Analysis: Qual. Review

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1.1 Theorems and Definitions.

1.1.1 Basic Measure Theory

**Definition.** Let $X$ be a set. A **topology on** $X$ is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in \mathcal{T}$.
2. If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.
3. If $\{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called **open sets**.

**Definition.** Let $X$ be a set. A **σ-algebra on** $X$ is a set $\mathcal{M} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in \mathcal{M}$.
2. If $E \in \mathcal{M}$, then $X \setminus E \in \mathcal{M}$.
3. If $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$, then $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}$.

Elements of $\mathcal{M}$ are called **measurable sets**.

**Definition.** Let $X$ be a set and $\mathcal{M}$ a σ-algebra on $X$. The pair $(X, \mathcal{M})$ is called a **measurable space**. A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$.
2. If $E_1, E_2, \ldots$ are pairwise disjoint elements of $\mathcal{M}$, then

$$\mu \left( \bigcup_{n \in \mathbb{N}} E_n \right) = \sum_{n \in \mathbb{N}} \mu(E_n).$$

The triple $(X, \mathcal{M}, \mu)$ is called a **measure space**. If $\mu(X) < \infty$, the measure $\mu$ is said to be **finite**; and if there is a partition $\{X_n\}_{n \in \mathbb{N}}$ of $X$ such that $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$, the measure $\mu$ is said to be **σ-finite**.
Proposition. (Properties of measures) Let \((X, \mathcal{M}, \mu)\) be a measure space, \(E, F \in \mathcal{M}\) such that \(E \subseteq F\), \(\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}\) such that \(E_n \subseteq E_{n+1}\), and \(\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}\) such that \(F_n \supseteq F_{n+1}\) and \(\mu(F_1) < \infty\). Then,

(a) \(\mu(E) \leq \mu(F)\).

(b) If \(\mu(E) < \infty\), then \(\mu(F \setminus E) = \mu(F) - \mu(E)\).

(c) \(\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \lim_{n \to \infty} \mu(E_n)\) and \(\mu\left(\bigcap_{n \in \mathbb{N}} F_n\right) = \lim_{n \to \infty} \mu(F_n)\).

Definition. Let \((X, \mathcal{M})\) and \((Y, \mathcal{N})\) be measurable spaces. A function \(h : X \to Y\) is said to be measurable if for all \(F \in \mathcal{N}\) it follows that \(h^{-1}(F) \in \mathcal{M}\). If \((Y, \mathcal{T})\) is instead a topological space, we only ask that \(h^{-1}(V) \in \mathcal{M}\) for any \(V \in \mathcal{T}\). (In fact, any topological space is a measurable one with the sigma algebra being the smallest one containing all open sets)

Definition. Let \((X, \mathcal{M})\) be a measurable space. A simple function \(s : X \to \mathbb{C}\) is a function such that there exist \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\) and \(E_1, E_2, \ldots, E_n \in \mathcal{M}\) such that \(X = \bigcup_{k=1}^{n} E_k\) and

\[s = \sum_{k=1}^{n} \alpha_k \chi_{E_k}\]

Furthermore, if \(\mu\) is a measure on \((X, \mathcal{M})\), we define the integral of \(s\) over \(X\) with respect to \(\mu\) as follows

\[\int_X s \, d\mu := \sum_{k=1}^{n} \alpha_k \mu(E_k)\]

Lemma. Let \((X, \mathcal{M})\) be a measurable space. If \(f : X \to [0, \infty]\) is measurable , there is a sequence \((s_n)_{n=1}^{\infty}\) of non-negative simple functions such that \(s_n \to f\) pointwise and \(s_n \leq s_{n+1}\) for all \(n \in \mathbb{N}\).

Definition. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f : X \to [0, \infty]\) a non-negative measurable function. Denote by \(S^+(X, \mathcal{M})\) the set of all non-negative simple functions. We define the integral of \(f\) over \(X\) with
respect to $\mu$ as follows
\[
\int_X f \, d\mu := \sup \left\{ \int_X s \, d\mu : s \in S^+(X, \mathcal{M}), \ s \leq f \right\}
\]

**Theorem.** *(Monotone Convergence Theorem)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable non-negative functions such that

(i) $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$

(ii) $f(x) := \lim_{n \to \infty} f_n(x)$ for all $x \in X$

Then, $f$ is measurable and
\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu
\]

**Theorem.** *(Fatou’s Lemma)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable non-negative functions. Then,
\[
\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu
\]

**Notation.** Let $X$ be a set. For a function $g : X \to [-\infty, \infty]$ we define functions $g^+, g^- : X \to [0, \infty]$ by
\[
g^+(x) := \max\{g(x), 0\} \quad \text{and} \quad g^-(x) := \max\{-g(x), 0\}
\]
Observe that $g = g^+ - g^-$. For a function $h : X \to \mathbb{C}$ we denote by $\text{Re}h, \text{Im}h : X \to \mathbb{R}$ the real and imaginary parts of $h$ respectively. Observe that $h = \text{Re}h + i\text{Im}h$.

**Definition.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $R$ be either $[-\infty, \infty]$ or $\mathbb{C}$. If $f : X \to R$ is measurable, we define the integral of $f$ over $X$ with respect to $\mu$ as follows
\[
\int_X f \, d\mu := \int_X (\text{Re}f)^+ \, d\mu - \int_X (\text{Re}f)^- \, d\mu + i \left( \int_X (\text{Im}f)^+ \, d\mu - \int_X (\text{Im}f)^- \, d\mu \right)
\]
Further, we say that $f$ is integrable with respect to $\mu$ if
\[
\int_X |f| \, d\mu < \infty
\]
Theorem. (Lebesgue Dominated Convergence Theorem) Let \((X, \mathcal{M}, \mu)\) be a measure space and \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions \(f_n : X \to \mathbb{C}\) such that

(i) \(f(x) := \lim_{n \to \infty} f_n(x)\) exists for all \(x \in X\)

(ii) There is an integrable function \(g : X \to [0, \infty)\) such that \(|f| \leq g\).

Then, each \(f_n\) and \(f\) are integrable. Moreover,

\[ \int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu \]

1.1.2 Lebesgue Measure

Definition. Let \(X\) be a set. A function \(\mu^* : \mathcal{P}(X) \to [0, \infty]\) is an outer measure if

(1) \(\mu^*(\emptyset) = 0\).

