply-connected and $\omega_{\hat{\mathcal{M}}} \cong \mathcal{O}_{\hat{\mathcal{M}}}$; perhaps we should just call it Calabi. In physics, non-Kählerness is remedied by something called H-flux.

The fact that our example involves twisted sheaves on $\hat{\mathcal{M}}$ is related to the fact that $\mathcal{M}$ is singular. According to Vafa and Witten [50], "what in classical geometry is a singularity may in string theory simply be a region in which stringy effects are large." They constructed examples of string theory models that appeared to be smooth despite being compactified on singular spaces and posed the question of how to understand them. Aspinwall, Morrison, and Gross [1] observed that all these examples involved a non-trivial Brauer class, which in physics corresponds to something called the B-field.

Lastly we mention *gauged linear sigma models*. A GLSM is a family of string theories, some of which can be described via Calabi–Yau 3-folds, or their twisted or non-commutative cousins, and for some of which the relationship to geometry is more obscure. Căldăraru, Distler, Hellerman, Pantev, and Sharpe [12] conjectured that homologically projectively dual varieties are connected by GLSMs, and in particular they studied the example of our X and Kuznetsov's $(\mathcal{M}, \mathcal{B})$ was studied. From this perspective, our example suggests that there may be a GLSM connecting the smooth, projective X and the twisted, non-Kähler $\hat{\mathcal{M}}$.

1.3 Matrix Factorizations

If $X \subset \mathbb{P}^m$ is a hypersurface of degree $d \leq m+1$, Orlov [44] has shown that

$$D(X) = \langle \mathcal{O}_X(-m+1+d), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, \mathrm{DMF}_{\mathrm{gr}}(X) \rangle \tag{1.3.1}$$

where $\mathrm{DMF}_{\mathrm{gr}}(X)$ is the derived category of graded matrix factorizations, which we will discuss in §2.6. This agrees with Kapranov's semi-orthogonal decomposition (1.1.3): a smooth, even-dimensional quadric has essentially only two matrix factorizations, which we will eventually see correspond to the spinor bundles.

But (1.3.1) also casts an interesting light on (1.1.1), (1.1.2), and our main theorem. The matrix factorizations of a smooth Q^{2n-2} just mentioned were of size 2^{n-1}. A corank 1 dimensional quadric has only one matrix factorization of size 2^{n-1}, so for the intersection of two or three quadrics, the fiber of $\mathcal{M} \to L$ over a quadric $Q \in L$ parametrizes matrix factorizations of Q; that is, $\mathcal{M}$ is a relative moduli space of matrix factorizations. A corank 2 quadric has two families of matrix factorizations of size 2^{n-1}, each parametrized by $\mathbb{P}^1$, so $\hat{\mathcal{M}}$ also parametrizes matrix factorizations—the two lines correspond to the two small resolutions of each ODP of $\mathcal{M}$.

These matrix factorizations of Q are parametrized by $\mathbb{P}^{n-1}$s on Q, but quite redundantly. The map

$$\begin{array}{c} Y = \{\text{pairs } (Q \in L, \mathbb{P}^{n-1} \subset Q)\} \\ \Big\downarrow \\ L \end{array}$$

factors through $\mathcal{M}$, and for the intersection of at most three quadrics, two points of Y parametrize the same matrix factorization if and only if they map to the same point of $\mathcal{M}$. For two quadrics, there is always a section $\mathcal{M} \to Y$. For three quadrics there is not typically a section, but when there is, the Brauer class α in (1.1.2) is trivial. For one quadric, $\mathcal{M}$ is just two points, so of course there is a section.

My hope is that for a higher-degree hypersurface or complete intersection X, $\mathrm{DMF}_{\mathrm{gr}}(X)$ is equivalent to the derived category of a variety or something close to one, such as twisted sheaves on a variety or complex manifold; that how far one has to stray from plain varieties is related to the rationality of X; and it can be studied using varieties of linear spaces on hypersurfaces.

Kuznetsov [32] has shown that for a cubic fourfold X containing a plane, there is a K3 surface S and a Brauer class $\alpha \in \mathrm{Br}(S)$, whose construction is very similar to our construction for the intersection of three quadrics, such that $\mathrm{DMF}_{\mathrm{gr}}(X) \cong D(S, \alpha)$. When X is rational, α is trivial. More generally, he has shown that for any cubic fourfold X, $\mathrm{DMF}_{\mathrm{gr}}(X)$ looks a lot like the derived category of a K3; that in cases where X is known to be rational, it actually is the derived category of an algebraic K3; and that it is close to being a birational invariant—that is, it is easy to understand how it changes under birational transformations. He conjectures that X is rational if and only if $\mathrm{DMF}_{\mathrm{gr}}(X)$ is the derived category of an algebraic K3.

I speculate that this might be realized as follows. Let F be the Fano variety of lines on a cubic fourfold X. F is 4-dimensional. For each line $\ell \subset X$, somehow produce a matrix factorization of X. This introduces an equivalence relation on lines: $\ell \sim \ell'$ if the corresponding matrix factorizations are isomorphic. The quotient $F/{\sim}$ is a K3, and the map $F \to F/{\sim}$ has a section if and only if X is rational. There is a universal matrix factorization on $X \times F$, and the pullback via this section gives an equivalence $D(F/{\sim}) \cong \mathrm{DMF}_{\mathrm{gr}}$.

More generally, I would hope that for hypersurfaces of higher degree and dimension, $F/{\sim}$ is a variety and $F \to F/{\sim}$ has a section if and only if X is rational. If this approach with linear spaces proves unworkable, one could approach the moduli space of matrix factorizations as a GIT problem, and see if fineness of the moduli space is related to rationality of the hypersurface. For complete intersections of hypersurfaces of the same degree, one could consider the linear system they span and the relative moduli space of matrix factorizations over it, as we did above for quadrics. For complete intersections of hypersurfaces of different degrees, it is less clear what to do.

Some recent papers of Macrì and Stellari lend a little weight to this speculation: in [4], they, Bernadara, and Mehrotra recover the Fano surface of lines on a cubic threefold as a moduli space of stable objects in $\mathrm{DMF}_{\mathrm{gr}}(X)$, and [37] they do the same for cubic fourfolds containing a plane.

Chapter 2

Techniques

2.1 Quadrics

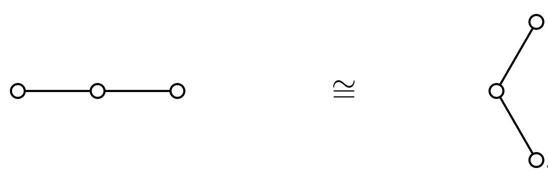
Our references in this section are Harris [21, Lecture 22].

2.1.1 Linear Spaces on Smooth Quadrics

Over $\mathbb{C}$, any two smooth quadrics of the same dimension are isomorphic. A smooth quadric Q contains $\mathbb{P}^k$s when $k \leq \frac{1}{2}\dim Q$. Let the variety of $\mathbb{P}^k$s on Q be called $OG(k+1, Q)$, for orthogonal Grassmannian. It is smooth. It is connected when $k < \frac{1}{2}\dim Q$ and has two connected components when $k = \frac{1}{2}\dim Q$ (so in particular Q is even-dimensional). In low dimensions, OG is fun to describe:

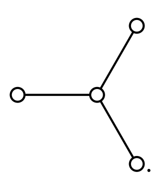
Via the Segre embedding, $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric in $\mathbb{P}^3$. The two families of lines are those of the form $\{\text{point}\} \times \mathbb{P}^1$ and those of the form $\mathbb{P}^1 \times \{\text{point}\}$. Thus we see that each point is contained in exactly one line from each family, and lines of the same family do not meet while lines of opposite families meet in a point. I imagine a hyperboloid of one sheet with its two rulings:

Via the Plücker embedding, the Grassmannian $G(2, 4)$ is a quadric in $\mathbb{P}^5$. Its points correspond to lines in $\mathbb{P}^3$. Planes of one family on $G(2,4)$ correspond to points in $\mathbb{P}^3$: for a fixed point, the lines through it form a plane on $G(2,4)$. Planes of the other family on $G(2,4)$ correspond to planes in $\mathbb{P}^3$: for a fixed plane, the lines on it form a plane on $G(2,4)$. Lines on $G(2,4)$ correspond to flags $\{\text{point} \in \text{plane}\}$ in $\mathbb{P}^3$: for a fixed flag, the lines passing through the point and lying on the plane form a line on $G(2,4)$. Thus we see that each line is contained in exactly one plane from each family, and planes of the same family meet in a point or are equal while planes of opposite families meet in a line or not at all. (This behavior is general: on a smooth Q^{2n}, each $\mathbb{P}^{n-1}$ is contained in exactly one $\mathbb{P}^n$ of each family, and $\mathbb{P}^n$s of the same family meet in even codimension while $\mathbb{P}^n$s of opposite families meet in odd codimension.) If we fix a non-degenerate bilinear form on $\mathbb{C}^4$, the Hodge star gives an automorphism of $G(2,4)$ that exchanges the two families of planes. All this reflects the fact that the Dynkin diagrams A_3 and D_3 are isomorphic [19, §23.3]:



From left to right, the dots of A_3 correspond to points, lines, and planes in $\mathbb{P}^3$. The middle dot of D_3 corresponds to points in Q^4, the upper dot to planes of one family, and the lower dot to planes of the other family.

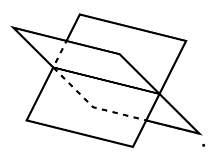
Each component of the variety of $\mathbb{P}^3$s on Q^6 is isomorphic it Q^6. This reflects the extra symmetry of D_4 [19, §20.3]:



The left dot corresponds to points, the upper dot to $\mathbb{P}^3$s of one family, and the lower dot to $\mathbb{P}^3$s of the other family, but there are automorphisms permuting these three dots, so the varieties of each are isomorphic. This symmetry is called *triality* and is related to the octonions.

2.1.2 Linear Spaces on Singular Quadrics

A singular quadric Q is a cone from a linear space, which is its singular locus, to a smooth quadric Q' of lower dimension. To understand $OG(k+1, Q)$, we begin with an example:



This quadric surface is a cone from a line to a smooth quadric 0-fold Q' (that is, two points). It contains planes as well as lines, so we do not have $k \leq \frac{1}{2} \dim Q$. For the variety of lines, consider the incidence correspondence

$$\begin{tikzcd}[column sep=small] & \{\text{flags line} \subset \text{plane} \subset Q\} \arrow[dl] \arrow[dr] & \\ \{\text{lines on } Q\} & & \{\text{planes on } Q\} = \{\text{points on } Q'\}. \end{tikzcd}$$

The space at the top is two disjoint planes and the southwest map glues them together at a point, namely the cone line. Thus the variety of lines is connected but not smooth.

In general, a $\mathbb{P}^k$ on Q must meet Q_{sing} if $k > \frac{1}{2} \dim Q'$. Let Λ be a $\mathbb{P}^k$ that meets Q_{sing} minimally (that is, in the least possible dimension), and let $l = \dim(\Lambda \cap Q_{\text{sing}})$ and

$m = \dim(\text{span of } \Lambda \text{ and } Q_{\text{sing}})$. Consider the incidence correspondence

$$\begin{array}{ccc} & \{\text{flags } \mathbb{P}^k \subset \mathbb{P}^m \subset Q \text{ with } \mathbb{P}^m \supset Q_{\text{sing}}\} & \\ \swarrow & & \searrow \\ OG(k+1, Q) & & \{\mathbb{P}^m\text{s on } Q \text{ containing } Q_{\text{sing}}\} = OG(k-l, Q'). \end{array} \tag{2.1.1}$$

If k is not maximal then $OG(k+1, Q)$ is connected but not smooth; the smooth locus consists of the $\mathbb{P}^k$s that meet Q_{sing} minimally, and the southwest map in (2.1.1) is a resolution of singularities. It has two irreducible components if a $\mathbb{P}^k$ that meets Q_{sing} minimally projects, via the projection from Q_{sing} to Q', to a linear space of dimesion $\frac{1}{2}\dim Q'$, and is irreducible if such a $\mathbb{P}^k$ projects to a linear space of smaller dimension.

2.1.3 Invariant Description

It will sometimes be more convenient to work with a quadratic form q on a vector space V than with its zero set $Q \subset \mathbb{P}V$. Quadratic forms q are in bijection with symmetric bilinear forms $b : V \otimes V \to \mathbb{C}$ via

$$b(v, v') = \frac{q(v+v') - q(v-v')}{4} \qquad\qquad q(v) = b(v, v).$$

Taking the adjoint $\tilde{b} : V \to V^*$ we see that bilinear forms are in bijection with symmetric matrices. To be quite concrete, if

$$q = \sum_{1 \le i \le j \le \dim V} a_{ij} x_i x_j$$

then the corresponding symmetric matrix is

$$\tilde{b} = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} & \cdots \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} & \cdots \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We define the *kernel* K of q or b to be the kernel of the matrix $\tilde{b}$. We say that q is *non-degenerate* if $K = 0$. We define the *rank* of Q or q or b to be the rank of $\tilde{b}$, and the *corank* as the dimension of the kernel. Over $\mathbb{C}$, two quadratic forms of the same rank are isomorphic.

A degenerate quadratic form q induces a non-degenerate one on V/K, which corresponds to what we said above about a singular quadric being a cone over a smooth quadric—the singular locus of Q is $\mathbb{P}K \subset \mathbb{P}V$, and the smooth quadric $Q' \subset \mathbb{P}(V/K)$ is the zero locus of the induced form. An isotropic subspace $W \subset V$, that is, one with $q|_W = 0$, corresponds to a linear space $\mathbb{P}W \subset Q$. The *orthogonal* of W is

$$W^{\perp} = \{v \in V : b(v, w) = 0 \text{ for all } w \in W\}.$$

Since W is isotropic, $W \subset W^{\perp}$. If q is non-degenerate then $\dim W + \dim W^{\perp} = \dim V$, and in general $\dim W + \dim W^{\perp} = \dim V + \dim(W \cap \ker q)$. The set of isotropic subspaces containing W is the set of subspaces of $W^{\perp}/W$ isotropic for the induced quadratic form. This generalizes what we said above about a $\mathbb{P}^{n-1}$ on a smooth Q^{2n} being contained in exactly two $\mathbb{P}^n$s.

Let T be the tautological bundle on the Grassmannian $G(k, V)$. Then q determines a section s of $\operatorname{Sym}^2 T^*$: if $W \in G(m, V)$ then $s(W) = q|_W \in \operatorname{Sym}^2 W^*$. The zero locus of s is $OG(k, q)$.

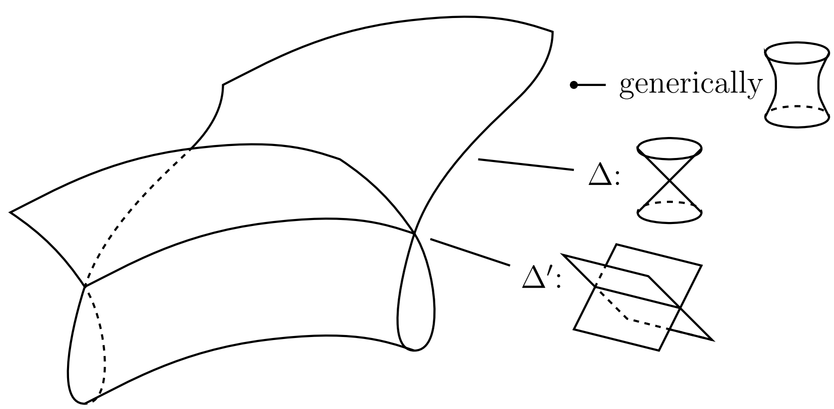
Proposition 2.1.1. *The section s just described is transverse to the zero section at $W \in OG(k, q)$ if and only if $W \cap K = 0$*

Proof. If q is non-degenerate this is well-known. If q is degenerate and $W \cap K = 0$, choose a subspace $U \subset V$ complementary to K with $U \supset W$. Then $q|_U$ is non-degenerate, so the restriction of s to $G(k, U) \subset G(k, V)$ is transverse to the zero section. Conversely, if $W \cap K \neq 0$ then from (2.1.1) we see that either W is a singular point of $OG(k, q)$ or the dimension of $OG(k, q)$ is too big. $\square$

2.1.4 The Space of Quadrics

The space of quadratic forms on V is $\operatorname{Sym}^2 V^*$, and the space of quadrics in $\mathbb{P}V$ is the projective space $\Phi := \mathbb{P}\operatorname{Sym}^2 V^*$. The singular quadrics form a hypersurface Δ of degree $\dim V$, given by the vanishing of the determinant of the matrix $\tilde{b}$ above. This hypersurface is not smooth; its singular locus Δ' consists of quadrics of corank 2 or more, which is codimension 3 in Φ and degree $\binom{\dim V+1}{3}$. This too is singular; its singular locus Δ'' consists of quadrics of corank 3 or more, which is codimension 6 in Φ. In general, the locus of quadrics of corank k or more is codimension $\binom{k+1}{2}$ in Φ, and its singular locus consists of quadrics of corank $k+1$ or more.

I imagine the whole situation as looking like this:



but note that Δ' is really codimension 3 and Δ'' is not pictured.

In §4.1 we will have occasion to consider the relative Grassmannian of isotropic k-dimensional spaces:

$$\Phi' := \{(W, Q) \in G(k, V) \times \Phi : \mathbb{P}W \subset Q\}.$$

This is a projective bundle over $G(k, V)$, hence is smooth and connected. Consider the map of vector bundles on $G(k, V) \times \Phi$

$$\begin{array}{ccc} \mathcal{O}_G \boxtimes \mathcal{O}_\Phi(-1) & \to & \operatorname{Sym}^2 T^* \boxtimes \mathcal{O}_\Phi \\ 1 \boxtimes q & \mapsto & q|_W \boxtimes 1. \end{array}$$

Then Φ' is the zero locus of the corresponding section s of $\operatorname{Sym}^2 T^* \boxtimes \mathcal{O}_\Phi(1)$, which is transverse to the zero section.

2.2 Spinor Bundles

Our reference in this section is Ottaviani [45].

2.2.1 Definition

On smooth quadrics there exist certain vector bundles called *spinor bundles*. On a $(2n-1)$-dimensional quadric there is one, of rank 2^{n-1}. On a $2n$-dimensional quadric there are two, both of rank 2^{n-1}. In low dimensions they coincide with other well-known bundles: On $Q^1 \cong \mathbb{P}^1$ it is $\mathcal{O}(1)$. On $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ they are $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. On Q^3, which is the Lagrangian Grassmannian $LG(2,4)$, it is the quotient bundle. On $Q^4 \cong G(2,4)$ they are the quotient bundle and the dual of the tautological bundle.