(2) If \(A \subseteq B \subseteq X\), then \(\mu^*(A) \leq \mu^*(B)\).

(3) If \(A_1, A_2, \ldots \subseteq X\), then

\[ \mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n) \]

Furthermore, we say that \(E \in \mathcal{P}(X)\) is \(\mu^*\)-measurable if for any \(A \subseteq X\) we have

\[ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap (X \setminus E)) \]

Remark. Intuitively, since \(A \cap E\) and \(A \cap (X \setminus E)\) are disjoint, we note that \(\mu^*\)-measurable sets are those that “break” any subset of \(X\) as expected with respect to \(\mu^*\).

Theorem. (Carathéodory construction) Let \(X\) be a set and \(\mu^*\) an outer measure. The \(\mu^*\)-measurable sets form a \(\sigma\)-algebra on \(X\).
Definition. Let $d \in \mathbb{N}$. An open box in $\mathbb{R}^d$ is a subset of the form

$$B := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

where for each $1 \leq k \leq d$ we have that $a_k \leq b_k$. We define its volume by

$$\text{vol}(B) := \prod_{k=1}^d (b_k - a_k)$$

Definition. Let $d \in \mathbb{N}$. For any $A \subseteq \mathbb{R}^d$ we denote by $B(A)$ to the set of all countable covers of $A$ by open boxes and we define $m^*_d : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ by

$$m^*_d(A) := \inf \left\{ \sum_{n=1}^\infty \text{vol}(B_n) : \{B_n\}_{n \in \mathbb{N}} \in B(A) \right\}$$

Theorem. The function $m^*_d : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ is an outer measure.

Definition. The Lebesgue measurable sets on $\mathbb{R}^d$, denoted by $\mathcal{L}_d$ are those $m^*_d$-measurable sets and the Lebesgue measure on $\mathbb{R}^d$, denoted by $\mu_d$, is the restriction of $m^*_d$ to $\mathcal{L}_d$.

Definition. If $(X, \mathcal{T})$ is a topological space, the $\sigma$-algebra $\mathfrak{B}_X$ generated by all the open sets of $X$ is called the Borel $\sigma$-algebra of $X$ and its elements are called Borel sets. If $\mathfrak{M}$ is another $\sigma$-algebra on $X$ and $\mu$ a measure on $(X, \mathfrak{M})$, we say that $\mu$ is a Borel measure if $\mathfrak{B}_X \subseteq \mathfrak{M}$.

Theorem. Let $d \in \mathbb{N}$. Then $\mathfrak{B}_{\mathbb{R}^d} \subseteq \mathcal{L}_d$ and therefore $\mu_d$ is a Borel measure.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say that $X$ is locally compact if every point of $X$ has an open neighborhood whose closure is compact.

Definition. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. A Borel measure $\mu$ is said to be regular if

1. $\mu(K) < \infty$ for all $K \subset X$ compact.
2. $\mu(E) = \inf\{\mu(U) : U \in \mathcal{T}, E \subseteq U\}$ for all $E \in \mathfrak{B}_X$
3. If $E \in \mathfrak{B}_X$ with $\mu(E) < \infty$ or if $E \in \mathcal{T}$, then
\begin{align}
(3.1) \quad \mu(E) &= \sup\{\mu(K) : K \text{ is compact }, K \subseteq E\} \\
(3.2) \quad \mu(E) &= \sup\{\mu(C) : X \setminus C \in \mathcal{T}, C \subseteq E\}
\end{align}

Condition (2) is known as \textit{outer regular}, (3.1) as \textit{strong inner regular} and (3.2) as \textit{weak inner regular}.

\textbf{Theorem.} Lebesgue measure \( m_d \) is regular.

\textbf{Definition.} Let \( X \) be a locally compact Hausdorff space. The set of \textit{continuous functions that vanish at infinity} is defined as follows

\[ C_0(X) := \{ f \in C(X) : \forall \varepsilon > 0 \ , \{ x \in X : |f(x)| \geq \varepsilon \} \text{ is compact} \} \]

The set of \textit{continuous functions with compact support} is

\[ C_c(X) := \{ f \in C(X) : \text{supp}(f) \text{ is compact} \} \]

\textbf{Theorem.} Let \( X \) be a locally compact Hausdorff space and equip \( C(X) \) with the \( \sup \) norm. Then, \( C_0(X) \) is closed in \( C(X) \) and \( C_c(X) \) is dense in \( C_0(X) \).

\textbf{Theorem.} (Lusin) Let \( X \) be a locally compact Hausdorff space and \( \mu \) a regular Borel measure. If \( f : X \to \mathbb{C} \) is measurable and \( A \subset X \) is such that

\begin{enumerate}
\item \( \mu(A) < \infty \)
\item \( f(x) = 0 \) for all \( x \notin A \)
\end{enumerate}

Then, for all \( \varepsilon > 0 \) there is \( g \in C_c(X) \) such that

\begin{enumerate}
\item \( \mu(\{ x \in X : f(x) \neq g(x) \}) < \varepsilon \)
\item \( \sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)| \)
\end{enumerate}

\subsection*{1.1.3 \( L^p \) spaces}

\textbf{Definition.} Let \( (X, \mathcal{M}, \mu) \) be a measure space and \( p \in [1, \infty) \). The space of \( p \)-\textit{integrable functions with respect to} \( \mu \) is

\[ L^p(X, \mathcal{M}, \mu) := \frac{\{\text{measurable functions } f : X \to \mathbb{C} : |f|^p \text{ is integrable}\}}{\{\text{measurable functions } f : X \to \mathbb{C} : f = 0 \text{ a.e [} \mu \} \}} \]
Theorem. Let \( (X, \mathcal{M}, \mu) \) be a measure space and \( p \in [1, \infty) \). Put

\[
\|f\|_p := \left( \int_X |f|^p \right)^{1/p}
\]

Then, \( \| \cdot \|_p : L^p(X, \mathcal{M}, \mu) \to [0, \infty) \) is a well defined norm (so from now on we write \( \|f\|_p := \|[f]\|_p \)) and \( L^p(X, \mathcal{M}, \mu) \) is complete with respect to this norm.

Theorem. (Hölder) Let \( (X, \mathcal{M}, \mu) \) be a measure space and \( p, q \in (1, \infty) \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( f \in L^p(X, \mathcal{M}, \mu) \) and \( g \in L^q(X, \mathcal{M}, \mu) \), then \( fg \in L^1(X, \mathcal{M}, \mu) \) and

\[
\|fg\|_1 \leq \|f\|_p \|g\|_q
\]

Definition. Let \( (X, \mathcal{M}, \mu) \) be a measure space. A measurable function \( f : X \to \mathbb{C} \) is said to be essentially bounded with respect to \( \mu \) if

\[
\|f\|_\infty := \inf\{\alpha > 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\} < \infty
\]

The space of essentially bounded functions is denoted by \( L^\infty(X, \mathcal{M}, \mu) \).

Theorem. Let \( (X, \mathcal{M}, \mu) \) be a measure space. Then, \( \| \cdot \|_\infty \) is a well defined norm and \( L^\infty(X, \mathcal{M}, \mu) \) is complete with respect to this norm.

Theorem. Let \( (X, \mathcal{M}, \mu) \) be a measure space. If \( f \in L^1(X, \mathcal{M}, \mu) \) and \( g \in L^\infty(X, \mathcal{M}, \mu) \), then \( fg \in L^1(X, \mathcal{M}, \mu) \) and

\[
\|fg\|_1 \leq \|f\|_p \|g\|_\infty
\]

1.1.4 Complex and Signed measures

Definition. Let \( (X, \mathcal{M}) \) be a measurable space. A function \( \nu : \mathcal{M} \to \mathbb{C} \) is a complex measure if whenever \( E_1, E_2, \ldots \) are pairwise disjoint elements in \( \mathcal{M} \), it follows that

\[
\nu \left( \bigcup_{n \in \mathbb{N}} E_n \right) = \sum_{n \in \mathbb{N}} \nu(E_n).
\]

As consequence of this definition we see that \( \nu(\emptyset) = 0 \) and that \( \mu \) may not assume infinite values. ▲
Definition. Let \((X, \mathcal{M})\) be a measurable space. A function \(\nu : \mathcal{M} \rightarrow (-\infty, \infty]\) (or \(\rightarrow [−\infty, \infty]\)), is a **signed measure** if whenever \(E_1, E_2, \ldots\) are pairwise disjoint elements in \(\mathcal{M}\), it follows that
\[
\nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \nu(E_n) \quad \text{and} \quad \nu(\emptyset) = 0
\]

Definition. Let \((X, \mathcal{M})\) be a measurable space and \(\nu\) a complex or signed measure. A measurable set \(E\) is said to be **\(\nu\)-null** if whenever \(F \subseteq E\) it follows that \(\nu(F) = 0\).

Definition. Let \((X, \mathcal{M})\) be a measurable space and \(\nu\) a signed measure. A measurable set \(E\) is said to be **\(\nu\)-positive** if whenever \(F \subseteq E\) it follows that \(\nu(F) \geq 0\). A measurable set \(E\) is said to be **\(\nu\)-negative** if whenever \(F \subseteq E\) it follows that \(\nu(F) \leq 0\).

Definition. Let \((X, \mathcal{M})\) be a measurable space. Two measures \(\mu, \lambda\) are **mutually singular**, denoted by \(\mu \perp \lambda\), if there is a measurable set \(A\) such that \(\mu(A) = 0 = \lambda(X \setminus A)\). If the measures are instead complex or signed measures, we require \(A\) to be \(\mu\)-null and \(X \setminus A\) is \(\lambda\)-null.

Definition. Let \((X, \mathcal{M})\) be a measurable space. A measure \(\mu\) is **absolutely continuous with respect to another measure** \(\lambda\), denoted by \(\mu \ll \lambda\), if whenever \(\lambda(E) = 0\) it follows that \(\mu(E) = 0\). If the measures are instead complex or signed measures, we require that \(\lambda\)-null implies \(\mu\)-null.

Proposition. Let \((X, \mathcal{M})\) be a measurable space and let \(\mu, \lambda\) be finite measures. Then \(\mu \ll \lambda\) if and only if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that if \(\lambda(E) < \delta\), then \(\mu(E) < \varepsilon\).

Theorem. (Hahn Decomposition) Let \((X, \mathcal{M})\) be a measurable space. If \(\mu\) is a signed measure, then there exist measurable sets \(P\) and \(N\) such that \(P\) is \(\mu\)-positive, \(N\) is \(\mu\)-negative and \(X = P \cup N\). The pair \((P, N)\) is called a **Hahn decomposition of** \(X\). Furthermore, a pair \((P_0, N_0)\) is a Hahn decomposition of \(X\) if and only if \(P \triangle P_0\) and \(N \triangle N_0\) are \(\mu\)-null.

Theorem. (Jordan Decomposition) Let \((X, \mathcal{M})\) be a measurable space. If \(\mu\) is a signed measure, then there exist measures \(\mu_+, \mu_-\) such that \(\mu = \mu_+ - \mu_-\) and \(\mu_+ \perp \mu_-\). Moreover, If \(\mu = \lambda_1 - \lambda_2\), then \(\mu_+ \leq \lambda_1\) and \(\mu_- \leq \lambda_2\), and if also \(\lambda_1 \perp \lambda_2\), then \(\mu_+ = \lambda_1\) and \(\mu_- = \lambda_2\).

Theorem. (Radon-Nikodym) Let \((X, \mathcal{M})\) be a measurable space, \(\mu\) a \(\sigma\)-finite measure and \(\nu\) a complex measure such that \(\nu \ll \mu\). Then, there exists
\[
\text{The function } f \text{ is called a \textbf{Radon-Nikodym derivative of } } \nu \text{ \textit{with respect to} } \mu, \text{ } f \text{ \textit{is unique a.e} } [\mu] \text{ and it’s denoted as } \\
f = \frac{d\nu}{d\mu}
\]

\textbf{Theorem. (Lebesgue Decomposition)} Let \((X, \mathcal{M})\) be a measurable space, \(\mu\) a \(\sigma\)-finite measure and \(\nu\) a complex measure. Then, there exist unique complex measures \(\nu_s\) and \(\nu_a\) such that \(\nu = \nu_s + \nu_a\), \(\nu_s \perp \mu\) and \(\nu_a \ll \mu\).