Ottaviani defined the spinor bundles as follows. For a $(2n-1)$-dimensional quadric Q, consider the incidence correspondence

$$\begin{tikzcd} & \{\text{flags point} \in \mathbb{P}^{n-1} \subset Q\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Q & & OG(n, Q) \end{tikzcd}$$

where as in the previous section $OG(n, Q)$ is the space of $\mathbb{P}^{n-1}$s on Q. The Picard group of OG is $\mathbb{Z}$. The square of its ample generator is the Plücker line bundle, i.e. the determinant of the quotient bundle, so the ample generator is called the Pfaffian line bundle [47, §12.3] and denoted Pf. Then the spinor bundle is $S := p_* q^* \operatorname{Pf}$.* For a $2n$-dimensional quadric Q, $OG(n+1, Q)$ has two connected components which we label arbitrarily as OG_+ and OG_-. Each has a Pfaffian line bundle, and by the same construction these give rise to two spinor bundles S_+ and S_-.

These bundles are called "spinor bundles" not because they admit a spin structure (although they do), but because they are intimately connected with the spin representations. When $\dim Q = 2n-1$, $\operatorname{Spin}_{2n+1} \mathbb{C}$ acts on Q and thus on OG. The line bundle Pf and thus the vector bundle S are equivariant for this action, and $\Gamma(S) = \Gamma(\operatorname{Pf})$ is the dual of the spin representation. In the space of spinors $\Gamma(\operatorname{Pf})^*$, the spinors in the affine cone of OG are called *pure spinors* [13]. When $\dim Q = 2n$, the two components $OG_\pm$ and the two vector bundles $S_\pm$ correspond to the two spin representations of $\operatorname{Spin}_{2n+2} \mathbb{C}$, and pure spinors are related to almost-complex structures.

2.2.2 Properties

Ottaviani showed that spinor bundles have the following properties:

- *Cohomology.* Let S be a spinor bundle on Q^{2n-1} or Q^{2n}. Then $H^0(S(t)) = 0$ for $t < 0$, $\dim H^0(S(1)) = 2^n$, and $H^i(S(t)) = 0$ for $0 < i < \dim Q$ and all $t \in \mathbb{Z}$. This last property is called being *arithmetically Cohen-Macaulay* (ACM).

*In fact Ottaviani works with the dual of this bundle. Following Kapranov, we will not take the dual.

- *Short exact sequences.* On Q^{2n-1} there is a short exact sequence

$$0 \to S(-1) \to \mathcal{O}_Q^{2^n} \to S \to 0.$$

On Q^{2n} there are short exact sequences

$$0 \to S_-(-1) \to \mathcal{O}_Q^{2^n} \to S_+ \to 0$$

$$0 \to S_+(-1) \to \mathcal{O}_Q^{2^n} \to S_- \to 0.$$

Recalling our low-dimensional coincidences, on $Q^1 \cong \mathbb{P}^1$ this is

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0;$$

on $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \mathcal{O}(-1,0) \to \mathcal{O}^2 \to \mathcal{O}(1,0) \to 0$$

$$0 \to \mathcal{O}(0,-1) \to \mathcal{O}^2 \to \mathcal{O}(0,1) \to 0;$$

and on $Q^4 \cong G(2,4)$

$$0 \to T \to \mathcal{O}^4 \to Q \to 0$$

$$0 \to Q^* \to \mathcal{O}^4 \to T^* \to 0$$

where T is the tautological bundle and Q the quotient bundle.

- *Dual.* $S^* = S(-1)$. If n is even then $S_+^* = S_+(-1)$ and $S_-^* = S_-(-1)$. If n is odd then $S_+^* = S_-(-1)$ and $S_-^* = S_+(-1)$.

- *Hyperplane sections.* If H is a hyperplane meeting Q^{2n} transversely then $S_\pm|_H$ is the spinor bundle S on $Q^{2n-1} = Q^{2n} \cap H$, and conversely any vector bundle E on Q^{2n} with $E|_H = S$ for all H meeting Q^{2n} transversely must be S_+ or S_-. If H

is a hyperplane meeting Q^{2n+1} transversely then $S|_H = S_+ \oplus S_-$, and a similar converse holds.

- *Horrocks' criterion.* It is a celebrated theorem of Grothendieck that every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles. Horrocks' criterion is a generalization to $\mathbb{P}^n$: every ACM vector bundle on $\mathbb{P}^n$ splits as a direct sum of line bundles.

  Ottaviani [46] generalized this to smooth quadrics: if E is a vector bundle on Q^{2n-1} such that both E and $E \otimes S$ are ACM, or a vector bundle on Q^{2n} such that E, $E \otimes S_+$, and $E \otimes S_-$ are ACM, then E splits as a direct sum of line bundles.

  Ballico [2] generalized this to singular quadrics: let Q be a singular linear section of Q^{2n-1} (or Q^{2n}) with Q_{sing} at least codimension 3 in Q; if E is a vector bundle on Q such that E and $E \otimes S|_Q$ (or $E \otimes S_\pm|_Q$) are ACM then E splits as a direct sum of line bundles.

- *Stability.* Spinor bundles are stable.

2.2.3 Resolution on the Ambient Projective Space

If $\mathcal{F}$ is a sheaf on $\mathbb{P}^m$ then one of the Beĭlinson spectral sequences [23, Prop. 8.28] is:

$$E_1^{p,q} = H^q(\mathcal{F}(p)) \otimes \Omega^{-p}(-p) \Rightarrow \begin{cases} \mathcal{F} & \text{if } p+q=0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $E_1^{p,q} = 0$ unless $-m \le p \le 0$ and $0 \le q \le m$.

If S is the spinor bundle on Q^{2n-1}, we will use this to resolve S on the ambient $\mathbb{P}^{2n}$. As we find the terms $E_1^{p,q}$, the reader may want to refer to (2.2.1) below, where they are assembled.

From Ottaviani's calculation of the cohomology of S and its twists,

$$E_1^{0,0} = \mathcal{O}^{2^n}$$

$$E_1^{p,0} = 0 \text{ for } p < 0$$

$$E_1^{p,q} = 0 \text{ for } 0 < p < 2n-1 \text{ and all } q.$$

Since $\dim Q = 2n - 1$,

$$E_1^{-2n,q} = 0 \text{ for all } q.$$

By Serre duality on Q^{2n-1} and the fact that $S^* = S(-1)$,

$$H^{2n-1}(S(p)) = H^0(S^*(-p-2n+1))^* = H^0(S(-p-2n))^*$$

so

$$E_1^{-2n,2n-1} = \Omega^{2n}(2n)^{2^n} = \mathcal{O}(-1)^{2^n}$$
$$E_1^{p,2n-1} = 0 \text{ for } p > -2n.$$

Thus the E_1 page looks like

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
\mathcal{O}(-1)^{2^n} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}^{2^n}
\end{array} \tag{2.2.1}$$

and all the differentials are necessarily zero until the E_{2n} page

$$\begin{tikzcd}
0 \arrow[ddddrrrrr] & 0 & 0 & \cdots & 0 & 0 \\
\mathcal{O}(-1)^{2^n} \arrow[ddddrrrrr, "\varphi"] & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \mathcal{O}^{2^n},
\end{tikzcd}$$

after which the E_{2n+1} page and all subsequent pages look like

$$\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
\ker\varphi & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \operatorname{coker}\varphi
\end{array}$$

so $\ker\varphi = 0$ and $\operatorname{coker}\varphi = S$. So we get a very nice resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^{2n}}^{2^n}(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^{2n}}^{2^n} \to S \to 0. \tag{2.2.2}$$

From this resolution it is easy to get back the cohomology of S: $\dim H^0(S(t)) = \dim H^{2n-1}(S(-2n-t)) = 2^n\binom{2n-1+t}{2n-1}$ for $t \geq 0$, and $H^i(S(t)) = 0$ for all other i and t.

Similarly, for $S_\pm$ on Q^{2n} we get

$$0 \to \mathcal{O}_{\mathbb{P}^{2n+1}}^{2^n}(-1) \to \mathcal{O}_{\mathbb{P}^{2n+1}}^{2^n} \to S_\pm \to 0.$$

2.2.4 Alternate Definitions

Besides Ottaviani's geometric construction of spinor bundles, we mention two others. Kapranov [27] constructed spinor bundles via Clifford algebras and used them to show that the derived category of a smooth quadric is generated by an exceptional collection

$$D(Q^{2n-1}) = \langle \mathcal{O}(-2n+2), \ldots, \mathcal{O}(-1), \mathcal{O}, S \rangle$$
$$D(Q^{2n}) = \langle \mathcal{O}(-2n+1), \ldots, \mathcal{O}(-1), \mathcal{O}, S_+, S_- \rangle$$

as we mentioned earlier (1.1.3).

Langer [33] constructed spinor bundles via an explicit matrix factorization of

$$x_1 x_2 + \cdots + x_{2n-1} x_{2n}$$

in even dimensions and

$$x_1 x_2 + \cdots + x_{2n-1} x_{2n} + x_{2n+1}^2$$

in odd dimensions, as we will discuss in §2.6.

We will generalize these constructions in Chapter 3.

2.3 Stable Sheaves

Our reference in this section is Huybrechts and Lehn [24]. We will define a notion called *slope stability* or *Mumford–Takemoto stability*. Mumford introduced this to study moduli spaces of vector bundles on curves. Takemoto found that to extend Mumford's work to variety of dimension $n \geq 2$, one should consider not only vector bundles but also torsion-free sheaves. (On a smooth curve, every torsion-free sheaf is a vector bundle.) At the end of the section we will mention *Gieseker stability*, which is better-suited to studying torsion sheaves.

The *degree* of a line bundle $\mathcal{L}$ on an n-dimensional variety $X \subset \mathbb{P}^m$ is the degree of the corresponding divisor D, that is, the number of points in which D meets a general

$\mathbb{P}^{m-n+1} \subset \mathbb{P}^n$. By Poincaré duality,

$$\deg \mathcal{L} = \int_X c_1(\mathcal{L}) \wedge \omega^{n-1}$$

where ω is the restriction to X of the Fubini–Study Kähler form on $\mathbb{P}^n$. We define the degree of a vector bundle by the same formula. If

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is a short exact sequence of vector bundles then $c_1(\mathcal{E}) = c_1(\mathcal{E}') + c_1(\mathcal{E}'')$, so degree is additive on short exact sequences. If X is smooth then a sheaf $\mathcal{F}$ can be resolved by a finite sequence of vector bundles

$$0 \to \mathcal{E}_k \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

so we can define $\deg \mathcal{E} = \deg \mathcal{E}_0 - \deg \mathcal{E}_1 + \cdots \pm \deg \mathcal{E}_k$, which is the correct definition when $\mathcal{E}$ is torsion-free. But this definition is impractical to use, and is not valid if X is singular.

The degree of a vector bundle can also be read off its Hilbert polynomial. If the Hilbert polynomial of X is

$$\chi(\mathcal{O}_X(t)) = \deg X \cdot \frac{t^n}{n!} + C \cdot \frac{t^{n-1}}{(n-1)!} + \cdots .$$

then that of a vector bundle $\mathcal{E}$ is

$$\chi(\mathcal{E}(t)) = \deg X \cdot \operatorname{rank} \mathcal{E} \cdot \frac{t^n}{n!} + (C \operatorname{rank} \mathcal{E} + \deg \mathcal{E}) \cdot \frac{t^{n-1}}{(n-1)!} + \cdots . \tag{2.3.1}$$

If $\mathcal{E}$ is a torsion-free sheaf then its Hilbert polynomial has the same leading term, so we can define its rank and degree by the same formula.

Since we are considering only torsion-free sheaves $\mathcal{E}$ we should consider only *saturated* subsheaves $\mathcal{F} \subset \mathcal{E}$, that is, ones where the quotient $\mathcal{F}/\mathcal{E}$ is torsion-free.

The *slope* of a torsion-free sheaf $\mathcal{E}$ is

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}.$$

If $\mathcal{F} \subset \mathcal{E}$ is a proper saturated subsheaf then $0 < \operatorname{rank} \mathcal{E} < \operatorname{rank} \mathcal{F}$, so $\operatorname{rank}(\mathcal{E}/\mathcal{F})$ is positive, so $\mu(\mathcal{F})$ and $\mu(\mathcal{E}/\mathcal{F})$ are defined. Since rank and degree are additive on short exact sequences, $\mu(\mathcal{E})$ is a weighted average of $\mu(\mathcal{F})$ and $\mu(\mathcal{E}/\mathcal{F})$, so there are three possiblities:

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) < \mu(\mathcal{E}/\mathcal{F}), \text{ or}$$
$$\mu(\mathcal{F}) = \mu(\mathcal{E}) = \mu(\mathcal{E}/\mathcal{F}), \text{ or}$$
$$\mu(\mathcal{F}) > \mu(\mathcal{E}) > \mu(\mathcal{E}/\mathcal{F}).$$

Now we say that $\mathcal{E}$ is *stable* if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for all such $\mathcal{F}$, and *semi-stable* if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. This definition is natural because the first Chern class, and hence the slope, is related to curvature, which decreases in sub-bundles and increases in quotient bundles [20, p. 79].

Every torsion-free sheaf has a unique *Harder–Narasimhan filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{E}$$

where the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semi-stable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) < \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$. Every semi-stable sheaf has a *Jordan–Hölder filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{E}$$

where the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are stable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = \mu(\mathcal{E})$. This filtration not unique, but the associated graded object $\bigoplus_{i=1} \mathcal{F}_i/\mathcal{F}_{i-1}$ is. Two semi-stable sheaves whose Jordan–Hölder filtrations have the same associated graded object are called *S-equivalent*. A sheaf is called *polystable* if it is a direct sum of stable sheaves of the same slope; every semi-stable sheaf is S-equivalent to a unique polystable sheaf. Every semi-stable sheaf contains a unique maximal polystable subsheaf called its *socle*. In both filtrations the $\mathcal{F}_i$ are necessarily saturated, and the socle is constructed from the first terms of Jordan–Hölder filtrations, so it too is saturated.

As an example, let us compute the slope of a spinor bundle. From (2.2.2) we can compute its Hilbert polynomial:

$$\chi(S(t)) = 2\operatorname{rank} S \cdot \binom{t+n}{n} = 2\operatorname{rank} S \cdot \frac{t^n}{n!} + \operatorname{rank} S \cdot (n+1) \cdot \frac{t^{n-1}}{(n-1)!} + \cdots .$$

Comparing this to the Hilbert polynomial of Q

$$\chi(\mathcal{O}_Q(t)) = \binom{t+n}{n} + \binom{t+n-1}{n} = 2 \cdot \frac{t^n}{n!} + n \cdot \frac{t^{n-1}}{(n-1)!} + \cdots$$

we see that $\mu(S) = 1$. We can also compute its degree directly from the first Chern class, as follows. A general plane section of Q is a smooth conic Q^1, and by our remarks in §2.2.2 the restriction of S to this plane section is the direct sum of $\operatorname{rank}(S)$ copies of the spinor bundle on Q^1. But $Q^1 \cong \mathbb{P}^1$, and its spinor sheaf is $\mathcal{O}_{\mathbb{P}^1}(1)$, whose degree is 1. Thus $\deg S = \operatorname{rank} S$, so again $\mu(S) = 1$.

We remark here that a stable sheaf $\mathcal{E}$ is simple, that is, $\operatorname{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}$, as follows. Let $\varphi : \mathcal{E} \to \mathcal{E}$ be an endomorphism. If $\operatorname{im} \varphi \neq 0$ and $\operatorname{im} \varphi \neq \mathcal{E}$ then $\mu(\operatorname{im} \varphi) < \mu(\mathcal{E})$, but this contradicts the fact that $\operatorname{im} \varphi = \mathcal{E} / \ker \mathcal{E}$ and $\mu(\ker \mathcal{E}) < \mu(\mathcal{E})$. Thus φ was either zero or an isomorphism, so by Schur's lemma, $\operatorname{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}$.

If $\mathcal{E}$ is a torsion sheaf then $\operatorname{rank} \mathcal{E} = 0$, so $\mu(\mathcal{E})$ is undefined. To remedy this, we introduce the *reduced Hilbert polynomial* $p(\mathcal{E})$, which is the Hilbert polynomial made monic, i.e. divided by its leading coefficient. For an n-dimensional sheaf $\mathcal{E}$ we have

$$p(\mathcal{E})(t) = t^n + \frac{1}{\deg X}(C + \mu(\mathcal{E})) \cdot nt^{n-1} + \cdots ,$$

so the reduced Hilbert polynomial is a refinement of the slope. We say that $p(\mathcal{F}) < p_(\mathcal{E})$ if the leading coefficient of $p(\mathcal{E}) - p(\mathcal{F})$ is positive, or equivalently if $p(\mathcal{F})(t) < p(\mathcal{E})(t)$ for $t \gg 0$. We say that a pure[*] sheaf $\mathcal{E}$ is *Gieseker stable* if $p_{\mathcal{F}} < p_{\mathcal{E}}$ for all proper subsheaves $\mathcal{F} \subset \mathcal{E}$, and semi-stable if $p_{\mathcal{F}} \leq p_{\mathcal{E}}$. For Gieseker stability there is also a Harder–Narasimhan filtration, a Jordan–Hölder filtration, and so on. It is not hard to see that slope stable implies Giesker stable, which implies Gieseker semi-stable, which implies slope semi-stable.

In a moduli space of sheaves, smooth points parametrize stable sheaves, while singular points parametrize S-equivalence classes of properly semi-stable sheaves.

2.4 Fourier–Mukai Transforms

Our reference in this section and the next is Huybrechts [23].

Let X and Y be compact complex manifolds and $\mathcal{S}$ a complex of sheaves on $X \times Y$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Then the *Fourier–Mukai transform* from Y to X with kernel $\mathcal{S}$ is the map

$$\begin{aligned} F_{\mathcal{S}}^{Y \to X} : D(Y) &\to D(X) \\ \mathcal{F} &\mapsto \pi_{X*}(\pi_Y^* \mathcal{F} \otimes \mathcal{S}) \end{aligned}$$

[*] This is a generalization of torsion-free: it means that if $\mathcal{F} \subset \mathcal{E}$ then $\dim(\operatorname{supp} \mathcal{F}) = \dim(\operatorname{supp} \mathcal{E})$.

where the pushforward and tensor are, of course, derived. Some authors prefer to call this an *integral transform* and reserve *Fourier–Mukai transform* for the case where $F_{\mathcal{S}}^{Y \to X}$ is an equivalence of categories, but we will follow Huybrechts and call them all Fourier–Mukai transforms.