**Proposition.** Let \((X, \mathcal{M})\) be a measurable space. We denote by \(M(X, \mathcal{M})\) to the set of all complex measures on \((X, \mathcal{M})\). For \(\mu \in M(X, \mathcal{M})\) and \(E \in \mathcal{M}\) let

\[
|\mu|(E) := \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \text{ is a pairwise disjoint cover of } E \right\}
\]

Then, \(|\mu|\) is a finite measure and \(M(X, \mathcal{M})\) is a Banach space with norm given by \(\|\mu\| := |\mu|(X)\).

**Theorem.** Let \((X, \mathcal{M})\) be a measurable space and \(\mu \in M(X, \mathcal{M})\). Then there is a function \(h : X \to \mathbb{C}\) such that \(|h| = 1\) a.e \([|\mu|]\) and

\[
\mu(E) = \int_E h \, d|\mu|
\]

\textbf{Definition.} Let \(X\) be a topological space and \(\mu \in M(X, \mathcal{M})\). We say that \(\mu\) is regular if \(|\mu|\) is regular.

\textbf{Theorem. (Riesz Representation Theorems)} Let \(X\) be a locally compact Hausdorff space.

(a) If \(\omega \in C_0(X)^*\), then there exist a unique regular measure \(\mu \in M(X, \mathcal{B}_X)\) such that

\[
\omega(f) = \int_X f \, d\mu \text{ and } \|\omega\| = \|\mu\|
\]

(b) If \(\omega \in C_c(X)^*\) is positive (i.e. \(\omega(f) \geq 0\) whenever \(f \geq 0\)), then there exist a unique regular positive measure \(\mu\) such that

\[
\omega(f) = \int_X f \, d\mu
\]
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2.1 Theorems and Definitions.

2.1.1 Fubini’s Theorem

Definition. Let \((X, \mathcal{M})\) and \((Y, \mathcal{N})\) be measurable spaces. A **measurable rectangle** is a set of the form \(E \times F\) where \(E \in \mathcal{M}\) and \(F \in \mathcal{N}\).

Definition. Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces. Then, we denote by \(\mathcal{M} \otimes \mathcal{N}\) the \(\sigma\)-algebra on \(X \times Y\) generated by the measurable rectangles.

Theorem. Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces. Then, there exist a unique measure \(\mu \otimes \nu\) on \(\mathcal{M} \otimes \mathcal{N}\) such that

\[(\mu \otimes \nu)(E \times F) = \mu(E)\nu(F) \quad \forall \quad (E \in \mathcal{M}, \ F \in \mathcal{N}).\]

Theorem. (Tonelli [Fubini]) Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces, \(f : X \times Y \to \mathbb{C}\) a \(\mathcal{M} \otimes \mathcal{N}\)-measurable function. Suppose that \(f\) is non-negative \([f\ is \ \mu \otimes \nu\text{-integrable}]\). Then,

(a) For all \(y \in Y\), \(x \mapsto f(x, y)\) is \(\mathcal{M}\)-measurable \([\mu\text{-integrable for a.e. } y \in Y]\).

(b) The function

\[y \mapsto \int_X f(x, y) \, d\mu(x)\]

is defined a.e \([\nu]\) and it’s \(\mathcal{N}\)-measurable \([\nu\text{-integrable}].\)

(c)

\[\int_Y \left( \int_X f(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{X \times Y} f \, d(\mu \otimes \nu)\]

Furthermore, items (a), (b) and (c) above also hold when interchanging \(y \leftrightarrow x\), \(Y \leftrightarrow X\), \(\mathcal{M} \leftrightarrow \mathcal{N}\) and \(\mu \leftrightarrow \nu\).

2.1.2 Differentiation

Definition. Let \(d \in \mathbb{N}\) and let \(\mu\) be a regular Borel measure on \(\mathbb{R}^d\). For \(r > 0\) and \(x \in \mathbb{R}^d\), we define

\[(Q_r \mu)(x) := \frac{\mu(B_r(x))}{m(B_r(x))}\]
where $m$ is the Lebesgue measure in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ set

$$D_\mu(x) := \lim_{r \to 0^+} (Q_r \mu)(x)$$

**Definition.** Let $d \in \mathbb{N}$ and let $\mu$ be a complex regular Borel measure on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we put

$$M_\mu(x) := \sup_{r > 0} (Q_r |\mu|(x))$$

**Lemma.** Let $d \in \mathbb{N}$ and let $\mu$ be a complex regular Borel measure on $\mathbb{R}^d$. The function $M_\mu$ is lower semicontinuous, that is $M_\mu^{-1}( (t, \infty) )$ is open for every $t \in \mathbb{R}$.

**Definition.** Let $d \in \mathbb{N}$. A **locally integrable functions** is a Lebesgue measurable function $f : \mathbb{R}^d \to \mathbb{C}$ such that $\int_K |f| \, dm < \infty$ for all compact subsets $K \subset \mathbb{R}^d$. We denote the set of locally integrable functions by $L^1_{\text{loc}}(\mathbb{R}^d)$.

**Definition.** For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, define

$$Mf(x) := \sup_{r > 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \, dm$$

**Definition.** Let $d \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. A point $x_0 \in \mathbb{R}^k$ is a **Lebesgue point of $f$** if

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x_0))} \int_{B_r(x_0)} |f - f(x_0)| \, dm = 0$$

**Theorem.** Let $d \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then, the complement of the set of Lebesgue points of $f$ has measure 0. That is, a.e. $[m] \in \mathbb{R}^d$ is a Lebesgue point.

**Theorem.** Let $d \in \mathbb{N}$ and let $\mu$ be a complex regular Borel measure on $\mathbb{R}^d$ such that $\mu \ll m$. If $f = \frac{d\mu}{dm}$ a.e. $[m]$, it follows that $f(x) = D_\mu(x)$ for all Lebesgue points of $f$. 

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Definition. Let $d \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We say that a sequence of subsets of $\mathbb{R}^d$, $(E_n)_{n \in \mathbb{N}}$, shrinks nicely to $x$ if there is a sequence $(r_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ and $\alpha > 0$ such that

1. $r_n \to 0$
2. $E_n \subset B_{r_n}(x)$
3. $m(E_n) \geq \alpha m(B_{r_n}(x))$

Theorem. If $d \in \mathbb{N}$, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ is a Lebesgue point of $f$ and $(E_n)_{n \in \mathbb{N}}$ shrinks nicely to $x$, then

$$\lim_{n \to \infty} \frac{1}{m(E_n)} \int_{E_n} f \, m = f(x)$$

Definition. A function $f : [a, b] \to \mathbb{C}$ is absolutely continuous if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_n, b_n)$ are disjoint intervals in $[a, b]$ and $\sum_{k=1}^n (b_k - a_k) < \delta$, it follows that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$.

Theorem. If $f : [a, b] \to \mathbb{C}$ is absolutely continuous, its derivative $f'$ exist a.e $[m]$, $f' \in L^1([a, b], m)$ and

$$f(x) = f(a) + \int_a^x f' \, dm$$

Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is non decreasing. The following are equivalent.

(i) $f$ is absolutely continuous.
(ii) If $m(E) = 0$, then $m(f(E)) = 0$.
(iii) $f'$ exist a.e $[m]$, $f' \in L^1([a, b], m)$ and

$$f(x) = f(a) + \int_a^x f' \, dm$$
2.1.3 Basic Functional Analysis

**Theorem.** (Hahn-Banach) Let $K = \mathbb{C}$ or $\mathbb{R}$. Let $E$ be a normed vector space over $K$ and let $M$ be a subspace of $E$. If $\omega_0 : M \to K$ is a bounded linear functional, there exist a linear functional $\omega : E \to K$ such that

(i) $\omega|_M = \omega_0$

(ii) $\|\omega\| = \|\omega_0\|

**Corollary I.** If $E$ is a normed vector space and $\xi_0 \in E \setminus \{0\}$, then there exist $\omega : E \to \mathbb{C}$ such that

(i) $\omega(\xi_0) = \|\xi_0\|

(ii) $\|\omega\| = 1$

**Corollary II.** If $E$ is a normed vector space and $\Phi : E \to E^{**}$ is given by

$[\Phi(\xi)](\omega) := \omega(\xi)$ for all $\xi \in E$ and $\omega \in E^*$ ,

then $\Phi$ is an injective isometry. We may, and do, identify $E$ as a subset of $E^{**}$.

**Theorem.** (Baire Category Theorem) Let $X$ be a complete metric space. Let $U_1, U_2, \ldots$, be open dense subsets of $X$. Then,

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X$$

**Theorem.** (Uniform Boundedness Principle) Let $E$ be a Banach space, $F$ a normed vector space and $S \subseteq L(E, F)$. Assume there is a dense $G_δ$ set $B \subseteq E$ (i.e. $B$ is a countable intersection of open sets) such that for any $\xi \in B$,

$$\sup_{a \in S} \|a(\xi)\| < \infty$$

Then,

$$\sup_{a \in S} \|a\| < \infty$$

**Theorem.** (Open Mapping Theorem) Let $E, F$ be Banach spaces and $a \in L(E, F)$ surjective. Then,

(i) There is $\delta > 0$ such that $B_\delta^E(0) \subseteq a(B_1^F(0))$. 

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(ii) If \( U \subset E \) is open, then \( a(U) \) is open.

(iii) If \( a \) is also injective, then \( a^{-1} \in L(F,E) \).

**Theorem. (Closed Graph)** If \( a : E \to F \) is linear, then \( a \) is bounded if and only if the graph of \( a \) is a closed subset of \( E \times F \).

### 2.1.4 Hilbert Spaces

**Definition.** Let \( \mathcal{H} \) be a vector space over \( \mathbb{C} \). A **scalar product** on \( \mathcal{H} \) is a function \( (\xi, \eta) \mapsto \langle \xi, \eta \rangle \) from \( \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) such that

1. It’s linear in \( \xi \) for each fixed \( \eta \).
2. \( \langle \xi, \eta \rangle = \langle \eta, \xi \rangle \).
3. \( \langle \xi, \xi \rangle \geq 0 \).
4. If \( \langle \xi, \xi \rangle = 0 \), then \( \xi = 0 \).

**Theorem. (Cauchy-Schwarz)**

\[ |\langle \xi, \eta \rangle|^2 \leq \langle \xi, \xi \rangle \langle \eta, \eta \rangle \]

**Remark.** A scalar product on \( \mathcal{H} \) induces a norm in \( \mathcal{H} \) by letting \( \| \xi \| := (\langle \xi, \xi \rangle)^{1/2} \), so Cauchy-Schwarz gives \( |\langle \xi, \eta \rangle| \leq \| \xi \| \| \eta \| \).

**Definition.** If \( \mathcal{H} \) is equipped with a scalar product, we say that \( \mathcal{H} \) is a **Hilbert space** if it is complete with respect to the induced norm.

**Theorem. (Parallelogram Law)** In any scalar product space we have

\[ \| \xi + \eta \|^2 + \| \xi - \eta \|^2 = 2(\| \xi \|^2 + \| \eta \|^2) \]

**Definition.** Let \( \mathcal{H} \) be a scalar product space. We say that \( \xi, \eta \in \mathcal{H} \) are **orthogonal**, denoted by \( \xi \perp \eta \), if \( \langle \xi, \eta \rangle = 0 \). For \( S, T \subseteq \mathcal{H} \), we say that \( S \) is **orthogonal to** \( T \), denoted by \( S \perp T \), if \( \xi \perp \eta \) for all \( \xi \in S \) and all \( \eta \in T \). Further, we put \( S^\perp := \{ \eta \in \mathcal{H} : \eta \perp \xi, \forall \xi \in S \} \).
Theorem. If $H$ is a Hilbert space and $S \subseteq H$, then $S^\perp$ is a closed subset of $H$.

Theorem. If $H$ is a Hilbert space and $K$ is a closed convex subset of $H$, then there is a unique $\xi_0 \in K$ such that

$$\text{dist}(0, K) = \|\xi_0\|$$

Theorem. Let $H$ be a Hilbert space and $M \subseteq H$ a closed subspace. Then, for every $\xi \in H$ there are unique $p(\xi) \in M$ and $q(\xi) \in M^\perp$ such that

(1) $\xi = p(\xi) + q(\xi)$.
(2) $p : H \to M$, $q : H \to M^\perp$ are linear with $\|p\|, \|q\| \leq 1$ and $p^2 = p$, $q^2 = q$.
(3) $\|\xi\|^2 = \|p(\xi)\|^2 + \|q(\xi)\|^2$
(4) $p(\xi)$ is the nearest point in $M$ to $\xi$, $q(\xi)$ is the nearest point in $M^\perp$ to $\xi$.