The analogy is with the integral transforms of classical analysis, such as the Fourier transform

$$f(y) \mapsto \mathcal{F}f(x) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i x y}\, dy$$

or the Laplace transform

$$f(y) \mapsto \mathcal{L}f(x) = \int_{0}^{\infty} f(y)\, e^{-xy}\, dy$$

or the Hilbert transform

$$f(y) \mapsto \mathcal{H}f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy.$$

The pullback π_Y^* corresponds to regarding $f(y)$ as a function of x and y. Tensoring with the kernel $\mathcal{S}$, which lives on $X \times Y$, corresponds to multiplying by one of the kernels $e^{-2\pi i x y}$, e^{-xy}, or $1/\pi(x-y)$, which are functions of x and y. The pushforward π_{X*} corresponds to integrating out y; recall that in de Rham cohomology, pushforward is integration along the fiber [8]. Fourier–Mukai transforms can also be viewed as a refinement of correspondences from the theory of motives.

In many of the most interesting examples, Y is a fine moduli space of sheaves on X and $\mathcal{S}$ is the universal sheaf. But more mundane functors are also Fourier–Mukai transforms: for example, if $f : Y \to X$ is a holomorphic map, let $\Gamma \subset X \times Y$ be its graph; then f_* is $F_{\mathcal{O}_\Gamma}^{Y \to X}$ and f^* is $F_{\mathcal{O}_\Gamma}^{X \to Y}$. It has long been suspected, and looks increasingly likely [49, 36], that every exact functor between derived categories is a Fourier–Mukai transform.

If $F = F_{\mathcal{S}}^{Y\to X}$ is an embedding (that is, fully faithful) then for any points $p, q \in Y$ and any $i \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{D(X)}(F\mathcal{O}_p, F\mathcal{O}_q[i]) = \mathrm{Hom}_{D(Y)}(\mathcal{O}_p, \mathcal{O}_q[i]) = \begin{cases} 0 & \text{if } p \neq q \\ \bigwedge^i T_p Y & \text{if } p = q. \end{cases}$$

Surprisingly, this is sufficient to guarantee that F is an embedding. In fact, we need even less:

Theorem (Bondal–Orlov [6])**.** *A Fourier–Mukai transform $F = F_{\mathcal{S}}^{Y\to X}$ is an embedding if an only if for any points $p, q \in Y$ and any $i \in \mathbb{Z}$ one has*

$$\mathrm{Hom}(F\mathcal{O}_p, F\mathcal{O}_q[i]) = \begin{cases} \mathbb{C} & \textit{if } p = q \textit{ and } i = 0 \\ 0 & \textit{if } p \neq q \textit{ or } i < 0 \textit{ or } i > \dim Y. \end{cases}$$

Observe that if Y is a moduli space of sheaves on X and $\mathcal{S}$ the universal bundle then $F\mathcal{O}_p$ is the sheaf that p parametizes.

If F is an equivalence then X and Y have the same dimension and F preserves the Serre functors: $F \circ S_Y = S_X \circ F$. Thus for any point $p \in Y$, $F\mathcal{O}_p \otimes \omega_X \cong F(\mathcal{O}_p \otimes \omega_Y) = F\mathcal{O}_p$. Surprisingly again, this is sufficient to guarantee that an embedding is an equivalence:

Theorem (Bridgeland [9])**.** *A Fourier–Mukai embedding $F = F_{\mathcal{S}}^{Y\to X}$ is an equivalence if and only if for any point $p \in Y$ one has*

$$F\mathcal{O}_p \otimes \omega_X \cong F\mathcal{O}_p.$$

Observe that if X is Calabi–Yau, the hypothesis is automatically satisfied.

2.5 Semi-Orthogonal Decompositions

A full triangulated subcategory $A \subset D(X)$ is called *admissible* if the inclusion functor $A \hookrightarrow D(X)$ has left and right adjoints. For example, the image of a Fourier–Mukai embedding is an admissible subcategory, for

$$F_{\mathcal{S}^*}^{X \to Y} \circ S_X \;\dashv\; F_{\mathcal{S}}^{Y \to X} \;\dashv\; S_Y \circ F_{\mathcal{S}^*}^{X \to Y}$$

where S_X and S_Y are again the Serre functors.

A key feature of an admissible subcategory is that if its left or right orthogonal is zero, the inclusion is an equivalence. The *left orthogonal* of A is the full subcategory

$${}^{\perp}A := \{E \in D(X) : \operatorname{Hom}(E, F[i]) = 0 \text{ for all } F \in A \text{ and } i \in \mathbb{Z}\}$$

and the *right orthogonal* $A^{\perp}$ is defined similarly; they are again admissible. If an object $E \in D(X)$ is left orthogonal to all the skyscraper sheaves $\mathcal{O}_p$, $p \in X$, then $E \cong 0$. If X is projective and F is right orthogonal to all the line bundles $\mathcal{O}_X(t)$, $t \ll 0$ then again $F \cong 0$ by the famous but poorly-named Theorem B. Thus to show that an admissible subcategory A is equivalent to $D(X)$, it suffices to show that it contains the skyscraper sheaves or, if X is projective, the negative line bundles.

This provides an example of an inadmissible subcategory: the triangulated subcategory generated by skyscraper sheaves (by taking shifts and cones of morphisms). Its left orthogonal is zero, but it does not contain $\mathcal{O}_X$.

If A_1 and A_2 are admissible subcategories of $D(X)$ with $A_1 \subset A_2^{\perp}$ then the triangulated subcategory they generate is again admissible. A *semi-orthogonal decomposition* of $D(X)$ is a sequence $A_1, A_2, \dots, A_k$ of admissible subcategories with $A_i \subset A_j^{\perp}$ when

$i < j$ (that is, there are no Homs from right to left) that generates $D(X)$. We write

$$D(X) = \langle A_1, A_2, \dots, A_k \rangle.$$

Sometimes an individual object $E \in D(X)$ appears in a semi-orthogonal decomposition. This indicates that E is *exceptional*; that is,

$$\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise,*} \end{cases}$$

or equivalently, if we view E as an object of $D(\text{point} \times X)$ then the Fourier–Mukai transform $D(\text{point}) \to D(X)$ is an embedding. Thus the subcategory generated by E, which just consists of objects of the form $\bigoplus E[i]^{n_i}$, is an admissible subcategory, so it is reasonable for an exceptional object to appear in a semi-orthogonal decomposition. A sequence $E_1, E_2, \dots, E_k$ of exceptional objects is an *exceptional collection* if there are no Homs or Exts from right to left.

For example, let $X \subset \mathbb{P}^n$ be a complete intersection of hypersurfaces of degrees $d_1, \dots, d_k$ with $d_1 + \cdots + d_k \leq n$, so X is Fano. Then

$$\mathcal{O}_X(-n + d_1 + \cdots + d_k), \dots, \mathcal{O}_X(-1), \mathcal{O}_X \tag{2.5.1}$$

is an exceptional collection, as follows. We know that $\mathcal{O}_{\mathbb{P}^n}(t)$ has no cohomology for $-n \leq t \leq -1$ and that $H^*(\mathcal{O}_{\mathbb{P}^n}) = H^*(\text{point})$. From the Koszul complex of X

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d_1 - \cdots - d_k) \to \\ \cdots \to \bigoplus_{i<j} \mathcal{O}_{\mathbb{P}^n}(-d_i - d_j) \to \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

*In particular, exceptional objects are simple and rigid.

we find that $\mathcal{O}_X(t)$ has no cohomology for $-n+d_1+\cdots+d_k \leq t \leq -1$ and $H^*(\mathcal{O}_X) = H^*(\text{point})$. Thus $\operatorname{Ext}^*(\mathcal{O}_X(s), \mathcal{O}_X(t)) = H^*(\mathcal{O}_X(t-s))$ is as required for an exceptional collection. Observe that the most negative line bundle in (2.5.1) is $\omega_X(1)$, so we couldn't have gone any further: $\operatorname{Ext}^{n-k}(\mathcal{O}_X, \omega_X) = H^0(\mathcal{O}_X) = \mathbb{C}$.

We conclude with an easy proof of Beĭlinson's theorem

$$D(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle,$$

which we mentioned in §1.1. We just saw that this is an exceptional collection. To show that it generates $D(\mathbb{P}^n)$, we could observe that it generates the skyscraper sheaves, via the Koszul complex of a point

$$0 \to \mathcal{O}(-n) \to \cdots \to \mathcal{O}(-3)^{\binom{n}{3}} \to \mathcal{O}(-2)^{\binom{n}{2}} \to \mathcal{O}(-1)^n \to \mathcal{O} \to \mathcal{O}_{\text{point}} \to 0,$$

or we could observe that it generates the negative line bundles, via the Koszul complex of the complete intersection of $n+1$ hyperplanes (which is empty)

$$0 \to \mathcal{O}(-n-1) \to \cdots \to \mathcal{O}(-2)^{\binom{n+1}{2}} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$$

and its twists

$$0 \to \mathcal{O}(-n-2) \to \cdots \to \mathcal{O}(-3)^{\binom{n+1}{2}} \to \mathcal{O}(-2)^{n+1} \to \mathcal{O}(-1) \to 0$$
$$0 \to \mathcal{O}(-n-3) \to \cdots \to \mathcal{O}(-4)^{\binom{n+1}{2}} \to \mathcal{O}(-3)^{n+1} \to \mathcal{O}(-2) \to 0$$
$$\cdots.$$

Observe that this theorem implies that every complex of sheaves on $\mathbb{P}^n$ is quasi-isomorphic to a complex of those $n+1$ line bundles.

2.6 Matrix Factorizations

An irreducible polynomial $w \in R := \mathbb{C}[x_0, \ldots, x_n]$ cannot be factored as the product of two polynomials. But it may be possible to give a *matrix factorization* of w: a pair of $N \times N$ matrices φ and ψ of polynomials such that $\varphi \cdot \psi = \psi \cdot \varphi = w \cdot 1_{N\times N}$. For example,

$$\begin{pmatrix} x_2 & x_4 & x_6 & 0 \\ -x_3 & x_1 & 0 & x_6 \\ -x_5 & 0 & x_1 & -x_4 \\ 0 & -x_5 & x_3 & x_2 \end{pmatrix} \begin{pmatrix} x_1 & -x_4 & -x_6 & 0 \\ x_3 & x_2 & 0 & -x_6 \\ x_5 & 0 & x_2 & x_4 \\ 0 & x_5 & -x_3 & x_1 \end{pmatrix} = (x_1x_2 + x_3x_4 + x_5x_6) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{2.6.1}$$

is one of Langer's matrix factorizations mentioned in §2.2.4. The number N is called the *size* of the matrix factorization.

We can view a matrix factorization as a sequence of maps of free modules

$$\cdots \to R^N \xrightarrow{\varphi} R^N \xrightarrow{\psi} R^N \xrightarrow{\varphi} R^N \xrightarrow{\psi} R^N \to \cdots$$

which is like a complex except that we have $d^2 = w$ rather than $d^2 = 0$, or as a sequence of maps of vector bundles on $\mathbb{A}^{n+1}$

$$\cdots \to \mathcal{O}^N \xrightarrow{\varphi} \mathcal{O}^N \xrightarrow{\psi} \mathcal{O}^N \xrightarrow{\varphi} \mathcal{O}^N \xrightarrow{\psi} \mathcal{O}^N \to \cdots$$

where again $d^2 = w$.

If w is homogeneous of degree d, a *graded matrix factorization* is one where we require the free modules and maps to be graded:

$$\cdots \to \bigoplus_{i=1}^N R(m_i) \xrightarrow{\varphi} \bigoplus_{i=1}^N R(n_i) \xrightarrow{\psi} \bigoplus_{i=1}^N R(m_i + d) \xrightarrow{\varphi(d)} \bigoplus_{i=1}^N R(n_i + d) \to \cdots,$$

or on $\mathbb{P}^n$,

$$\cdots \to \bigoplus \mathcal{O}(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}(n_i) \xrightarrow{\psi} \bigoplus \mathcal{O}(m_i + d) \xrightarrow{\varphi(d)} \bigoplus \mathcal{O}(n_i + d) \to \cdots.$$

The example (2.6.1) above was in fact graded, with $m_i = -1$ and $n_i = 0$. We are interested in hypersurfaces in $\mathbb{P}^n$, so we will confine our interest to graded matrix factorizations.

Following Orlov [44] and Kontsevich, we can form a category $\mathrm{MF}_{\mathrm{gr}}(w)$ of graded matrix factorizations and chain maps between them. If $X \subset \mathbb{P}^n$ is the hypersurface defined by w, we will sometimes call this category $\mathrm{MF}_{\mathrm{gr}}(X)$. It is not an abelian category, because the kernel and cokernel of a map of vector bundles of the form $\bigoplus \mathcal{O}(m_i)$ may not be of the same form, but it is an exact category, which means essentially that we can say what a short exact sequence is. There is an exact functor $\operatorname{coker} : \mathrm{MF}_{\mathrm{gr}}(X) \to \mathrm{Coh}(X)$, sending a matrix factorization

$$\cdots \to \bigoplus \mathcal{O}(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}(n_i) \xrightarrow{\psi} \bigoplus \mathcal{O}(m_i + d) \xrightarrow{\varphi(d)} \bigoplus \mathcal{O}(n_i + d) \to \cdots .$$

to the sheaf $\operatorname{coker} \varphi$. This sheaf has many nice properties:

Proposition 2.6.1.

1. $\operatorname{coker} \varphi$ *is supported on* X*.*

2. *It has a two-term resolution on* $\mathbb{P}^n$

$$0 \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}_{\mathbb{P}^n}(n_i) \to \operatorname{coker} \varphi \to 0.$$

   Taking cohomology of this, we see immediately that $\operatorname{coker} \varphi$ *is arithmetically Cohen-Macaulay.*

3. *It has a resolution on* X *to the left*

$$\cdots \to \bigoplus \mathcal{O}_X(n_i - d) \xrightarrow{\psi(-d)} \bigoplus \mathcal{O}_X(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}_X(n_i) \to \operatorname{coker} \varphi \to 0$$

   and one to the right:

$$0 \to \operatorname{coker} \varphi \to \bigoplus \mathcal{O}_X(m_i + d) \xrightarrow{\varphi(d)} \bigoplus \mathcal{O}_X(n_i + d) \xrightarrow{\psi(d)} \bigoplus \mathcal{O}_X(m_i + 2d) \to \cdots .$$

4. *Its dual on X is* $\operatorname{coker}(\varphi^*(-d))$, *where*

$$\varphi^* : \mathcal{O}_{\mathbb{P}^n}(-n_i) \to \mathcal{O}_{\mathbb{P}^n}(-m_i)$$

is the transpose. This is the dual both in the ordinary and the derived senses; that is, $\mathcal{H}om(\operatorname{coker}\varphi, \mathcal{O}_X) = \operatorname{coker}(\varphi^*(-d))$, *and* $\mathcal{E}xt^i_X(\operatorname{coker}\varphi, \mathcal{O}_X) = 0$ *for* $i > 0$.

5. *It is reflexive.*

Proof.

1. Restrict

$$\bigoplus \mathcal{O}(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}(n_i) \to \operatorname{coker}\varphi \to 0$$

to $U := \mathbb{P}^n \setminus X$ to get

$$\bigoplus \mathcal{O}_U(m_i) \xrightarrow{\varphi|_U} \bigoplus \mathcal{O}_U(n_i) \to \operatorname{coker}\varphi|_U \to 0.$$

But $\varphi \cdot \psi = w \cdot 1$, and w does not vanish on U, so $\varphi|_U$ is an isomorphism, so $\operatorname{coker}\varphi|_U = 0$. Thus the support of $\operatorname{coker}\varphi$ is contained in X, and property 4 will show that it is all of X.

2. $\psi \circ \varphi = w \cdot 1$ is injective, so φ is injective.

3. We will show that the complex

$$\cdots \to \bigoplus \mathcal{O}_X(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}_X(n_i) \xrightarrow{\psi} \bigoplus \mathcal{O}_X(m_i + d) \to \cdots$$

is exact by showing that the corresponding complex of free (R/w)-modules is exact. It is enough to check exactness in one place. Let $x \in \bigoplus R(n_i)$ and let $\bar{x}$ be its residue class in $\bigoplus (R/w)(n_i)$, which corresponds to the middle vector bundle above. If $\psi\bar{x} = 0$ then $\psi x = wy$ for some $y \in \bigoplus R(m_i + d)$, so $wx = \varphi\psi x = \varphi w y = w\varphi y$, so $x = \varphi y$, so $\bar{x} = \varphi\bar{y}$.

4. We just saw that $\operatorname{coker}\varphi$ is quasi-isomorphic to the complex of vector bundles

$$\cdots \to \bigoplus \mathcal{O}_X(n_i - d) \xrightarrow{\psi(-d)} \bigoplus \mathcal{O}_X(m_i) \xrightarrow{\varphi} \bigoplus \mathcal{O}_X(n_i) \to 0,$$

whose dual is

$$0 \to \bigoplus \mathcal{O}_X(-n_i) \xrightarrow{\varphi^*} \bigoplus \mathcal{O}_X(-m_i) \xrightarrow{\psi^*(d)} \bigoplus \mathcal{O}_X(n_i - d) \to \cdots .$$

Since $\varphi^* \cdot \psi^*$ is again a graded matrix factorization of w, this is quasi-isomorphic to $\ker(\varphi^*) = \operatorname{coker}(\varphi^*(-d))$.

5. Because $\varphi^* \cdot \psi^*$ is a graded matrix factorization of w, we have $(\operatorname{coker}\varphi)^{**} = (\operatorname{coker}(\varphi^*)(-d))^* = \operatorname{coker}(\varphi^{**})(-d)(d) = \operatorname{coker}\varphi$. $\square$

In our example (2.6.1), $\operatorname{coker}\varphi$ and $\operatorname{coker}\psi$ are the two spinor bundles on Q^4, and property 2 above is (2.2.2).

We form the derived category $\mathrm{DMF}_{\mathrm{gr}}(X)$ from $\mathrm{MF}_{\mathrm{gr}}(X)$ by modding out chain homotopies.* It is a triangulated category. When $d \leq n+1$, so X is Fano or Calabi–Yau, Orlov shows that

$$D(X) = \langle \mathcal{O}_X(-n+d), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, \mathrm{DMF}_{\mathrm{gr}}(X) \rangle.$$

Notice that if $d = n+1$, so X is Calabi–Yau, the list of line bundles is empty, so in fact $D(X) = \mathrm{DMF}_{\mathrm{gr}}(X)$. Unfortunately the embedding $\mathrm{DMF}_{\mathrm{gr}}(X) \to D(X)$ is more complicated than the functor coker above.