Corollary. Let $H$ be a Hilbert space and $M \subseteq H$ a closed subspace. Then,

$$H = M \oplus M^\perp$$

Theorem. (Riesz) Let $H$ be a Hilbert space and $\omega \in H^*$. Then, there is a unique $\eta \in H$ such that

$$\omega(\xi) = \langle \xi, \eta \rangle \forall \xi \in H$$

Moreover, $\|\omega\| = \|\eta\|$.

Theorem. Let $H$ be a Hilbert space and $a \in L(H)$. Then there is a unique $a^* \in L(H)$ such that

$$\langle a \xi, \eta \rangle = \langle \xi, a^* \eta \rangle \forall \xi, \eta \in H$$

Furthermore, for any $a, b \in L(H)$ and $\lambda \in \mathbb{C}$

(1) $\|a^*\| = \|a\|$.
(2) $(a + \lambda b)^* = a^* + \lambda b^*$.
(3) $(ab)^* = b^* a^*$. 

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(4) \( \text{id}^* = \text{id} \).
(5) \((a^*)^* = a\).
(6) \(\|a^*a\| = \|a\|^2\).

**Definition.** Let \( \mathcal{H} \) be a Hilbert space. A family \( (\xi_i)_{i \in I} \) of elements in \( \mathcal{H} \) is: **orthogonal** if \( \xi_i \perp \xi_j \) for all \( i \neq j \) in \( I \); **orthonormal** if in also \( \|x_i\| = 1 \) for all \( i \in I \); and a **Hilbert basis** if in addition \( \text{span}(\{\xi_i : i \in I\}) \) is dense in \( \mathcal{H} \).

**Definition.** Let \( \mathcal{H} \) be a Hilbert space and \( (\xi_i)_{i \in I} \) a family of elements in \( \mathcal{H} \). We say that

\[
\sum_{i \in I} \xi_i \to \xi \in \mathcal{H}
\]

if for all \( \varepsilon > 0 \) there is a finite set \( F \subset I \), such that for every finite set \( S \subset I \) with \( F \subset S \), we have

\[
\left\| \sum_{i \in S} \xi_i - \xi \right\| < \varepsilon
\]

**Theorem.** Let \( \mathcal{H} \) be any Hilbert space.

(a) \( \mathcal{H} \) has an orthonormal basis.

(b) Any other orthonormal basis of \( \mathcal{H} \) has the same cardinality.

(c) For an orthonormal basis \( (\xi_i)_{i \in I} \), the map \( \ell^2(I) \to \mathcal{H} \) given by

\[
(x_i)_{i \in I} \mapsto \sum_{i \in I} x_i \xi_i
\]

is an isometric isomorphism with inverse given by

\[
\xi \mapsto (\langle \xi, \xi_i \rangle)_{i \in I}
\]

**Theorem.** Let \( \mathcal{H} \) be a Hilbert space and \( (\xi_i)_{i \in I} \) a family of elements in \( \mathcal{H} \) such that \( \text{span}(\{\xi_i : i \in I\}) = M \). Then,

(a) For all \( \xi \in \mathcal{H} \),

\[
\sum_{i \in I} |\langle \xi, \xi_i \rangle|^2 \leq \|\xi\|^2
\]

(b) The orthogonal projection \( p : \mathcal{H} \to M \) is

\[
p\xi = \sum_{i \in I} \langle \xi, \xi_i \rangle \xi_i
\]
2.1.5 Basic Fourier Analysis

**Definition.** For \( f, g \in L^1(\mathbb{R}, m) \) we define the **convolution** \( f * g \) by

\[
(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y) \, dm(y)
\]

**Theorem.** If \( f, g \in L^1(\mathbb{R}, m) \) are Lebesgue measurable, then

(a) \((x, y) \mapsto f(y)g(x - y)\) is Lebesgue measurable

(b) \( y \mapsto f(y)g(x - y) \) is integrable for a.e. \( x \)

(c) \( \|f * g\|_1 \leq \|f\|_1 \|g\|_1 \)

**Definition.** Put \( \overline{m} := (2\pi)^{-1/2}m \), and let \( f \in L^1(\mathbb{R}, \overline{m}) \). We define the **Fourier Transform of** \( f \), denoted by \( \hat{f} \), by

\[
\hat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x) \, d\overline{m}(x)
\]

**Proposition.** *(Properties of \( \hat{f} \)) For \( f \in L^1(\mathbb{R}, \overline{m}) \), \( \alpha \in \mathbb{R} \) and \( \beta \in (0, \infty) \) we have

(a) If \( g(x) = e^{ix} f(x) \), then \( \hat{g}(t) = \hat{f}(t - \alpha) \)

(b) If \( g(x) = f(x - \alpha) \), then \( \hat{g}(t) = e^{-it\alpha} \hat{f}(t) \)

(c) If \( g \in L^1(\mathbb{R}, \overline{m}) \), then \( \overline{f * g} = \hat{f} \cdot \hat{g} \)

(d) If \( g(x) = f(-x) \), then \( \hat{g}(t) = \hat{f}(t) \)

(e) If \( g(x) = f(\beta^{-1}x) \), then \( \hat{g}(t) = \beta \hat{f}(t) \)

(f) If \( g(x) = -ix f(x) \) is such that \( g \) is in \( L^1(\mathbb{R}, \overline{m}) \), then \( \hat{f}' \) exists and is equal to \( \hat{g} \).

**Theorem.** The map \( f \mapsto \hat{f} \) is a contractive linear map from \( L^1(\mathbb{R}, \overline{m}) \) to \( C_0(\mathbb{R}) \); that is \( \hat{f} \in C_0(\mathbb{R}) \) and \( \|\hat{f}\|_{\infty} \leq \|f\|_1 \).
Theorem. (Fourier Inversion) Suppose that \( f \in L^1(\mathbb{R}, m) \) and that \( \hat{f} \in L^1(\mathbb{R}, m) \). If
\[
g(x) := \int_{\mathbb{R}} e^{itx} \hat{f}(t) \, dm(t),
\]
then \( g \in C_0(\mathbb{R}) \) and \( g = f \) a.e.

Corollary. If \( f \in L^1(\mathbb{R}, m) \) and \( \hat{f} = 0 \), then \( f = 0 \) a.e.

Theorem. (Plancherel) There is a mapping \( F : L^2(\mathbb{R}, m) \rightarrow L^2(\mathbb{R}, m) \) such that
1. \( F(f) = \hat{f} \) for all \( f \in L^1(\mathbb{R}, m) \cap L^2(\mathbb{R}, m) \).
2. \( \|F(f)\|_2 = \|f\|_2 \) for all \( f \in L^2(\mathbb{R}, m) \).
3. \( F \) is a Hilbert space isomorphism of \( L^2(\mathbb{R}, m) \) onto itself.
4. If
\[
\varphi_A(t) := \int_{-A}^{A} e^{-itx} f(x) \, dm(x) \quad \text{and} \quad \psi_A(x) := \int_{-A}^{A} e^{itx} F(f)(t) \, dm(t),
\]
Then
\[
\lim_{A \to \infty} \|\varphi_A - F(f)\|_2 = 0 \quad \text{and} \quad \lim_{A \to \infty} \|\psi_A - f\|_2 = 0.
\]

Corollary. If \( f \in L^2(\mathbb{R}, m) \) and \( F(f) \in L^1(\mathbb{R}, m) \), then
\[
f(x) = \int_{\mathbb{R}} e^{itx} F(f)(t) \, dm(t) \text{ a.e.}
\]

Remark. \( L^1(\mathbb{R}, m) \) is a Banach algebra with multiplication given by convolution.

Theorem. To every non-zero complex homomorphism \( \omega : L^1(\mathbb{R}, m) \rightarrow \mathbb{C} \) (i.e. Banach algebra homomorphism) corresponds a unique \( t \in \mathbb{R} \) such that \( \omega(f) = \hat{f}(t) \) for all \( f \in L^1(\mathbb{R}, m) \).
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3.1 Theorems and Definitions.

3.1.1 Basic Complex Analysis

**Definition.** For $\Omega$ an open subset of $\mathbb{C}$, $a \in \Omega$, we say that $f : \Omega \to \mathbb{C}$ is **complex differentiable at** $a$ is the following limit exits

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

In such case, the limit is denoted by $f'(a)$. Equivalently, if $f := u + iv$, $z = x + iy$, then $f$ is complex differentiable at $a := x_0 + iy_0$ if $u$ and $v$ satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Furthermore,

$$f'(a) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

**Lemma.** If $f'(a)$ exists, then $f$ is continuous at $a$.

**Definition.** Let $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \to \mathbb{C}$. We say that $f$ is **representable by power series on** $\Omega$ if for all $a \in \Omega$ and for all $r > 0$ with $B_r(a) \subseteq \Omega$, there is a sequence $(c_n)_{n \in \mathbb{N}_0}$ in $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n \forall z \in B_r(a)$$

**Definition.** Let $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \to \mathbb{C}$. We say that $f$ is **weakly representable by power series on** $\Omega$ if for all $z_0 \in \Omega$, there is $a \in \Omega$ such that $f$ is given by a power series about $a$ on some neighborhood of $z_0$.

**Theorem.** For a sequence $(c_n)_{n \in \mathbb{N}_0}$ in $\mathbb{C}$, we define $R \in [0, \infty]$ by

$$\frac{1}{R} := \limsup_{n \to \infty} |c_n|^{1/n}$$

Then,
(a) If $|z - a| > R$, then $c_n(z - a)^n \not\to 0$ as $n \to \infty$.

(b) For all $s \in (0, R)$, for all $a \in \mathbb{C}$, the series $\sum_{n \in \mathbb{N}_0} c_n(z - a)^n$ converges uniformly and absolutely on $B_s(a)$.

**Theorem.** Let $\Omega \subseteq \mathbb{C}$ open. If $f : \Omega \to \mathbb{C}$ is representable by power series on $\Omega$, then $f$ is complex differentiable on $\Omega$ and $f'$ is also representable by power series on $\Omega$. Further,

$$
\text{if } f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n, \text{ then } f'(z) = \sum_{n=1}^{\infty} nc_n(z - a)^{n-1}.
$$

**Corollary.** If $f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n$, then $f^{(k)}(a) = k!c_k$ for all $n \in \mathbb{N}_0$.

**Lemma.** Let $(X, \mu)$ be a complex measure space, $\Omega \subseteq \mathbb{C}$ open, and $\varphi : X \to \mathbb{C}$ measurable with $\varphi(X) \cap \Omega = \emptyset$. Then,

$$
f(z) := \int_X \frac{1}{\varphi(x) - z} d\mu(x)
$$

is representable by power series on $\Omega$ with coefficients around $a \in \Omega$ given by

$$
c_n = \int_X \frac{1}{(\varphi(x) - a)^{n+1}} d\mu(x)
$$

**Definition.** Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a $C^1$ curve and $f$ a complex valued function defined on $\text{Ran}(\gamma)$. We define the line integral of $f$ along $\gamma$ by

$$
\int_{\gamma} f(\zeta) \, d\zeta := \int_{\alpha}^{\beta} f(\gamma(t)) \, \gamma'(t) \, dt
$$

**Definition.** Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise $C^1$ closed curve, and let $\Omega := \mathbb{C} \setminus \text{Ran}(\gamma)$. The winding number of $\gamma$ around $z \in \Omega$ is defined by

$$
\text{Ind}_\gamma(z) := \int_{\gamma} \frac{1}{\zeta - z} \, d\zeta
$$

**Theorem.** Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise $C^1$ closed curve, and let $\Omega := \mathbb{C} \setminus \text{Ran}(\gamma)$. The function $\text{Ind}_\gamma : \Omega \to \mathbb{Z}$ is continuous and vanishes on the unbounded component of $\Omega$. 

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**Theorem.** Let $\Omega \subseteq \mathbb{C}$ open, $f$ holomorphic on $\Omega$ and $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise $C^1$ closed curve such that $\text{Ind}_\gamma = 0$ on $\mathbb{C} \setminus \Omega$. Then,

(a) (Cauchy’s Theorem)
\[ \int_\gamma f(\zeta) \, d\zeta = 0 \]

(b) (Cauchy’s Formula) If $a \in \Omega \setminus \text{Ran}(\gamma)$
\[ \text{Ind}_\gamma(z) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \, d\zeta \]

**Theorem.** Let $\Omega \subset \mathbb{C}$ be a region (i.e. open and connected), $f$ holomorphic on $\Omega$, and $Z(f) := \{ z \in \Omega : f(z) = 0 \}$. Then,

(a) $Z(f) = \Omega$ or $Z(f)$ has no limit point in $\Omega$

(b) If $Z(f)$ has no limit point in $\Omega$, then $Z(f)$ is at most countable and for all $a \in Z(f)$ there is a unique $n \in \mathbb{N}$ and a holomorphic function $g$ on $\Omega$ with $g(a) \neq 0$, such that
\[ f(z) = (z - a)^n g(z) \quad \forall \ z \in \Omega \]

The number $n$ is known as the order of the zero $a$.