Matrix factorizations were introduced by Eisenbud [18] in commutative algebra in 1980, but in recent years they have re-emerged in homological mirror symmetry. Indeed,

*The reader may wonder why we don't invert quasi-isomorphisms. It is because they don't make any sense for matrix factorizations: since $d^2 \neq 0$, we can't talk about cohomology, much less chain maps that induce isomorphisms on cohomology.

Orlov does not use the phrase "matrix factorization", instead calling $\mathrm{DMF}_{\mathrm{gr}}$ the category of "graded D-branes of type B in Landau–Ginzburg models".

2.7 Twisted Sheaves

Our reference in this section is Căldăraru's thesis [11].

If $\mathcal{E}$ is a holomorphic vector bundle on a complex manifold X, we can form its projectivization $\mathbb{P}\mathcal{E}$. Locally in the analytic topology, every projective bundle is the projectivization of a vector bundle, but globally this is not so.* To see the obstruction, consider the short exact sequence of groups

$$1 \to \mathbb{C}^* \to GL_r\mathbb{C} \to PGL_r\mathbb{C} \to 1$$

which gives rise the long exact sequence

$$\cdots \to H^1(X, \mathbb{C}^*) \to H^1(X, GL_r\mathbb{C}) \to H^1(X, PGL_r\mathbb{C}) \to H^2(X, \mathbb{C}^*).$$

What we mean here is cohomology of the sheaves of holomorphic functions with values in those groups,* so it would be better to call the first and last terms $H^i(X, \mathcal{O}^*)$.

From the map of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/r \arrow[r] \arrow[d] & SL_r\mathbb{C} \arrow[r] \arrow[d] & PSL_r\mathbb{C} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL_r\mathbb{C} \arrow[r] & PGL_r\mathbb{C} \arrow[r] & 1
\end{tikzcd}$$

we see that the obstruction for a rank $r-1$ projective bundle lies in the image of $H^2(X, \mathbb{Z}/r) \to H^2(X, \mathcal{O}^*)$, hence is torsion. It is an old theorem of Serre that topologically, every torsion element of $H^2(X, \mathcal{O}^*)$ occurs as the obstruction of some projective

*Surprisingly, if X is projective, a projective bundle that is locally trivial in the Zariski topology is the projectivization of a vector bundle, by Hartshorne's exercise II.7.10(c).

*For sheaves of non-abelian groups only H^0 and H^1 are defined, and the latter is only a pointed set, not a group. The long exact sequence extends as far as we have given it only for central extensions.

bundle, and a recent theorem of de Jong that this is also true holomorphically. But a class of order r need not be the obstruction of a bundle of rank $r-1$. The torsion subgroup of $H^2(X, \mathcal{O}^*)$ is called the Brauer group and denoted $\mathrm{Br}(X)$.

From the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$$

we see that topologically, and holomorphically if $H^{2,0}(X) = 0$, $\mathrm{Br}(X)$ is the torsion subgroup of $H^3(X, \mathbb{Z})$, which is where a K-theorist would be more accustomed to finding the twisting.

So a general projective bundle is the projectivization of an *α-twisted vector bundle*, $\alpha \in \mathrm{Br}(X)$, which means the following (up to a suitable notion of isomorphism): an open cover $\{U_i\}$ of X, vector bundles $\mathcal{E}_i$ on U_i, and isomorphisms on the pairwise intersections $\varphi_{ij} : \mathcal{E}_i|_{U_i \cap U_j} \to \mathcal{E}_j|_{U_i \cap U_j}$ that "don't quite match up" on the triple intersections—that is, $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} \in \mathrm{Aut}(\mathcal{E}_i|_{U_i \cap U_j \cap U_k})$ is not the identity, but rather multiplication by a non-vanishing function α_{ijk}, and the resulting cocycle represents $\alpha \in H^2(X, \mathcal{O}^*)$.

We define α-twisted sheaves similarly. We can take the direct sum of two α-twisted sheaves. We can take the tensor product of an α- and a β-twisted sheaf to get an $(\alpha \cup \beta)$-twisted sheaf. If $\mathcal{F}$ is α-twisted and $\mathcal{G}$ is β-twisted then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is $(\alpha^{-1} \cup \beta)$-twisted. If $f : Y \to X$, we can pull back an α-twisted sheaf to get an $f^*\alpha$-twisted sheaf and push forward an $f^*\alpha$-twisted sheaf to get an α-twisted sheaf. The category of α-twisted sheaves has enough injectives, and we can form its derived category. If $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$ then a complex $\mathcal{S}$ of $(\alpha \boxtimes \beta)$-twisted sheaves on $X \times Y$ gives a Fourier–Mukai transform

$$F_{\mathcal{S}}^{Y \to X} : D(Y, \beta^{-1}) \to D(X, \alpha)$$

and the embedding and equivalence criteria of §2.4 are still valid.

Twisted sheaves arise naturally in moduli problems. A moduli space $\mathcal{M}$ of stable sheaves on X is called *fine* if there is a *universal sheaf* on $X \times \mathcal{M}$: that is, one whose restriction to each slice $X \times [\mathcal{F}] \cong X$ is $\mathcal{F}$. But a moduli space of stable sheaves is not typically fine—locally analytically on $\mathcal{M}$, a universal sheaf always exists, and on pairwise intersections we can glue, but on triple intersections, there is no reason to expect the gluings to match up. A priori, over a slice $X \times [\mathcal{F}]$, the cocycle $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij}$ could take values in $\operatorname{Aut}(\mathcal{F})$, but since $\mathcal{F}$ is stable, it is simple, so in fact it only takes values in $\mathbb{C}^*$. Thus we get a $(1 \boxtimes \alpha)$-twisted pseudo-universal bundle on $X \times \mathcal{M}$, where $\alpha \in H^2(\mathcal{M}, \mathcal{O}^*)$.

For a second perspective on twisted sheaves, let us make a short foray into non-commutative varieties. If X is a Noetherian scheme and $\mathcal{A}$ a bundle of commutative $\mathcal{O}_X$-algebras then one can construct a scheme $\operatorname{Spec} \mathcal{A}$ over X, whose fiber over a point $x \in X$ is Spec of the fiber $\mathcal{A}|_x$. The category of coherent sheaves on $\operatorname{Spec} \mathcal{A}$ is equivalent to the category of finitely generated $\mathcal{A}$-modules. If $\mathcal{A}$ is not commutative then $\operatorname{Spec} \mathcal{A}$ does not exist, but we know what its category of coherent sheaves would be if it did: the category of finitely generated $\mathcal{A}$-modules. Thus we regard the pair $(X, \mathcal{A})$ as a "non-commutative scheme". Many properties of honest schemes can be translated into this context: for example, we say that $(X, \mathcal{A})$ is smooth if $\mathcal{A}$ has finite homological dimension.

By Morita theory, a ring R and a matrix algebra $M_r(R)$ over it have equivalent categories of modules. Similarly, if X is a scheme or complex manifold and $\mathcal{E}$ a vector bundle on X then $\mathcal{O}_X$ and the bundle of $\mathcal{O}_X$-algebras $\mathcal{H}om(\mathcal{E},\mathcal{E})$ have equivalent categories of modules. An *Azumaya algebra* is a bundle of $\mathcal{O}_X$-algebras that is locally (in the étale or analytic topology) isomorphic to a bundle of matrix algebras. A scheme or complex manifold equipped with an Azumaya algebra should be viewed as the mildest non-commutative cousin of X.

Since every automorphism of a matrix algebra is inner,

$$\mathrm{Aut}(M_r\mathbb{C}) = GL_r\mathbb{C}/\{\text{its center}\} = PGL_r\mathbb{C},$$

so Azumaya algebras are bundles with structure group $PGL_r\mathbb{C}$, just as projective bundles were. In fact, if $\mathcal{E}$ is an α-twisted vector bundle then $\mathcal{A} := \mathcal{H}om(\mathcal{E},\mathcal{E})$, which is untwisted, is an Azumaya algebra, and the category of α-twisted sheaves is equivalent to the category of $\mathcal{A}$-modules.

As we mentioned in §1.2, twisted sheaves are related to the B-field in string theory.

2.8 Ordinary Double Points

An *ordinary double point* (ODP) or *node* is a singularity of an algebraic variety that looks locally analytically like the affine cone on a smooth quadric. In a curve or a surface, the only way to resolve an ODP is to blow it up, but in a 3-fold we have more choices. A 3-fold ODP looks locally analytically like

$$\{xz = yw\} \subset \mathbb{C}^4,$$

the affine cone on $Q^2 = \mathbb{P}^1 \times \mathbb{P}^1$. We can blow up the singular point to produce the "big resolution" $\tilde{X} \to X$, which introduces an exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle

$\mathcal{O}(-1,-1)$. Or we can blow up one of the planes on $\{xy = zw\}$, such as $x = z = 0$ or $x = w = 0$, to produce a "small resolution", which introduces an exceptional curve $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)^2$. The two families of planes give different small resolutions, which we call $X_+ \to X$ and $X_- \to X$. If we blow up the exceptional line in a small resolution we obtain the big resolution:

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X_+ \arrow[dr] & & X_- \arrow[dl] \\ & X. & \end{tikzcd}$$

Any smooth 3-fold containing a $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)^2$ can play the role of X_+ in this diagram: we can contract the $\mathbb{P}^1$ to give a space X with an ODP, of which one small resolution is the original X_+ and the other is a different space X_-. This process of turning X_+ into X_- is an example of a *flop*. Bondal and Orlov [6] showed that such flops leave the derived category unchanged: $D(X_+) \cong D(X_-)$. Bridgeland [10] showed that this is true of all 3-fold flops. Two smooth Calabi–Yau 3-folds (or more generally two minimal models) that are birational can be connected by a sequence of flops, hence are derived equivalent.

An advantage of the small resolutions $\pi : X_\pm \to X$ is that they preserve the canonical class: $K_{X_\pm} = \pi^* K_X$. By contrast, the big resolution $\pi : \tilde{X} \to X$ has $X_{\tilde{X}} = \pi^* K_X + 2E$ where E is the exceptional divisor. This difference in the canonical classes is called the *discrepancy*; a *crepant resolution*[*] is one with no discrepancy, such as a small resolution. One is especially keen to preserve the canonical class when working with Calabi–Yau 3-folds, i.e. those with zero canonical class.

[*]This coinage is one of Miles Reid's characteristic groaners.

We should mention Kawamata's conjecture on K-equivalence and derived equivalence [29]. Two varieties X and Y are called K-equivalent if there is a smooth variety Z and maps

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

such that f and g are birational and $f^*K_X = g^*K_Y$. He conjectures that birational varieties X and Y are K-equivalent if and only if $D(X) \cong D(Y)$. Bridgeland's result on flops gives evidence for this conjecture.

A disadvantage of small resolutions is that we might start with a projective variety X and end up with a small resolution of its ODPs $\hat{X}$ that is not projective, or even Kähler.* To produce the small resolution, we worked locally analytically, blowing up a divisor that passed through the ODP. We can analytically continue this divisor to all of X and blow that up to get a projective variety, but if the analytically continued divisor wraps around and meets the ODP again, this blow-up is not the small resolution.

An obstruction to $\hat{X}$ being Kähler is a number called the *defect*, which arises as follows. If it is Kähler, the integral of a Kähler form ω over any exceptional line ℓ is positive, so the class $[\ell] \in H_2(\hat{X}, \mathbb{C})$ pairs positively with $[\omega] \in H^2(\hat{X}, \mathbb{C})$, hence is not zero. Let n be the number of ODPs, and consider the long exact sequence of the pair $(\hat{X}, \text{exceptional lines})$, which includes the following:

$$0 \to H_3(\hat{X}, \mathbb{C}) \to H_3(X, \mathbb{C}) \to \mathbb{C}^n \to H_2(\hat{X}, \mathbb{C}) \to H_2(X, \mathbb{C}) \to 0.$$

*Since $\hat{X}$ is birational to a projective variety, it is Moishezon—that is, its function field has the expected transcendence degree, namely 3—and for Moishezon manifolds, the notions of Kähler and projective are equivalent.

From this we see that $b_2(\hat{X}) = b_2(X) + \delta$ and $b_3(\hat{X}) = b_3(X) - n + \delta$ for some $0 \leq \delta \leq n$. This δ is the defect. If $\hat{X}$ is Kähler, we have argued that the map $\mathbb{C}^n \to H_2(\hat{X}, \mathbb{C})$ is non-zero, so $\delta > 0$. For hypersurfaces and double solids, δ is well-understood.

A *double solid* is a double cover of $\mathbb{P}^3$ branched over a surface of degree $2d$. Clemens [14] showed that when the branching surface has only ODPs, the defect can be calculated as follows. In the space of polynomials of degree $3d-4$, or indeed any degree, those that vanish at a point $p \in \mathbb{P}^3$ form a codimension 1 subspace. Thus one would expect that the codimension of the subspace of those that vanish on the ODPs of the branching surface would be the number n of ODPs, but in fact it is $n - \delta$. Taking cohomology of

$$0 \to \mathcal{I}_{\mathrm{ODPs}}(3d-4) \to \mathcal{O}_{\mathbb{P}^3}(3d-4) \to \mathcal{O}_{\mathrm{ODPs}}(3d-4) \to 0,$$

we find that $\delta = H^1(\mathcal{I}_{\mathrm{ODPs}}(3d-4))$.

Werner [52] showed that the defect of a hypersurface of degree d in $\mathbb{P}^4$ is the analogous number for polynomials of degree $2d-5$, and made a thorough study of the Kählerness of small resolutions of both hypersurfaces and double solids.

Chapter 3

Spinor Sheaves

In this chapter we will define spinor sheaves on singular quadrics, generalizing the spinor bundles on smooth quadrics that we discussed in §2.2.

Given a linear space Λ on a quadric Q defined by a polynomial q, we will construct a module over the Clifford algebra of q, and from this a matrix factorization of q, and from this two reflexive sheaves S and T on Q, which we will call spinor sheaves. None of these steps is new, but the composition is new. We treat smooth and singular quadrics uniformly, but even on smooth quadrics our construction has advantages over Ottaviani's, with which it is difficult to do homological algebra (as we saw in §2.2.3), Kapranov's, with which it is difficult to do geometry, and Langer's, with which it is difficult to vary the quadric in a family.

In §3.1 we give the details of the construction and see how close S and T are to being vector bundles. In §3.2 we describe how they depend on Λ. In §3.3 we study their dual sheaves. In §3.4 we study how they restrict to a hyperplane section of Q and pull back to a cone on Q, and prove an analogue of Horrocks' criterion. In §3.5 we show that they are stable when Λ is maximal and properly semi-stable otherwise.

3.1 The Construction

Let V be a complex vector space equipped with a quadratic form q of rank at least 2, so the corresponding quadric hypersuface $Q \subset \mathbb{P}V$ is reduced. Let $C\ell$ be the Clifford algebra of q, which can be defined either as a quotient of the tensor algebra

$$\begin{aligned} C\ell &= \mathcal{T}(V)/\langle v^2 = q(v)\rangle \\ &= \mathcal{T}(V)/\langle vv' + v'v = 2b(v, v')\rangle \end{aligned}$$

or as a deformation of the exterior algebra

$$C\ell = \bigwedge V \qquad\qquad v\xi = v \wedge \xi + v \lrcorner \xi.^{*} \tag{3.1.1}$$

In this chapter we will prefer the former description, and in the next the latter. If $\{v_1, \dots, v_n\}$ is a basis for V then

$$\{v_{i_1} \cdots v_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis for $C\ell$.

Given an isotropic subspace $W \subset V$, choose a basis $w_1, \dots, w_m$ and let I be the left ideal $I = C\ell \cdot w_1 \cdots w_m$. Since W is isotropic, choosing a different basis just rescales the generator $w_1 \cdots w_m$ by the determinant of the change-of-basis matrix, so I is independent of this choice. Since $C\ell$ is $\mathbb{Z}/2$-graded, we can write $I = I_{\text{ev}} \oplus I_{\text{odd}}$. We will always consider I in the category of graded $C\ell$-modules and maps that respect the grading.

From the module I, define a map of vector bundles

$$\begin{aligned} \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{ev}} &\xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{odd}} \\ v \otimes \xi \quad &\mapsto \quad 1 \otimes v\xi; \end{aligned}$$

*Here $v\lrcorner : \bigwedge^i V \to \bigwedge^{i-1} V$ is the anti-derivation determined by $v \lrcorner v' = b(v, v')$.

here we are regarding $\mathcal{O}_{\mathbb{P}V}(-1)$ as the tautological line bundle, that is, as a sub-bundle of $\mathcal{O}_{\mathbb{P}V} \otimes V$.* Define $\psi : \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\mathrm{odd}} \to \mathcal{O}_{\mathbb{P}V} \otimes I_{\mathrm{ev}}$ similarly. Then the compositions

$$\mathcal{O}_{\mathbb{P}V} \otimes I_{\mathrm{ev}} \xrightarrow{\varphi(1)} \mathcal{O}_{\mathbb{P}V}(1) \otimes I_{\mathrm{odd}} \xrightarrow{\psi(2)} \mathcal{O}_{\mathbb{P}V}(2) \otimes I_{\mathrm{ev}}$$
$$\mathcal{O}_{\mathbb{P}V} \otimes I_{\mathrm{odd}} \xrightarrow{\psi(1)} \mathcal{O}_{\mathbb{P}V}(1) \otimes I_{\mathrm{ev}} \xrightarrow{\varphi(2)} \mathcal{O}_{\mathbb{P}V}(2) \otimes I_{\mathrm{odd}}$$

are just multiplication by q, so we have a matrix factorization of q.

Finally, let $S = \operatorname{coker} \varphi$ and $T = \operatorname{coker} \psi$, which we understand fairly well from §2.6: They are reflexive sheaves on Q. They have resolutions on $\mathbb{P}V$

$$\begin{gathered} 0 \to \mathcal{O}_{\mathbb{P}V}^N(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}V}^N \to S \to 0 \\ 0 \to \mathcal{O}_{\mathbb{P}V}^N(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}V}^N \to T \to 0, \end{gathered} \tag{3.1.2}$$

where $N = \dim I_{\mathrm{ev}} = \dim I_{\mathrm{odd}} = 2^{\operatorname{codim} W - 1}$, from which it is easy to compute their cohomology. There are short exact sequences on Q

$$0 \to T(-1) \to \mathcal{O}_Q^N \to S \to 0$$
$$0 \to S(-1) \to \mathcal{O}_Q^N \to T \to 0.$$

We ask how far S and T are from being vector bundles. Let $K \subset V$ be the kernel of q and recall that the singular locus of Q is $\mathbb{P}K$.

Proposition 3.1.1. *The restriction of S to $\mathbb{P}K \cap \mathbb{P}W$ is trivial of rank $2^{\operatorname{codim} W - 1}$. If* $\operatorname{codim} W > 1$ *then elsewhere on Q, S is locally free of rank $2^{\operatorname{codim} W - 2}$. In particular, if Q is smooth then S is a vector bundle. The same is true of T.*

Proof. In the next section we will see that S and T are equivariant for the action of a group G_{ev} on Q that acts transitively on Q_{sm} when $\operatorname{rank} Q > 2$ and on the two

*It is interesting to note φ's resemblance to the Thom class in K-theory [35, Appendix C].

components of Q_{sm} when $\operatorname{rank} Q = 2$; so S and T are vector bundles there. Thus for all $v \in K$ and one v in each component of Q_{sm}, we want to know the rank of the linear maps $I_{\mathrm{ev}} \xrightarrow{v\cdot} I_{\mathrm{odd}}$ and $I_{\mathrm{odd}} \xrightarrow{v\cdot} I_{\mathrm{ev}}$ given by left multiplication by v.

If $v \in W \cap K$ then both maps are zero: any $v \in K$ commutes with elements of $C\ell_{\mathrm{ev}}$ and anti-commutes with elements of $C\ell_{\mathrm{odd}}$, and any $v \in W$ annihilates the generator $w_1 \cdots w_m$ of I, so any $v \in W \cap K$ annihiliates I.

If $v \notin W$, choose a basis for V starting with v and ending with $w_1, \ldots, w_m$. Then any element of I can be written uniquely as $(v\xi + \eta)w_1 \cdots w_m$ where v and $w_1, \ldots, w_m$ do not appear in ξ and η. Now

$$v \cdot (v\xi + \eta)w_1 \cdots w_m = v\eta w_1 \cdots w_m,$$

which is zero if and only if $\eta = 0$, so both maps have rank $2^{\operatorname{codim} W - 2}$. □

If $\operatorname{codim} W = 1$ then $Q = \mathbb{P}W \cup \mathbb{P}W'$, where W' is the other maximal isotropic subspace, and it is easy to check that $S = \mathcal{O}_{\mathbb{P}W}$ and $T = \mathcal{O}_{\mathbb{P}W'}$ when $\dim V$ is odd and vice versa when $\dim V$ is even.

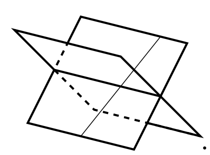
3.2 Dependence on the Subspace

In this section we will first summarize how S and T depend on W and are how they are related for Ws of various dimensions, then prove the corresponding statements about graded $C\ell$-modules, and finally show that the functor from modules to sheaves is fully faithful.

If q is non-degenerate then S and T are rigid, so varying W continuously leaves them unchanged. If $\dim W < \frac{1}{2}\dim V$, so W belongs to a connected family, then $S \cong T$. If

$\dim W = \frac{1}{2}\dim V$, so W belongs to one of two families, then $S \ncong T$, and switching W to the other family interchanges S and T. If W is maximal then when $\dim V$ is odd, $S \cong T$ is the spinor bundle discussed in §2.2, and when $\dim V$ is even, S and T are the two spinor bundles. If W' is codimension 1 in a maximal W then $S' \cong T' \cong S \oplus T$, if codimension 2 then $S' \cong T' \cong (S \oplus T)^{\oplus 2}$, and in general $S' \cong T' \cong (S \oplus T)^{\oplus 2^{\dim W/W' - 1}}$. In short then, on smooth quadrics, maximal linear spaces give the old spinor bundles, and non-maximal ones give direct sums of them.

If q is degenerate then S and T are not rigid in general, since by Proposition 3.1.1 we can recover $W \cap K$ from them; but varying W continuously while keeping $W \cap K$ fixed leaves them unchanged. Let $\pi : V \to V/K$ be the projection, and recall that q descends to a non-degenerate form on V/K. If $\dim \pi(W) < \frac{1}{2}\dim V/K$ then $S \cong T$. If $\dim \pi(W) = \frac{1}{2}\dim V/K$ then $S \ncong T$, and switching $\pi(W)$ to the other family (still keeping $W \cap K$ fixed) interchanges them. For example, consider a line $\mathbb{P}W$ on a corank 2 quadric surface:



If $\mathbb{P}W$ is not the cone line, it lies on one plane or the other and meets the cone line in a point, and these data determine the isomorphism classes of S and T: varying the line while keeping the point fixed leaves S and T unchanged, switching to the other plane interchanges them, and varying the point deforms them.

Our earlier comments on direct sums are generalized as follows: if W' is codimension 1 in W then there are exact sequences

$$0 \to S \to S' \to T \to 0 \qquad\qquad 0 \to T \to T' \to S \to 0 \tag{3.2.1}$$

which is split if and only if $W' \cap K = W \cap K$. That is, if $W \cap K$ shrinks we get interesting extensions, but if $\pi(W)$ shrinks we just get direct sums. So while on smooth quadrics only maximal Ws were interesting, on singular quadrics non-maximal Ws may be interesting, but only if $\pi(W)$ is maximal.

To prove all this, we introduce the following group action: Let G be the subgroup of the group of units $C\ell^\times$ generated by the unit vectors, that is, by those $u \in V$ with $q(u) = 1$, and let G act on V by reflections:

$$u \cdot v = -uvu^{-1} = v - 2b(v,u)u.$$

This preserves q, for

$$q(-uvu^{-1}) = (-uvu^{-1})^2 = uv^2u^{-1} = uq(v)u^{-1} = q(v).$$

The spinor sheaves S and T can be made equivariant for the action of $G_{\text{ev}} := G \cap C\ell_{\text{ev}}$ on Q, as we see from the commutative diagram

$$\begin{tikzcd}
 & \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{ev}} \arrow[r, "\varphi"] \arrow[ddd] & \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{odd}} \arrow[ddd] & \\
v \otimes \xi \arrow[d, mapsto] & & & 1 \otimes \xi \arrow[d, mapsto] \\
gvg^{-1} \otimes g\xi & & & 1 \otimes g\xi \\
 & \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{ev}} \arrow[r, "\varphi"] & \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{odd}}. &
\end{tikzcd} \tag{3.2.2}$$

If q is non-degenerate then G is $\operatorname{Pin}(V,q)$, the central extension of the orthogonal group $O(V,q)$ by $\mathbb{Z}/2$ [35, §I.2], and G_{ev} is $\operatorname{Spin}(V,q)$. When $m < \frac{1}{2}\dim V$, G_{ev} acts transitively on the variety of m-dimensional isotropic subspaces, and when $m = \frac{1}{2}\dim V$, G_{ev} acts transitively on each of its connected components and G_{odd} interchanges them.

If q is degenerate, the natural map $G \to O(V,q)$ is not surjective, since $O(V,q)$ acts transitively on K while G acts as the identity on K. If $U \subset V$ is a subspace complementary to K then $q|_U$ is non-degenerate, and G contains $\mathrm{Pin}(U, q|_U)$. From this it is not hard to see that G can take W to W' if $W \cap K = W' \cap K$, and that G_{ev} can if in addition $\pi(W)$ and $\pi(W')$ lie in the same family.

Now if $g \in G$ takes an isotropic subspace $W \subset V$ to another one $W' = gWg^{-1}$, then right multiplication by g^{-1} takes I to I':

$$C\ell \cdot w_1 \cdots w_m \cdot g^{-1} = C\ell \cdot (\pm g w_1 g^{-1}) \cdots (\pm g w_m g^{-1})$$

so if $g \in G_{\mathrm{ev}}$ then $I \cong I'$, and if $g \in G_{\mathrm{odd}}$ then $I \cong I'[1]$.* Thus we have proved:

Proposition 3.2.1.

- *If* $\dim \pi(W) < \frac{1}{2} \dim V/K$ *then* $I \cong I[1]$.
- *Suppose that* $W \cap K = W' \cap K$. *If* $\pi(W)$ *and* $\pi(W')$ *lie in the same family then* $I \cong I'$. *If they lie in opposite families then* $I \cong I'[1]$.

Inversely,

Proposition 3.2.2.

- *If* $\dim \pi(W) = \frac{1}{2} \dim V/K$ *then* $I \ncong I[1]$.
- *If* $W \cap K \neq W' \cap K$ *then* I *is not isomorphic to* I' *or* $I'[1]$.

Proof. For the first statement, let $\dim V/K = 2k$. Then there is a basis $e_1, \ldots, e_n$ of V in which

$$q = x_1 x_{k+1} + \cdots + x_k x_{2k}$$

*$I'[1]$ is just I' with the grading shifted by one, that is, with the odd and even pieces interchanged. In particular, $I'[2] = I'$.

and $W = \operatorname{span}(e_{k+1}, \ldots, e_{2k}, e_{2k+1}, \ldots, e_{2k+l})$, where $l = \dim(W \cap K)$. Observe that $\xi := e_1 \cdots e_k$ annihilates every element of the associated basis of I except $e_{k+1} \cdots e_{2k} e_{2k+1} \cdots e_{2k+l}$. Thus if $\dim W$ is even then ξ annihilates I_{odd} but not I_{ev}, and vice versa if $\dim W$ is odd. For the second statement, we saw in the proof of Proposition 3.1.1 that $W \cap K = V \cap \operatorname{Ann} I$, where the latter intersection takes place in $C\ell$. $\square$

Proposition 3.2.3. *Suppose that $W' \subset W$ is codimension 1. Then for any $w \in W \setminus W'$ the sequence*

$$0 \to I \to I' \xrightarrow{\cdot w} I[1] \to 0$$

is exact; it is split if and only if $W \cap K = W' \cap K$.

Proof. To see that the sequence is exact, choose a basis $w_1, \ldots, w_m$ for W', and extend this to a basis for V ending with $w, w_1, \ldots, w_m$; then just as we argued in the proof of Proposition 3.1.1, an element of I' can be written as $(\xi + \eta w) w_1 \cdots w_m$ where w and $w_1, \ldots, w_m$ do not appear in ξ and η, and if this times w equals zero then $\xi = 0$.

If $W \cap K = W' \cap K$ then $\pi(w) \notin \pi(W')$, so there is a $v \perp W'$ with $b(v, w) = \frac{1}{2}$. Since $v \perp W'$, we have $I \cdot v \subset I'$. Since $b(v, w) = \frac{1}{2}$, the map $I[1] \xrightarrow{\cdot v} I'$ splits $I' \xrightarrow{\cdot w} I[1]$:

$$\xi w w_1 \cdots w_m \cdot v w = \xi w w_1 \cdots w_m (1 - w v) = \xi w w_1 \cdots w_m.$$

Inversely, if $W \cap K \neq W' \cap K$ then I' and $I \oplus I[1]$ have different annihilators, hence are not isomorphic. $\square$

To see that what we have proved about modules implies what we have claimed about spinor sheaves, we study the functor that sends a graded $C\ell$-module M to a sheaf S on

Q. It is indeed a functor, for a homogeneous map $f : M \to M'$ induces a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}V}(-1) \otimes M_{\mathrm{ev}} \arrow[r] \arrow[d, "1 \otimes f_{\mathrm{ev}}"] & \mathcal{O}_{\mathbb{P}V} \otimes M_{\mathrm{odd}} \arrow[d, "1 \otimes f_{\mathrm{odd}}"] \\
\mathcal{O}_{\mathbb{P}V}(-1) \otimes M'_{\mathrm{ev}} \arrow[r] & \mathcal{O}_{\mathbb{P}V} \otimes M'_{\mathrm{odd}}
\end{tikzcd}$$

and hence a map on cokernels. It is exact. In fact, it is fully faithful:

Proposition 3.2.4. *The natural map* $\mathrm{Hom}_{C\ell}(M, M') \to \mathrm{Hom}_Q(S, S')$ *is an isomorphism.*

Proof. The inverse is essentially the map $\mathrm{Hom}_Q(S, S') \to \mathrm{Hom}_{\mathbb{C}}(M_{\mathrm{odd}}, M'_{\mathrm{odd}})$, where the second object is vector space homomorphisms, given by taking global sections. This is injective because S is generated by global sections. The composition $\mathrm{Hom}_{C\ell}(M, M') \to \mathrm{Hom}_{\mathbb{C}}(M_{\mathrm{odd}}, M'_{\mathrm{odd}})$ sends f to f_{odd}. This too is injective: a map f of graded $C\ell$-modules is determined by f_{odd}, for if $m \in M_{\mathrm{ev}}$ and $v \in V$ has $q(v) = 1$ then $f(m) = v^2 f(m) = vf(vm)$.

It remains to check that a linear map $M_{\mathrm{odd}} \to M'_{\mathrm{odd}}$ induced by a sheaf map $S \to S'$ is induced by a module map $M \to M'$. Applying $\mathrm{Hom}_Q(-, S')$ to (3.1.2), we have

$$0 \to \mathrm{Hom}_Q(S, S') \to M^*_{\mathrm{odd}} \otimes \Gamma(S') \xrightarrow{\varphi^*} M^*_{\mathrm{ev}} \otimes \Gamma(S'(1)).$$

Taking global sections of (3.1.2) and its twist by $\mathcal{O}(1)$, we can augment this to

$$
\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & & M_{\mathrm{ev}}^* \otimes M'_{\mathrm{ev}} \arrow[d, "\varphi'_*"] \\
 & & M_{\mathrm{odd}}^* \otimes M'_{\mathrm{odd}} \arrow[r, "\varphi^*"] \arrow[d, equal] & M_{\mathrm{ev}}^* \otimes M'_{\mathrm{odd}} \otimes V^* \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_Q(S, S') \arrow[r] & M_{\mathrm{odd}}^* \otimes \Gamma(S') \arrow[r, "\varphi^*"] & M_{\mathrm{ev}}^* \otimes \Gamma(S'(1)) \arrow[d] \\
 & & & 0
\end{tikzcd}
$$

where the bottom row and the right column are exact. Thus $\mathrm{Hom}_Q(S, S')$ is the set of $A \in \mathrm{Hom}_{\mathbb{C}}(M_{\mathrm{odd}}, M'_{\mathrm{odd}})$ for which there is a $B \in \mathrm{Hom}_{\mathbb{C}}(M_{\mathrm{ev}}, M'_{\mathrm{ev}})$ with $A\varphi = \varphi' B$; here we are thinking of A and B as matrices of complex numbers and φ and φ' as matrices of linear forms. Since φ'_* is injective, such a B is unique. Multiplying $A\varphi = \varphi' B$ by ψ on the right and ψ' on the left, we have $\psi' A q = q B \psi$, so $\psi' A = B\psi$. Thus

$$\begin{pmatrix} B & \\ & A \end{pmatrix} \begin{pmatrix} & \psi \\ \varphi & \end{pmatrix} = \begin{pmatrix} & \psi' \\ \varphi' & \end{pmatrix} \begin{pmatrix} B & \\ & A \end{pmatrix}.$$

Now $\left(\begin{smallmatrix} & \psi \\ \varphi & \end{smallmatrix}\right)$ is a matrix of linear forms, and plugging in any $v \in V$ we get the map $M \to M$ given by left multiplication by v. The vs generate $C\ell$, so $\left(\begin{smallmatrix} B & \\ & A \end{smallmatrix}\right)$ is in fact a homomorphism of $C\ell$-modules, not just of vector spaces. Since the matrix is block diagonal, it respects the grading. $\square$

3.3 The Dual

From §2.6, we know that the dual of S is $\operatorname{coker}\varphi^*(-2)$, so

$$0 \to \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\mathrm{odd}}^* \xrightarrow{\varphi^*(-1)} \mathcal{O}_{\mathbb{P}V} \otimes I_{\mathrm{ev}}^* \to S^*(1) \to 0 \tag{3.3.1}$$

where I_{odd}^* and I_{ev}^* are the dual vector spaces. This resolution makes us suspect that $S^*(1)$ is a spinor sheaf. In fact it is:

Proposition 3.3.1. *If* $\operatorname{codim} W$ *is odd then* $S^* \cong S(-1)$ *as* G_{ev}*-equivariant sheaves. If* $\operatorname{codim} W$ *is even then* $S^* \cong T(-1)$.

Proof. Let $^\top$ be the anti-automorphism of $C\ell$ determined by $(v_1 \cdots v_k)^\top = v_k \cdots v_1$. Then $I^\top$ is the right ideal $w_1 \cdots w_m \cdot C\ell$. The dual vector space I^* is a right $C\ell$-module via the action $(f \cdot \xi)(-) = f(\xi -)$. We will compare these two right modules.

The natural filtration of the tensor algebra descends to $C\ell$

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\dim V} = C\ell$$

and the associated graded pieces are $F_i/F_{i-1} = \bigwedge^i V$. In particular $C\ell/F_{\dim V - 1}$ is 1-dimensional, so by choosing a generator we get a linear form $\operatorname{tr} : C\ell \to \mathbb{C}$. The pairing $C\ell \otimes C\ell \to \mathbb{C}$ given by $\xi \otimes \eta \mapsto \operatorname{tr}(\xi\eta)$ is non-degenerate. If $v \in V$ and $\xi \in C\ell$ then $\operatorname{tr}(v\xi) = \pm \operatorname{tr}(\xi v)$.

I claim that I^* is generated by $\operatorname{tr}|_I$ and is isomorphic as an *ungraded* module to $I^\top$. Since $\dim I^* = \dim I^\top$, it suffices to check that $w_1 \cdots w_m$ and $\operatorname{tr}|_I$ have the same annihilator. If $\operatorname{tr}|_I \cdot \xi = 0$, that is, if $\operatorname{tr}(\xi\eta w_1 \cdots w_m) = 0$ for all $\eta \in C\ell$, then $\operatorname{tr}(w_1 \cdots w_m \xi\eta) = 0$ for all η, so $w_1 \cdots w_m \xi = 0$; and conversely.

Now $\operatorname{tr}|_I$ has degree $\dim V \pmod 2$, and $w_1 \cdots w_m$ has degree $\dim W \pmod 2$, so I^* is isomorphic to $I^\top$ or $I^\top[1]$ according as $\operatorname{codim} W$ is even or odd. So if $\operatorname{codim} W$ is even then (3.3.1) becomes

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{odd}}^{\top} & \to & \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{ev}}^{\top} \\ v \otimes \xi & \mapsto & 1 \otimes \xi v, \end{array}$$

so from the isomorphism

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{odd}} \arrow[r] \arrow[d, "1 \otimes {}^{\top}"'] & \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{ev}} \arrow[d, "1 \otimes {}^{\top}"] \\ \mathcal{O}_{\mathbb{P}V}(-1) \otimes I_{\text{odd}}^{\top} \arrow[r] & \mathcal{O}_{\mathbb{P}V} \otimes I_{\text{ev}}^{\top} \end{tikzcd}$$

we see that $S^*(1) \cong T$. Similarly, if $\operatorname{codim} W$ is odd then $S^*(1) \cong S$.