**Corollary.** Let $\Omega \subset \mathbb{C}$ be a region, $f, g$ holomorphic on $\Omega$ and $A \subset \Omega$ with limit points in $\omega$. If $f|_A = g|_A$, then $f = g$.

**Definition.** If $\Omega \subseteq \mathbb{C}$ is open and $f : \Omega \setminus \{a\} \to \mathbb{C}$ holomorphic, we say that $f$ has a singularity at $a$. Further,

1. $a$ is **removable** if there is a holomorphic function $g$ on $\Omega$ such that $g|_{\Omega \setminus \{a\}} = f$.

2. $a$ is a **pole** if there is $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, with $c_n \neq 0$ and a holomorphic function $g$ on $\Omega$, such that
\[ f(z) = \sum_{k=1}^{n} \frac{c_z}{(z - a)^k} + g(z) \quad \forall \ z \in \Omega \setminus \{a\} \]

The number $n$ is known as the order of the pole $a$. 

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(3) $a$ is an **essential singularity** if for all $r > 0$

$$f((B_r(a) \cap \Omega) \setminus \{a\})$$

is dense in $\mathbb{C}$.

\[\text{\textbf{Theorem.} (Cauchy Estimates) Let } a \in \mathbb{C}, r > 0. \text{ If } f \text{ is holomorphic on } B_r(a) \text{ and } |f(z)| \leq M \text{ for all } z \in B_r(a), \text{ then}

$$|f^{(n)}(a)| \leq \frac{n!M}{r^n} \ \forall n \in \mathbb{N}$$

\]

\[\text{\textbf{Theorem.} (Liouville’s Theorem) Any bounded entire function is constant.}

\[\text{\textbf{Theorem.} (Maximum Modulus) Let } \Omega \subset \mathbb{C} \text{ be a region, } f \text{ holomorphic on } \Omega \text{ and } B_r(a) \subset \Omega \text{ for some } a \in \Omega, r > 0. \text{ Then}

$$|f(a)| \leq \sup_{\theta} |f(a + re^{i\theta})|$$

Equality occurs if and only if $f$ is constant in $\Omega$.

\[\text{\textbf{Corollary.} Let } \Omega \subset \mathbb{C} \text{ be a region. If } f \text{ is holomorphic on } \Omega \text{ and } |f| \text{ has a local maximum on } \Omega, \text{ then } f \text{ is constant.}

\[\text{\textbf{Theorem.} (Open Mapping) Let } \Omega \subset \mathbb{C} \text{ be open, } f \text{ holomorphic on } \Omega \text{ and } a \in \Omega \text{ such that } f(a) \neq 0. \text{ Then, there is an open neighborhood } V \subset \Omega \text{ of } a \text{ such that}

\begin{enumerate}
  \item [(a)] $f|_V$ is injective.
  \item [(b)] $W := f(V)$ is open.
  \item [(c)] $(f|_V)^{-1} : W \to V$ is holomorphic.
\end{enumerate}

\[\text{\textbf{Corollary I.} Let } \Omega \subset \mathbb{C} \text{ be open, } f \text{ holomorphic on } \Omega. \text{ If } f \text{ is non constant, then } f \text{ is an open map.}

\[\text{\textbf{Corollary II.} Let } \Omega \subset \mathbb{C} \text{ be open, } f \text{ holomorphic on } \Omega. \text{ If } f \text{ is injective, then } f(\Omega) \text{ is open, } f^{-1} \text{ is holomorphic and } f'(z) \neq 0 \text{ for all } z \in \Omega.

\[\text{\textbf{Theorem.} (Morera) Let } \Omega \subset \mathbb{C} \text{ be open and } f : \Omega \to \mathbb{C} \text{ continuous. Suppose that for every closed triangle } \Delta \subset \Omega \text{ we have}

$$\int_{\partial \Delta} f(\zeta) \, d\zeta = 0$$

Then, $f$ is holomorphic on $\Omega$.\]

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Definition. Let $\Omega \subset \mathbb{C}$ be open. A complex valued function $f$ is said to be **meromorphic** [a function with isolated singularities] on $\Omega$ if there is a subset $A \subseteq \Omega$ with no limit points in $\Omega$ such that $f$ is holomorphic on $\Omega \setminus A$ and at each $a \in A$ $f$ as a pole [a pole or an essential singularity]. If $a \in A$, we set

$$\text{Res}(f, a) := \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \, d\zeta,$$

where $\gamma(t) := a + re^{it}$ for $t \in [0, 2\pi]$ and $r > 0$ is such that $\overline{B_r(a)} \cap A = \{a\}$ and $\overline{B_r(a)} \subseteq \Omega$. One checks that $\text{Res}(f, a)$ is independent of the $r$ chosen. $\▲$

Theorem. (Residue) Let $\Omega \subset \mathbb{C}$ be open and $f$ a meromorphic (or a function with isolated singularities) with set of singularities given by $A$. If $\Gamma$ is a cycle in $\Omega \setminus A$ such that $\text{Ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus A$, then

$$\frac{1}{2\pi i} \int_{\Gamma} f(\zeta) \, d\zeta = \sum_{a \in A} \text{Ind}_\Gamma(a) \text{Res}(f, a).$$

Theorem. Let $\Omega \subset \mathbb{C}$ be open and $\Gamma$ is a cycle in $\Omega$ such that

1. $\text{Ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$
2. $\text{Ind}_\Gamma(z) \in \{0, 1\}$ for $z \in \Omega \setminus \text{Ran}(\Gamma)$.

Let $U := \{z : \text{Ind}_\Gamma = 1\}$ and $f$ a holomorphic function on $\Omega$ that is not zero on any unbounded component and with no zeros in $\text{Ran}(\Gamma)$. Define

$$N_f := \# \text{number of zeros of } f \text{ (counting multiplicity) in } U$$

Then,

(a) (Argument Principle)

$$N_f = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta$$

(b) (Rouché) If $g$ is holomorphic on $\Omega$ and $|f(z) - g(z)| < |f(z)|$ on $\text{Ran}(\Gamma)$, then $N_f = N_g$. 

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