These isomorphisms are equivariant, as follows. In (3.2.2), an element $g \in G_{\text{ev}}$ acts on I by left multiplication by g, so it acts on I^* by right multiplication by g^{-1} and on $I^{\top}$ by right multiplication by $g^{\top}$. But from the definition of G we see that $g^{\top} g = 1$. $\square$

3.4 Linear Sections and Cones

In this section we are interested in restricting spinor sheaves to linear sections of Q and pulling them back to cones over Q, but in order to give the clearest proof, we begin with a too-general statement:

Lemma 3.4.1. *Suppose that a linear map $f : U \to V$ is transverse to W (that is, $W + \operatorname{im} f = V$). Let $Q' \subset \mathbb{P}U$ be the quadric cut out by f^*q, and let f also denote the rational map $Q' - - \to Q$, which is regular away from $\mathbb{P}(\ker f)$. Let S' be the spinor sheaf on Q' corresponding to $f^{-1}W$; then*

$$S'|_{(Q' \setminus \mathbb{P}(\ker f))} \cong \begin{cases} f^*S & \text{if } \dim U - \dim V \text{ is even} \\ f^*T & \text{if } \dim U - \dim V \text{ is odd.} \end{cases}$$

Proof. Note that transversality is equivalent to the condition $\operatorname{codim} f^{-1}W = \operatorname{codim} W$, which is clearly necessary. Since W can be moved without changing S, it is not much restriction.

Let I' be the $C\ell(U)$-module corresponding to $f^{-1}W$, and as usual I the $C\ell(V)$-module corresponding to W. Since f induces a map $C\ell(U) \to C\ell(V)$, I is also a $C\ell(U)$-module. Since pullback is right exact, we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}U}(-1) \otimes I_{\text{ev}} &\to \mathcal{O}_{\mathbb{P}U} \otimes I_{\text{odd}} \quad \to f^*S \to 0 \\ u \otimes \xi \quad &\mapsto \quad 1 \otimes f(u)\xi \end{aligned}$$

so it suffices to show that I is isomorphic as a $C\ell(U)$-module to $I'[\dim U - \dim V]$.

First suppose that f is injective. Choose a basis $u_1, \dots, u_l$ for $f^{-1}W$ and extend their images to a basis $f(u_1), \dots f(u_l), w_{l+1}, \dots, w_m$ of W. Then the map

$$I' = C\ell(U) \cdot f(u_1) \cdots f(u_l) \xrightarrow{\cdot w_{l+1} \cdots w_m} C\ell(V) \cdot f(u_1) \cdots f(u_l) \cdot w_{l+1} \cdots w_m = I[m-l]$$

is injective, and $\dim I = \dim I'$, so it is an isomorphism.

Next suppose that f is surjective. Any splitting $s : V \to U$ of f is compatible with the quadratic form, so by our previous argument $I \cong I'[\dim V - \dim U]$ as $C\ell(V)$-modules, and thus as $C\ell(U)$-modules.

In a general, f can be decomposed as a surjection followed by an injection. $\square$

If we wanted to generalize spinor bundles to singular quadrics knowing only that every singular quadric Q' is a cone over a smooth quadric Q, we would pull the spinor bundles from Q back to Q'_{sm} and then try to extend them across the singular locus; this lemma says that there is a unique extension, which is one of our spinor sheaves. Or if we knew only that every singular quadric Q is a linear section of a smooth quadric Q', we would restrict a spinor bundle on Q to Q'; this is another of our spinor sheaves. In particular, from our discussion of hyperplane sections in §2.2.2 we find that when q is non-degenerate and W is maximal, our S and T are indeed the usual spinor bundles. Also, we can generalize Ballico's Horrocks' criterion [2]:

Theorem. *Suppose that the singular locus Q_{sing} is at least codimension 3 in Q. Let W be any isotropic subspace and S and T the corresponding spinor sheaves. Then a vector bundle E on Q is a direct sum of line bundles if and only if E, $E \otimes S$, and $E \otimes T$ are ACM. (Recall that $S \cong T$ if $\operatorname{rank} Q$ is odd.)*

Proof. The "only if" statement is immediate from (3.1.2). For the "if" statement, observe that if $W \cap K = 0$ and $\pi(W)$ is maximal then by the previous proposition, S and T are exactly Ballico's vector bundles. If $W \cap K = 0$ and $\pi(W)$ is not maximal then S and T are direct sums of Ballico's bundles. Now induct on $\dim(W \cap K)$: if $W \cap K \neq 0$, choose a codimension 1 subspace $W' \subset W$ with $\pi(W') = \pi(W)$; then from the sequences (3.2.1) we see that if $E \otimes S$ and $E \otimes T$ are ACM then $E \otimes S'$ and $E \otimes T'$ are as well. □

3.5 Stability

In §2.3 we saw that any two spinor sheaves on Q have the same slope, and indeed the same reduced Hilbert polynomial. If W is maximal then we will show in a moment that S and T are stable. If W' is codimension 1 in W then (3.2.1) gives a Jordan–Hölder filtration $0 \subset S \subset S'$ with $S/0 = S$ and $S'/S = T$, so S' is properly semi-stable and S-equivalent to the polystable sheaf $S \oplus T$, as is T'. If W'' is codimension 1 in W' the Jordan–Hölder filtration is slightly more complicated, but S'' and T'' are S-equivalent to $S \oplus S \oplus T \oplus T$. In general the S-equivalence class of a spinor sheaf depends only on the dimension of the isotropic space.

To show that S and T are stable when W is maximal, we will need to know that they are simple:

Lemma 3.5.1. *If W is maximal then I is irreducible.*

Proof. If $\dim V/K = 2k$ is even, there is a basis $e_1, \dots, e_n$ of V in which

$$q = x_1 x_{k+1} + \cdots + x_k x_{2k}$$

and $W = \mathrm{span}(e_{k+1}, \dots, e_n)$. Let $\xi = e_{k+1} \cdots e_n$ be the generator of I. Then any $\xi' \in I$ different from zero is of the form

$$\xi' = \alpha e_{i_1} \cdots e_{i_l} \xi + \text{terms of the same or shorter length},$$

where $\alpha \in \mathbb{C}$ is not zero and $1 \leq i_1 < \cdots < i_l \leq k$. I claim that $e_{i_l+k} \cdots e_{i_1+k} \xi' = \alpha\xi$. To see this, observe that if $1 \leq i, j \leq k$ then e_i anti-commutes with e_{j+k} when $i \neq j$ and that $e_{i+k}\xi = 0$, so left multiplication by $e_{i_l+k} \cdots e_{i_1+k}$ annihilates any basis vector not containing $e_{i_1} \cdots e_{i_l}$; and $e_{i+k} e_i \xi = (1 - e_i e_{i+k})\xi = \xi$. Thus any non-zero element of I generates I, so I is irreducible.

If $\dim V/K = 2k + 1$ is odd, there is a basis $e_0, \dots, e_n$ of V in which

$$q = x_0^2 + x_1 x_{k+1} + \cdots + x_k x_{2k}$$

and $W = \mathrm{span}(e_{k+1}, \dots, e_n)$. Let $\xi = e_{k+1} \cdots e_n$ be the generator of I. Let $J \subseteq I$ be a graded submodule. By an argument similar to the one given above, for any non-zero $\xi' \in J$ there are $1 \leq i_1 < \cdots < i_l \leq k$ such that $e_{i_l} \cdots e_{i_1} \xi' = (\alpha + \beta e_0)\xi$, where $\alpha, \beta \in \mathbb{C}$

are not both zero. Since J is graded, it contains both $\alpha\xi$ and $\beta e_0\xi$. If $\alpha \neq 0$ then $\xi \in J$; if $\beta \neq 0$ then $e_0 \cdot \beta e_0 \xi = \beta\xi$, so again $\xi \in J$, so $J = I$. □

Proposition 3.5.2. *If W is maximal then S is simple, that is,* $\mathrm{Hom}_Q(S, S) = \mathbb{C}$*. If* $\dim V/K$ *is even,* $\pi(W)$ *is maximal in* V/K*, and* $W \cap K$ *is codimension 1 in* K*, then again* S *is simple. Otherwise* S *is not simple.*

Proof. The first statement is immediate from the previous lemma, Schur's lemma, and Proposition 3.2.4.

For the second statement, let $W' = W + K$. Let J be a proper submodule of I, and consider the short exact sequence

$$0 \to I' \to I \xrightarrow{\cdot v} I'[1] \to 0$$

where $v \in K \setminus W$. Since I' is irreducible, either $J \cap I' = 0$ or $J \supset I'$. If $J \cap I' = 0$ then Jv is isomorphic to J; since $I'[1]$ is irreducible, either $Jv = 0$, so $J = 0$, or $Jv = I'[1]$, so $I = I' \oplus J = I' \oplus I'[1]$, which we know is not true. If $J \supset I'$ then again either $Jv = 0$, so $J = I'$, or $Jv = I'[1]$, so $J = I$. Thus the only proper submodule of I is I', and the only proper quotient is $I'[1]$. Since these are not isomorphic, any homomorphism $I \to I$ is an isomorphism or zero, so again by Schur's lemma $\mathrm{Hom}_{C\ell}(I, I) = \mathbb{C}$.

For the third statement, if $\pi(W)$ is not maximal in V/K then S is a direct sum, hence is not simple. If $W \cap K$ is codimension 2 or more in K, choose $W'' \supset W' \supset W$ with $\pi(W'') = \pi(W') = \pi(W)$; then the composition $S \twoheadrightarrow T' \twoheadrightarrow S'' \hookrightarrow S' \hookrightarrow S$ is neither zero nor an isomorphism. If $W \cap K$ is codimension 1 in K and $\dim V/K$ is odd, let $W' = W + K$; then the composition $S \twoheadrightarrow T' \cong S' \hookrightarrow S$ is neither zero nor an isomorphism. □

Theorem. *Suppose that* $\operatorname{rank} Q > 2$*. If* W *is maximal then* S *is stable.*

Proof. Suppose that a subsheaf $\mathcal{F} \subset S$ is invariant under the action of G_{ev} introduced in §3.2. If $\operatorname{rank} Q > 2$ then G_{ev} acts transitively on the smooth locus Q_{sm}, so $\mathcal{F}$ is a vector bundle there. Let $p \in Q_{\mathrm{sm}}$ and $H \subset G_{\mathrm{ev}}$ be the stabilizer of p; then according to Ottaviani [45], the representation of H on the fiber $S|_p$ is irreducible (recall that G_{ev} contains $\mathrm{Spin}(U, q|_U)$ for any U complementary to K). Thus either $\operatorname{rank} \mathcal{F} = 0$, so $\mathcal{F} = 0$ since S is reflexive and hence pure, or $\operatorname{rank} \mathcal{F} = \operatorname{rank} S$.

Thus S has no invariant proper saturated subsheaves. The Harder–Narasimhan filtration is unique, hence invariant, so S is semi-stable. Similarly, the socle of S is unique, hence invariant, hence is S; that is, S is a direct sum of stable sheaves. But S is simple, hence indecomposable, so S is stable. $\square$

If $\operatorname{rank} Q = 2$ then Q is a union of hyperplanes H and H' and S is either $\mathcal{O}_H$ or $\mathcal{O}_{H'}$, which are torsion and thus not eligible for Mumford–Takemoto stability. But they have no proper saturated subsheaves and thus are Gieseker stable. We excluded from the beginning the non-reduced case $\operatorname{rank} Q = 1$.

Chapter 4

Intersections of Quadrics

In this chapter we will work with the following objects: V is a $2n$-dimensional complex vector space, $\Phi = \mathbb{P}\operatorname{Sym}^2 V^*$ is the space of quarics in $\mathbb{P}V$, $\Delta \subset \Phi$ is the hypersurface of singular quarics, and $\Delta' \subset \Delta$ is the locus of quadrics of corank at least 2. Recall from §2.1.4 that Δ is degree $2n$, Δ' is codimension 3 in Φ, and $\Delta_{\text{sing}} = \Delta'$.

Next, X is a complete intersection of two, three, or four quadrics in $\mathbb{P}V$, L is the line, plane, or 3-plane that those quadrics span in Φ, and $\mathcal{M}$ is the double cover of L branched $L \cap \Delta$. In §1.1 we said that X was "generic", but now we will be more precise: we require that L be transverse to Δ_{sm} and Δ', and that the linear space $L^\perp \subset \mathbb{P}\operatorname{Sym}^2 V$ of codimension two, three, or four be transverse to the Veronese embedding $\nu : \mathbb{P}V \hookrightarrow \mathbb{P}\operatorname{Sym}^2 V$.*

For two and three quadrics, L avoids Δ', so $L \cap \Delta$ is smooth, so $\mathcal{M}$ is smooth. For four quadrics, L meets Δ' in finitely many points.

Proposition 4.0.3. *A point $Q_0 \in L \cap \Delta'$ is an ordinary double point of $L \cap \Delta$.*

Proof. Since Q_0 is a smooth point of Δ', it is corank 2, so let ℓ be its cone line. Harris [21, Theorem 22.33] shows that the tangent cone to Δ at Q_0 is

$$TC_{Q_0}\Delta = \{Q \in \Phi : Q \text{ is tangent to } \ell\}.$$

*Since Δ is projectively dual to the image of ν, these requirements can probably be simplified.

Being tangent to ℓ, that is, having $Q \cap \ell$ be a degenerate quadric in ℓ, is a codimension 1 condition and is degree 2. Thus $TC_{Q_0}\Delta$ is a quadric in Φ. Its singular locus is

$$\{Q \in \Phi : Q \supset \ell\}$$

which is the tangent space to Δ' at Q_0. Since L is transverse to Δ', $TC_{Q_0}\Delta \cap L$ is a quadric cone in L with vertex Q_0, so indeed Q_0 is an ODP. □

Thus $\mathcal{M}$ has ordinary double points, so we may consider small resolutions $\hat{\mathcal{M}}$ of $\mathcal{M}$.

Proposition 4.0.4. *No small resolution $\hat{\mathcal{M}}$ of $\mathcal{M}$ is Kähler.*

Proof. By our discussion in §2.8, it suffices to show that $H^1(\mathcal{I}_{L\cap\Delta'}(3n-4)) = 0$. Józefiak [26] gives a very nice resolution of the structure sheaf of Δ', related to the Eagon–Northcott complex:

$$0 \to \mathcal{O}_\Phi(-2n-1)^{\binom{2n}{2}} \to \mathcal{O}_\Phi(-2n)^{4n^2-1} \to \mathcal{O}_\Phi(-2n+1)^{\binom{2n+1}{2}} \to \mathcal{O}_\Phi \to \mathcal{O}_{\Delta'} \to 0.$$

Since L meets Δ' transversely, this remains exact when restricted to L, so

$$0 \to \mathcal{O}_L(n-5)^{\binom{2n}{2}} \to \mathcal{O}_L(n-4)^{4n^2-1} \to \mathcal{O}_L(n-3)^{\binom{2n+1}{2}} \to \mathcal{I}_{L\cap\Delta'}(3n-4) \to 0$$

is exact. If X is not empty then $n \geq 3$, so we find that $H^1(\mathcal{I}_{\Delta'\cap L}(3n-4)) = 0$ as desired. □

In §4.1 we construct these small resolutions $\hat{\mathcal{M}} \to \mathcal{M}$ and an α-twisted pseudo-universal sheaf $\mathcal{S}$ on each. In §4.2 we show that $\mathcal{S}$ gives an embedding of $D(\hat{\mathcal{M}}, \alpha^{-1})$ in $D(X)$. In §4.3 we will finish the proof of our main theorem

$$D(X) = \langle \mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, D(\hat{\mathcal{M}}, \alpha^{-1})\rangle.$$

4.1 Construction of the Pseudo-Universal Bundle and Small Resolution

4.1.1 Varying the Quadric and Isotropic Subspace

Having defined spinor sheaves on one quadric and understood how they depend on a choice of isotropic space, we wish to study what happens in a family of quadrics.

First we vary the Clifford algebra. Define a bundle $\mathcal{A}$ of graded algebras on the space Φ of quadrics:

$$\begin{aligned}
\mathcal{A}_0 &= \mathcal{O}_\Phi \otimes \textstyle\bigwedge^0 V \\
\mathcal{A}_1 &= \mathcal{O}_\Phi \otimes \textstyle\bigwedge^1 V \\
\mathcal{A}_2 &= \mathcal{O}_\Phi \otimes \textstyle\bigwedge^2 V \;\oplus\; \mathcal{O}_\Phi(1) \otimes \textstyle\bigwedge^0 V \\
\mathcal{A}_3 &= \mathcal{O}_\Phi \otimes \textstyle\bigwedge^3 V \;\oplus\; \mathcal{O}_\Phi(1) \otimes \textstyle\bigwedge^1 V \\
\mathcal{A}_4 &= \mathcal{O}_\Phi \otimes \textstyle\bigwedge^4 V \;\oplus\; \mathcal{O}_\Phi(1) \otimes \textstyle\bigwedge^2 V \;\oplus\; \mathcal{O}_\Phi(2) \otimes \textstyle\bigwedge^0 V \\
&\cdots
\end{aligned}$$

The multiplication $\mathcal{A}_1 \otimes \mathcal{A}_1 \to \mathcal{A}_2$ is like (3.1.1) above, determined by

$$\begin{aligned}
V \otimes V &\to \textstyle\bigwedge^2 V & \qquad \mathcal{O}_\Phi(-1) \otimes V \otimes V &\to \mathcal{O}_\Phi \\
v \otimes v' &\mapsto v \wedge v' & q \otimes v \otimes v' &\mapsto q(v, v').
\end{aligned}$$

Observe that $\operatorname{rank} \mathcal{A}_0 = 1$, $\operatorname{rank} \mathcal{A}_1 = 2n$, and the ranks of the graded pieces grow for a while but eventually stabilize: for $k \geq 2n-1$ we have $\operatorname{rank} \mathcal{A}_k = 2^{2n-1}$ and $\mathcal{A}_{k+2} = \mathcal{A}_k(1)$. The fiber of $\mathcal{A}$ over a point $Q \in \Phi$ is not the Clifford algebra but the $\mathbb{Z}$-graded Clifford algebra that Kapranov considers in [27]:

$$\mathcal{A}|_Q = \mathcal{T}(V)[h]/\langle v^2 = q(v)h\rangle \qquad\qquad \deg h = 2.$$

This algebra is Koszul dual to the coordinate ring of Q. When $k \geq 2n-1$, its k^{th} graded piece is isomorphic to the odd or even piece of the usual Clifford algebra. More generally, if X is a complete intersection of quadrics $Q_1, \dots, Q_m$ and $L \subset \Phi$ is the linear space they span then $\Gamma(\mathcal{A}|_L)$ is the generalized $\mathbb{Z}$-graded Clifford algebra that Kapranov considers in [28]:

$$\Gamma(\mathcal{A}|_L) = \mathcal{T}(V)[h_1, \dots, h_m]/\langle v^2 = q_1(v)h_1 + \dots + q_m(v)h_m\rangle.$$

This is Koszul dual to the coordinate ring of X.

Next we vary the ideal. Let $\Phi' = \{(W, Q) \in G(n, V) \times \Phi : \mathbb{P}W \subset Q\}$ be the relative Grassmannian of isotropic $\mathbb{P}^{n-1}$s, which we discussed in §2.1.4, and $p : \Phi' \to \Phi$ be the projection. Let $\mathcal{I} \subset p^*\mathcal{A}$ be the bundle of left ideals which, over a point $(W, Q) \in \Phi'$, is generated by the line $\bigwedge^n W \subset \bigwedge^n V \subset \mathcal{A}_n|_Q$. If $k \geq 2n-1$, the fiber of $\mathcal{I}_k$ over a point (W, Q) is I_{odd} or I_{ev} from the previous chapter, and $\mathcal{I}_{k+2} = \mathcal{I}_k \otimes p^*\mathcal{O}_\Phi(1)$.

Last we vary the spinor sheaf: on $\mathbb{P}V \times \Phi'$, let $\mathcal{S}$ be the cokernel of the map

$$\mathcal{O}_{\mathbb{P}V}(-1) \boxtimes \mathcal{I}_{2n-1} \to \mathcal{O}_{\mathbb{P}V} \boxtimes \mathcal{I}_{2n}$$

given by $v \otimes \xi \mapsto v\xi$. Its restriction to a slice $\mathbb{P}V \times (W, Q)$ is the spinor sheaf from the previous chapter.

4.1.2 Restriction to the linear system L

Since $L^\perp$ is transverse to ν, X avoids the singularities of each $Q \in L$, so the restriction of $\mathcal{S}$ to $X \times p^{-1}L \subset \mathbb{P}V \times \Phi'$ is a vector bundle. We would like it to be the universal bundle for our moduli problem, but many points of $p^{-1}L$ parametrize the same bundle on X.

The branched cover $\mathcal{M}$ of L is the Stein factorization of $p^{-1}L \to L$:

$$
\begin{tikzcd}
 & p^{-1}L \arrow[r, hook] \arrow[dl] \arrow[dd] & \Phi' \arrow[dd, "p"] \\
\mathcal{M} \arrow[dr] & & \\
 & L \arrow[r, hook] & \Phi.
\end{tikzcd}
$$

If L is a line or a plane, the points in a fiber of $p^{-1}L \to \mathcal{M}$ all parametrize the same spinor sheaf, and points of different fibers parametrize different spinor sheaves, so what we want is a section of $p^{-1}L \to \mathcal{M}$. When L is a line there is a section; when L is a plane there are only local sections; when L is a 3-plane, these local sections will let us resolve the singularities of $\mathcal{M}$.

4.1.3 Two Quadrics

If L is a line, Reid [48] shows that X contains $\mathbb{P}^{n-2}$s; let Π be one of them. On a smooth $Q \in L$, there are two $\mathbb{P}^{n-1}$s containing Π, one from each family. On a singular Q there is only one, namely the span of Π and the cone point. Thus from Π we get a section of $p^{-1}L \to \mathcal{M}$. We obtain the desired universal bundle on $X \times \mathcal{M}$ by pulling back $\mathcal{S}$ from $\mathbb{P}V \times \Phi'$. Different choices of Π give us different universal bundles on $X \times \mathcal{M}$, but different universal bundles can only differ by the pullback of a line bundle from $\mathcal{M}$, which reflects Reid's result that the space of $\mathbb{P}^{n-2}$s on X is isomorphic to $\mathrm{Pic}^0(\mathcal{M})$.

4.1.4 Three Quadrics

If L is a plane then X does not generally contain a $\mathbb{P}^{n-2}$,* so we will not be able to produce a global section of $p^{-1}L \to \mathcal{M}$ as we did above, but we will be able to produce local sections, as follows. What was important about Π above was that it lay on every $Q \in L$ and that it avoided the cone points of the singular Qs. Now we will let $\Pi(Q)$ vary with Q. To be precise,

Lemma 4.1.1. *L can be covered by analytic open sets U on which there are maps $\Pi : U \to G(n-1, V)$ with $\Pi(Q) \subset Q_{sm}$ for each $Q \in U$.*

Proof. Let

$$\Phi'' = \{(W, Q) \in G(n-1, V) \times \Phi : \mathbb{P}W \subset Q\}$$

be the relative Grassmanian of isotropic $\mathbb{P}^{n-2}$s and $p' : \Phi'' \to \Phi$. I claim that p' is a submersion at a point (W, Q) if $\mathbb{P}W \subset Q_{\mathrm{sm}}$.

In §2.1.4 we saw that $\Phi'' \subset G \times \Phi$ is the zero set of a transverse section s of the vector bundle $\operatorname{Sym}^2 T^* \boxtimes \mathcal{O}_\Phi(1)$, where T is the tautological bundle on G. Thus there is an exact sequence

$$0 \to T_{(W,Q)}\Phi'' \to T_W G \oplus T_Q \Phi \xrightarrow{ds} \operatorname{Sym}^2 W^* \to 0.$$

By applying the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & T_{W,Q}\Phi' \arrow[r, equal] \arrow[d, hook] & T_{W,Q}\Phi' \arrow[r] \arrow[d, "dp"] & 0 \\
0 \arrow[r] & T_W G \arrow[r] & T_W G \oplus T_Q \Phi \arrow[r] & T_Q \Phi \arrow[r] & 0
\end{tikzcd}$$

*The case where $n = 3$ and X contains a line is much studied [38, 15, 25]; in that case the K3 surfaces X and $\mathcal{M}$ are birational and $\mathcal{M}$ is fine. For the four quadrics in $\mathbb{P}^7$, the Calabi–Yau 3-folds X and $\mathcal{M}$ are not birational in general but are so when X contains a plane [40].

we get an exact sequence

$$0 \to \ker dp \to T_W G \xrightarrow{ds} \operatorname{Sym}^2 W^* \to \operatorname{coker} dp \to 0$$

and see that dp is surjective if and only if $ds : T_W G \to \operatorname{Sym}^2 W^*$ is surjective. But by Proposition 2.1.1 this is true if and only if $\mathbb{P}W \subset Q_{\mathrm{sm}}$.

Now let $\Phi_0'' \subset \Phi''$ be the open subset on which p' is a submersion. By the implicit function theorem, $\Phi_0'' \to \Phi$ has local sections. The promised map Π is gotten by restricting such a local section to L and composing with the projection $\Phi'' \to G(n-1, V)$. (While in this proof it was convenient to talk about the subspace $W \subset V$, outside we will prefer to think projectively and talk about $\Pi(Q) = \mathbb{P}W \subset Q$.) $\square$

Now on a smooth $Q \in L$ there are two $\mathbb{P}^{n-1}$s containing $\Pi(Q)$, and on a singular Q there is only one, so we get local sections of $p^{-1}L \to \mathcal{M}$. If we pull back $\mathcal{I}_k$, $k \geq 2n-1$, from $\mathbb{P}V \times \Phi'$ to $U \times \mathcal{M}$ via these local sections, §3.2 tells us how to glue together the resulting local bundles on a pairwise intersection $U_i \cap U_j$, perhaps after shrinking the Us. In fact §3.2 glues $\mathcal{I}_k$ to $\mathcal{I}_{k+2l}$ for some $l > 0$, but these are just twists of each other, hence are locally isomorphic. But we could not have glued for $k < 2n-1$.

We cannot expect these gluings to match up on a triple intersection $U_i \cap U_j \cap U_{k'}$, so we do not get an honest bundle on $\mathcal{M}$ but only a twisted bundle. The spinor sheaves in question are simple by Proposition 3.5.2, so as we argued in §2.7, the twisting class α lies in $H^2(\mathcal{M}, \mathcal{O}_{\mathcal{M}}^*)$.

No confusion should result if this twisted bundle on $\mathcal{M}$ is also called $\mathcal{I}_k$. The map

$$\mathcal{O}_X(-1) \boxtimes \mathcal{I}_{2n-1} \to \mathcal{O}_X \boxtimes \mathcal{I}_{2n} \to \mathcal{S} \to 0,$$

is compatible with the gluing, so we also get a $(1 \boxtimes \alpha)$-twisted bundle $\mathcal{S}$ on $X \times \mathcal{M}$.

The order of α is at most two, as follows. In the proof of Proposition 3.3.1, the maps

$$C\ell \twoheadrightarrow I \qquad\qquad C\ell \twoheadrightarrow I^*$$

$$\xi \mapsto \xi w_1 \cdots w_m \qquad\qquad \xi \mapsto (\eta \mapsto \operatorname{tr}(\xi\eta))$$

had the same kernel, so $I^\top \cong I^*$ or $I^*[1]$. When we work with $\mathcal{A}$ and $\mathcal{I}$, these become

$$\mathcal{A}_{2n} \otimes \mathcal{O}_{G(n,V)}(1) \twoheadrightarrow \mathcal{I}_{3n} \qquad\qquad \mathcal{A}_{2n} \twoheadrightarrow \mathcal{H}om(\mathcal{I}_{2n}, \textstyle\bigwedge^{2n} V \otimes \mathcal{O}_\Phi(n))$$

so $\mathcal{I}_{2n}^*$ is isomorphic to $\mathcal{I}_{3n}$ twisted by a line bundle. The former is α^{-1}-twisted while the latter is α-twisted, so $\alpha = \alpha^{-1}$.

4.1.5 Four Quadrics

If L is a 3-plane then $\mathcal{M}$ has ODPs, and the points in the fiber of $p^{-1}L \to \mathcal{M}$ over a singular point of $\mathcal{M}$ do not all parametrize the same bundle on X, but our construction from the previous subsection will resolve both problems. Lemma 4.1.1 is still valid. On a smooth $Q \in L$ there are two $\mathbb{P}^{n-1}$s containing $\Pi(Q)$, on a corank 1 Q there is one, and on a corank 2 Q there is a whole line of them: for each point in the cone line of Q, we can take its span with $\Pi(Q)$.

Thus we have local *rational* sections of $p^{-1}L \to \mathcal{M}$ that are regular away from the singular points of $\mathcal{M}$.

Proposition 4.1.2. *The graph*

$$\Gamma = \{(W, Q) \in G(n-1, V) \times U : \Pi(Q) \subset \mathbb{P}W \subset Q\}$$

of such a rational section is smooth.

Proof. The map $\Pi : U \to G(n, V)$ determines a sub-bundle $\mathcal{E} \subset \mathcal{O}_U \otimes V$. Let $\mathcal{E}^\perp \subset \mathcal{O}_U \otimes V$ the fiberwise orthogonal. Because $\Pi(Q)$ avoids Q_{sing}, $\mathcal{E}^\perp$ is a vector bundle. Because $\Pi(Q) \subset Q$, $\mathcal{E} \subset \mathcal{E}^\perp$. Let P be the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}/\mathcal{E}^\perp)$ and $\pi : P \to U$.

The universal quadric in $\mathbb{P}V \times \Phi$ is the zero set of a transverse section s of $\mathcal{O}_{\mathbb{P}V}(2) \boxtimes \mathcal{O}_\Phi(1)$. Because L is transverse to the strata of $\Delta \subset \Phi$, s remains transverse when restricted to $\mathbb{P}V \times L$, and thence to $\mathbb{P}V \times U$. We will show that s remains transverse when restricted to $\mathbb{P}\mathcal{E}^\perp \setminus \mathbb{P}\mathcal{E} \subset U \times \mathbb{P}V$, and from there it descends to a section of $\mathcal{O}_P(2) \otimes \pi^*\mathcal{O}_U(1)$ on P, again transverse, with zero set Γ.

At a point $(l, Q) \in \mathbb{P}\mathcal{E}^\perp \setminus \mathbb{P}\mathcal{E} \subset \mathbb{P}V \times U$ there are exact sequences of tangent spaces

$$\begin{tikzcd}
0 \arrow[r] & T_l\Pi(Q)^\perp \arrow[r] \arrow[d, hook] & T_{(l,Q)}\mathbb{P}\mathcal{E}^\perp \arrow[r] \arrow[d, hook] & T_Q U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_l\mathbb{P}V \arrow[r] & T_l\mathbb{P}V \oplus T_Q U \arrow[r] & T_Q U \arrow[r] & 0.
\end{tikzcd}$$

Let f be the fiber of $\mathcal{O}_{\mathbb{P}V}(2) \boxtimes \mathcal{O}_U(1)$ over (l, Q). The map $ds : T_l\mathbb{P}V \oplus T_Q U \to f$ is surjective. Suppose that the composition $T_l\Pi(Q)^\perp \to f$ is zero. Then $l \subset \ker q|_{\Pi(Q)^\perp} = \ker q + \Pi(Q)$, but we assumed that $l \cap \Pi(Q) = 0$, so $l \subset \ker q$, so the map $T_l\mathbb{P}V \to f$ was also zero. Thus ds descends to a map $T_Q U \to f$ that is still surjective, so $T_{(l,Q)}\mathbb{P}\mathcal{E}^\perp \to f$ is surjective.

The projection $\mathbb{P}\mathcal{E}^\perp \setminus \mathbb{P}\mathcal{E} \to P$ gives

$$\begin{tikzcd}
0 \arrow[r] & T_l\Pi(Q)^\perp \arrow[r] \arrow[d] & T_{(l,Q)}\mathbb{P}\mathcal{E}^\perp \arrow[r] \arrow[d] & T_Q U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_l(\Pi(Q)^\perp/\Pi(Q)) \arrow[r] & T_{(\bar{l},Q)}P \arrow[r] & T_Q U \arrow[r] & 0
\end{tikzcd}$$

Now the section s over $\mathbb{P}\mathcal{E}^\perp$ descends to one over P, so ds descends to $T_{(\bar{l},Q)}P \to f$, which is surjective since the composite $T_{(l,Q)}\mathbb{P}\mathcal{E}^\perp \to T_{(\bar{l},Q)}P \to f$ is. $\square$

As we have said, $\Gamma \to \mathcal{M}$ contracts some lines over the singular points of $\mathcal{M}$, which are isolated, and is an isomorphism elsewhere. Thus we can cut out the ODPs of $\mathcal{M}$

and glue in Γ to produce a small resolution $\hat{\mathcal{M}} \to \mathcal{M}$. The two small resolutions of each ODP are again visible here: if $\Phi''_0 \subset \Phi''$ is as in the proof of Lemma 4.1.1, the fiber of $\Phi'' \to \Phi$ over a corank 2 quadric has two irreducible components, and the fiber of $\Phi''_0 \to \Phi$ (the map of which we took local sections) has two connected components.

Now the points of $\hat{\mathcal{M}}$ are in one-to-one correspondence with the isomorphism classes of spinor sheaves on X. As in the previous subsection we get twisted bundles $\mathcal{I}_k$ on $\hat{\mathcal{M}}$ for $k \geq 2n-1$, and $\mathcal{S}$ on $X \times \hat{\mathcal{M}}$.

Since twisted bundles can also be represented as modules over Azumaya algebras, it is natural to ask: if we take an Azumaya algebra on $\hat{\mathcal{M}}$ corresponding to our twisting α, is its pushforward to L the same as Kuznetsov's sheaf of algebras mentioned in §1.1?* The lemma below will satisfy our curiosity now, and will be indispensable in §4.3. Since $\mathcal{I}_k$, $k \geq 2n-1$, is α-twisted, $\mathcal{H}om_{\hat{\mathcal{M}}}(\mathcal{I}_k, \mathcal{I}_k)$ is a corresponding Azumaya algebra. It equals $\mathcal{H}om_{\hat{\mathcal{M}}}(\mathcal{I}_k, \mathcal{I}_{k+2n})(-n)$, whose pushforward to L we will see is $\mathcal{A}_{2n}(-n)|_L$, which is Kuznetsov's sheaf of algebras.

Lemma 4.1.3. *For $k, j \geq 2n-1$, let $\mathcal{H} = \mathcal{H}om_{\hat{\mathcal{M}}}(\mathcal{I}_k, \mathcal{I}_{k+j})$. Let $\pi : \hat{\mathcal{M}} \to L$ and consider the natural map $\pi^*(\mathcal{A}_j|_L) \to \mathcal{H}$. The adjunct map $\mathcal{A}_j|_L \to \pi_*\mathcal{H}$ is an isomorphism, and $R^i\pi_*\mathcal{H} = 0$ for $i > 0$.*

Proof. We will assume that k and j are even; the proof is entirely similar if one or both is odd. Over a smooth $Q \in L$ the result is clear, since if W and W' are maximal isotropic subspaces of opposite families, it is well-known that $C\ell_{\text{ev}} = \operatorname{End}_{\mathbb{C}}(I_{\text{ev}}) \oplus \operatorname{End}_{\mathbb{C}}(I'_{\text{ev}})$; but to show it in general will take some work.

*In that section I lied a little by saying that Kuznetsov worked with a sheaf of algebras on $\mathcal{M}$; in fact he worked with its pushforward to L.

First we show that $\pi_*\mathcal{H}$ is a vector bundle and that there is no higher pushforward. Let $Q \in L$ be corank 2 and $\ell = \pi^{-1}Q$, which is naturally identified with the cone line of Q. Let J be the ideal of the $\mathbb{P}^n$ spanned by $\Pi(Q)$ and the cone line. Then $\mathcal{O}_\ell \otimes J_{\mathrm{ev}} \subset \mathcal{I}_k|_\ell$. The map

$$\mathcal{O}_\ell(-1) \otimes \mathcal{I}_k|_\ell \to \mathcal{O}_\ell \otimes J_{\mathrm{odd}}$$

given by $v \otimes \xi \mapsto v\xi$ is surjective, and the kernel is $\mathcal{O}_\ell(-1) \otimes J_{\mathrm{ev}}$. Since there are no extensions of $\mathcal{O}_\ell$ by $\mathcal{O}_\ell(-1)$, we have

$$\mathcal{I}_k|_\ell = \mathcal{O}_\ell \otimes J_{\mathrm{ev}} \ \oplus \ \mathcal{O}_\ell(1) \otimes J_{\mathrm{odd}}.$$

Since $\mathcal{I}_{k+j}|_\ell = \mathcal{I}_k|_\ell$ and $\dim J_{\mathrm{ev}} = \dim J_{\mathrm{odd}} = 2^{n-2}$, we have $\mathcal{H}|_\ell = (\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell^2 \oplus \mathcal{O}_\ell(1))^{2^{2n-4}}$. Since $\dim H^0(\mathcal{H}|_\ell) = 2^{2n-2} = \operatorname{rank} \mathcal{H}$ and $H^1(\mathcal{H}|_\ell) = 0$, the pushforward of $\mathcal{H}$ from $\hat{\mathcal{M}}$ to $\mathcal{M}$ is a vector bundle and there is no higher pushforward. Since $\mathcal{M} \to L$ is flat and finite, the conclusion follows.

The question is local on L, so rather than working with $\pi : \hat{\mathcal{M}} \to L$ we work with $\pi : P \to U$ as in the proof of the previous proposition, and with $\Gamma \subset P$, which was the zero set of a transverse section s of $\mathcal{O}_P(2) \otimes \pi^*\mathcal{O}_U(1)$, and with the vector bundle $\mathcal{E} \subset \mathcal{O}_U \otimes V$.

Let $\mathcal{J} \subset \mathcal{A}|_U$ be the ideal generated by $\bigwedge^{n-1}\mathcal{E} \subset \mathcal{A}_{n-1}|_U$. Then $\mathcal{I}_k \subset \pi^*\mathcal{J}_k$. Dualizing, $\pi^*\mathcal{J}_k^* \twoheadrightarrow \mathcal{I}_k^*$. In this paragraph we will show that the adjunct map $\mathcal{J}_k^* \to \pi_*\mathcal{I}_k^*$ is an isomorphism. The sequence maps on P

$$\cdots \to \mathcal{O}_P(-1) \otimes \pi^*\mathcal{J}_{k-1} \to \mathcal{O}_P \otimes \pi^*\mathcal{J}_k \to \mathcal{O}_P(1) \otimes \pi^*\mathcal{J}_{k+1} \to \cdots$$

determined by $v \otimes \xi \mapsto v\xi$ is a matrix factorization of s, so its restriction to Γ is exact. The image in $\pi^*\mathcal{J}_k$ is $\mathcal{I}_k$, so by the yoga of matrix factorizations there is an exact sequence

$$0 \to \mathcal{O}_P(-1) \otimes \pi^* \mathcal{J}_{k+1}^* \to \mathcal{O}_P \otimes \pi^* \mathcal{J}_k^* \to \mathcal{I}_k^* \to 0.$$

Pushing this down to U, we find that $\mathcal{J}_k^* = \pi_* \mathcal{I}_k^*$, as claimed.

On Γ we have a diagram of vector bundles

$$\begin{tikzcd} \pi^*(\mathcal{A}_j|_U) \arrow[r] \arrow[d] & \mathcal{I}_k^* \otimes \mathcal{I}_{k+j} \arrow[d] \\ \pi^*(\mathcal{J}_k^* \otimes \mathcal{J}_{k+j}) \arrow[r] & \mathcal{I}_k^* \otimes \pi^* \mathcal{J}_{k+j}. \end{tikzcd}$$

On U, we have the adjunct

$$\begin{tikzcd} \mathcal{A}_j|_U \arrow[r] \arrow[d] & \pi_*(\mathcal{I}_k^* \otimes \mathcal{I}_{k+j}) \arrow[d] \\ \mathcal{J}_k^* \otimes \mathcal{J}_{k+j} \arrow[r] & \pi_* \mathcal{I}_k^* \otimes \mathcal{J}_{k+j}. \end{tikzcd}$$

The right vertical map is injective on each fiber, and we have just seen that the bottom map is an isomorphism, so the top map is injective on each fiber. Since $\mathcal{A}_j|_U$ and $\pi_*(\mathcal{I}_k^* \otimes \mathcal{I}_{k+j})$ have the same rank, it is an isomorphism, as claimed. $\square$

4.1.6 Five or More Quadrics

If L is a 4-plane, the singular locus of $\mathcal{M}$ is a curve C, which we can identify with its image in L. To construct a small resolution as above, for each corank 2 $Q \in C$ we would have to choose (continuously) one of the two families of $\mathbb{P}^{n-2}$s on Q_{sm}, or equivalently one of the two families of $\mathbb{P}^n$s on Q. But this is impossible; the associated double cover of C has no section, as follows. Consider $\{(\Lambda, Q) \in G(n+1, V) \times \Phi : \Lambda \subset Q\}$, the relative Grassmannian of isotropic $\mathbb{P}^n$s, which is smooth and connected. The image of the projection to Φ is the locus Δ' of quadrics of corank at least 2, since a corank 1 quadric contains only $\mathbb{P}^{n-1}$s. Since Δ' is codimension 3, the preimage of a general 4-plane $L \subset \Phi$ is irreducible by Bertini's theorem.

Not only does our construction of a small resolution of $\mathcal{M}$ fail, but $\mathcal{M}$ has no small resolution whatsoever, as follows. Let $Q \in C$ be any corank 2 quadric, and choose a general 3-plane $L' \subset L$ through Q. The preimage of this in $\mathcal{M}$ is a 3-fold with ODPs; the two small resolutions of each are in natural bijection with the two families of $\mathbb{P}^n$s on Q. Thus a small resolution of $\mathcal{M}$ would give a continuous choice of a family of $\mathbb{P}^n$s on each $Q \in C$, which we just saw is impossible.

A small resolution of $\mathcal{M}$ when $\dim L \geq 5$ would give one when $\dim L = 4$, so this too is impossible. Note that when $\dim L \geq 7$ the singular locus of $\mathcal{M}$ is no longer smooth.

4.2 Embedding of $D(\hat{\mathcal{M}}, \alpha^{-1})$

In this section we will show that the Fourier–Mukai transform

$$F_{\mathcal{S}} : D(\hat{\mathcal{M}}, \alpha^{-1}) \to D(X)$$

is an embedding when $n \geq 4$, where $\mathcal{S}$ is the α-twisted sheaf on $X \times \hat{\mathcal{M}}$ constructed in the §4.1.5. By our discussion in §2.4, it suffices to show that:

- If S_1 and S_2 are distinct spinor sheaves on quadrics $Q_1, Q_2 \in L$ then $\mathrm{Ext}^i_X(S_1|_X, S_2|_X) = 0$ for all i.

- If S is a spinor sheaf on a quadric $Q \in L$ then $\mathrm{Hom}_X(S|_X, S|_X) = \mathbb{C}$ and $\mathrm{Ext}^i_X(S|_X, S|_X) = 0$ for $i < 0$ and $i > 3$.

4.2.1 Hom and Ext between spinor sheaves from different quadrics

First suppose that S_1 and S_2 are spinor sheaves on different quadrics Q_1 and Q_2. We have resolutions

$$0 \to \mathcal{O}_{\mathbb{P}V}^N(-2) \to \mathcal{O}_{\mathbb{P}V}^N(-1) \to S_1^* \to 0$$
$$0 \to \mathcal{O}_{\mathbb{P}V}^N(-1) \to \mathcal{O}_{\mathbb{P}V}^N \to S_2 \to 0.$$

Choose $Q_3, Q_4 \in L$ transverse to each other and to Q_1 and Q_2. Since $Q_1 \cap \cdots \cap Q_4 = X$, $Q_3 \cap Q_4$ avoids the points where S_1 and S_2 fail to be vector bundles. Restrict one of the resolutions above to $Q_3 \cap Q_4$ and tensor it with the other to get a resolution

$$0 \to \mathcal{O}_{Q_3\cap Q_4}^{N^2}(-3) \to \mathcal{O}_{Q_3\cap Q_4}^{2N^2}(-2) \to \mathcal{O}_{Q_3\cap Q_4}^{N^2}(-1) \to (S_1^* \otimes S_2)|_X \to 0.$$

We required $n \geq 4$, so from our discussion around (2.5.1) we know that $\mathcal{O}_{Q_3\cap Q_4}(t)$ has no cohomology for $-3 \leq t \leq -1$, so $\mathrm{Ext}_X^*(S_1|_X, S_2|_X) = H^*((S_1^* \otimes S_2)|_X) = 0$.

4.2.2 Hom between spinor sheaves from the same quadric

Next suppose that S and S' are two spinor bundles on the same quadric Q_1. From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}V}^N(-1) \to \mathcal{O}_{\mathbb{P}V}^N \to S' \to 0.$$

we see that

$$0 = H^*(S'(-1)) = H^*(S'(-2)) = \cdots = H^*(S'(-2n+2)). \qquad (4.2.1)$$

Applying $\mathrm{Hom}_{Q_1}(-, S')$ to the exact sequence

$$0 \to S(-2) \to \mathcal{O}_Q^N(-1) \to \mathcal{O}_Q^N \to S \to 0$$

and its twists we see that

$$\mathrm{Ext}^i_{Q_1}(S, S') = \mathrm{Ext}^{i+2}_{Q_1}(S, S'(-2)) = \cdots = \mathrm{Ext}^{i+2n-2}_{Q_1}(S, S'(-2n+2)). \tag{4.2.2}$$

In particular,

$$0 = \mathrm{Ext}^{<2}_{Q_1}(S, S'(-2)) = \mathrm{Ext}^{<4}_{Q_1}(S, S'(-4)) = \mathrm{Ext}^{<6}_{Q_1}(S, S'(-6)).$$

Choose $Q_2, Q_3, Q_4 \in L$ transverse to each other and to Q_1. Tensor S' with the Koszul complex of $Q_2 \cap Q_3 \cap Q_4$ to get

$$0 \to S'(-6) \to S'(-4)^3 \to S'(-2)^3 \to S' \to S'|_X \to 0.$$

Applying $\mathrm{Hom}(S, -)$ and using the facts above, we find that $\mathrm{Hom}(S, S'|_X) = \mathrm{Hom}(S, S')$. The spinor sheaves considered here have $\mathrm{Hom}(S, S) = \mathbb{C}$ by Proposition 3.5.2, and from its proof it is immediate that $\mathrm{Hom}(S, S') = 0$ if $S \neq S'$. Of course $\mathrm{Hom}(S|_X, S'|_X) = \mathrm{Hom}(S, S'|_X)$.

4.2.3 Ext between different spinor sheaves from the same quadric

If S and S' are distinct spinor bundles on Q_1 then either they are both vector bundles or they fail to be so at distinct points, so we can let $E = S^* \otimes S'$ and rewrite (4.2.2) as

$$H^i(E) = H^{i+2}(E(-2)) = \cdots = H^{i+2n-2}(E(-2n+2)).$$

Above we saw that $H^0(E) = 0$, and since $\dim Q_1 = 2n-2$, we see that $E, E(-2), \ldots,$ $E(-2n+2)$ have no cohomology. Thus

$$0 \to E(-6) \to E(-4)^3 \to E(-2)^3 \to E \to E|_X \to 0$$

is a resolution of $E|_X$ by sheaves with no cohomology, so $\mathrm{Ext}^*_X(S|_X, S'|_X) = H^*(E|_X) = 0$.

4.2.4 Ext of a spinor sheaf with itself

If S on Q_1 fails to be a vector bundle at some point, choose a $Q_2 \in L$ that is transverse to Q_1 and avoids that point. If we now let $E = (S^* \otimes S')|_{Q_1 \cap Q_2}$, we only get

$$H^i(E) = H^{i+2}(E(-2)) = \cdots = H^{i+2n-4}(E(-2n+4)).$$

Since $\dim Q_1 \cap Q_2 = 2n-3$, we have

$$H^{>1}(E) = H^{>3}(E(-2)) = H^{>5}(E(-4)) = 0.$$

Thus from

$$0 \to E(-4) \to E^2(-2) \to E \to E|_X \to 0$$

we find that $\mathrm{Ext}^i_X(S|_X, S|_X) = H^i(E|_X) = 0$ for $i > 3$.

4.3 Semi-Orthogonal Decomposition of $D(X)$

Recall that our goal is to prove that

$$D(X) = \langle \mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X(-1), \mathcal{O}_X, F_{\mathcal{S}} D(\hat{\mathcal{M}}, \alpha^{-1}) \rangle.$$

If we were only interested in the Calabi–Yau case $n = 4$ then by the equivalence criterion discussed in §2.4 we would be done, but in the Fano case $n > 4$ there is a little more work to do.

4.3.1 Semi-Orthogonality

From our discussion around (2.5.1), we know that $\mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X$ is an exceptional collection.

Let S be a spinor sheaf on a quadric $Q_1 \supset X$ and choose $Q_2, Q_3, Q_4 \in L$ transverse to each other and to Q_1. We know that $\mathcal{O}_{Q_2 \cap Q_3 \cap Q_4}(t)$ has no cohomology for $-2n+7 \leq t \leq -1$. From the resolution

$$0 \to \mathcal{O}^N_{Q_2 \cap Q_3 \cap Q_4}(-2) \to \mathcal{O}^N_{Q_2 \cap Q_3 \cap Q_4}(-1) \to S^*|_X \to 0,$$

we see that $S^*(t)|_X$ has no cohomology for $-2n+9 \leq t \leq 0$.

Now $F_{\mathcal{S}} D(\hat{\mathcal{M}}, \alpha^{-1}) \perp \mathcal{O}_X(t)$ for $-2n+9 \leq t \leq 0$, as follows. Let G be the right adjoint to $F_{\mathcal{S}}$. If $p \in \hat{\mathcal{M}}$ then $F_{\mathcal{S}} \mathcal{O}_p$ is a spinor sheaf S restricted to X, so

$$0 = H^i(S^*(t)|_X) = \operatorname{Hom}(F_{\mathcal{S}} \mathcal{O}_p, \mathcal{O}_X(t)[i]) = \operatorname{Hom}(\mathcal{O}_p, G\mathcal{O}_X(t)[i]).$$

Since $G\mathcal{O}_X(t)$ is right orthogonal to all the skyscraper sheaves, by Serre duality it is left orthogonal to them, hence is 0. Thus for any object $B \in D(\hat{\mathcal{M}}, \alpha)$ we have $\operatorname{Hom}(F_{\mathcal{S}} B, \mathcal{O}_X(t)) = \operatorname{Hom}(B, G\mathcal{O}_X(t)) = 0$.

4.3.2 Generation

By §2.5, it suffices to show that $\mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X, D(\hat{\mathcal{M}}, \alpha^{-1})$ generate $\mathcal{O}_X(t)$ for all $t < -2n+9$.

Recall that $\mathcal{I}_k$, $k \geq 2n-1$ are α-twisted sheaves on $\hat{\mathcal{M}}$. We will take the Fourier–Mukai transform of $\mathcal{I}_{2n}^*(n-4)$. We will want to have this diagram visible:

$$\begin{tikzcd}
 & & X \times \hat{\mathcal{M}} \arrow[dl] \arrow[dr] & \\
 & X \times L \arrow[dl] \arrow[dr] & & \hat{\mathcal{M}} \arrow[dl] \\
X & & L. &
\end{tikzcd}$$

The kernel $\mathcal{S}$ on $X \times \hat{\mathcal{M}}$ is quasi-isomorphic to the complex

$$0 \to \mathcal{O}_X(1) \boxtimes \mathcal{I}_{2n+1} \to \mathcal{O}_X(2) \boxtimes \mathcal{I}_{2n+2} \to \mathcal{O}_X(3) \boxtimes \mathcal{I}_{2n+3} \to \cdots .$$

If $k \geq 2n-1$ and $l \geq 0$ then $\mathcal{I}_k(l) = \mathcal{I}_{k+2l}$, so tensoring with the pullback of $\mathcal{I}_{2n}^*(n-4)$ gives

$$0 \to \mathcal{O}_X(1) \boxtimes \mathcal{H}om_{\hat{\mathcal{M}}}(\mathcal{I}_{2n}, \mathcal{I}_{4n-1})(-3) \to \mathcal{O}_X(2) \boxtimes \mathcal{H}om_{\hat{\mathcal{M}}}(\mathcal{I}_{2n}, \mathcal{I}_{4n})(-3) \to \cdots .$$

By Lemma 4.1.3, pushing down to $X \times L$ gives

$$0 \to \mathcal{O}_X(1) \boxtimes \mathcal{A}_{2n-1}(-3)|_L \to \mathcal{O}_X(2) \boxtimes \mathcal{A}_{2n}(-3)|_L \to \cdots . \tag{4.3.1}$$

Now on $X \times \Phi$, the complex of vector bundles

$$0 \to \mathcal{O}_X \boxtimes \mathcal{A}_0 \to \mathcal{O}_X(1) \boxtimes \mathcal{A}_1 \to \mathcal{O}_X(2) \boxtimes \mathcal{A}_2 \to \cdots$$

is exact, as can be seen fiberwise. Thus (4.3.1) is quasi-isomorphic to

$$0 \to \mathcal{O}_X(-2n+2) \boxtimes \mathcal{A}_0(-3)|_L \to \cdots \to \mathcal{O}_X \boxtimes \mathcal{A}_{2n-2}(-3)|_L \to 0.$$

Recall that each $\mathcal{A}_i$ is a sum of line bundles $\mathcal{O}_\Phi(t)$ with $t \geq 0$, so $\mathcal{A}_i(-3)|_L$ is Γ-acyclic, so pushing this complex down to X gives

$$0 \to \mathcal{O}_X(-2n+2) \otimes \Gamma(\mathcal{A}_0(-3)|_L) \to \cdots \to \mathcal{O}_X \otimes \Gamma(\mathcal{A}_{2n-2}(-3)|_L) \to 0.$$

But $\Gamma(\mathcal{A}_i(-3)|_L) = 0$ for $i < 6$ and $\Gamma(\mathcal{A}_6(-3)|_L) = \mathbb{C}$, so in fact $F_{\mathcal{S}}(\mathcal{I}^*_{2n}(n-4))$ is quasi-isomorphic to

$$0 \to \mathcal{O}_X(-2n+8) \to \mathcal{O}_X(-2n+9) \otimes \Gamma(\mathcal{A}_7(-3)|_L) \to \cdots \to \mathcal{O}_X \otimes \Gamma(\mathcal{A}_{2n-2}(-3)|_L) \to 0.$$

Thus from $F_{\mathcal{S}}(\mathcal{I}^*_{2n}(n-4))$ and $\mathcal{O}_X(-2n+9), \ldots, \mathcal{O}_X$ we can generate $\mathcal{O}_X(-2n+8)$. Similarly, from $F_{\mathcal{S}}(\mathcal{I}^*_{2n+1}(n-4))$ we can generate $\mathcal{O}_X(-2n+7)$, and similarly all the negative line bundles, so we are done.

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