Banach Space Ultraproducts.

Alonso Delfín
University of Oregon.

December 5, 2019

Abstract

Although ultraproducts are probably most associated with logic, the definition is purely set-theoretic. Here, we’ll give this definition from scratch (this includes defining what an ultrafilter is) and explain how to modify it to get an ultraproduct construction for Banach spaces.

1 A Brief Review on Filters and Ultrafilters

Definition 1.1. A filter on a set $X$ is $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

1. $X \in \mathcal{F}$.
2. $\emptyset \notin \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $B$ is such that $A \subseteq B$, then $B \in \mathcal{F}$.
4. If $A$ and $B$ are both in $\mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 1.2. Let $X$ be any set. Below we have three examples of filters on $X$.

(i) The trivial filter on $X$ is $X$ itself.
(ii) Let $x \in X$. The principal filter generated by $x$ is $\mathcal{F}_x = \{ A \subseteq X : x \in A \}$.
(iii) If $X$ has infinite cardinality, the cofinite filter on $X$ is $\mathcal{F} := \{ A \subseteq X : X \setminus A \text{ is finite} \}$

Definition 1.3. Let $\mathcal{F}$ be a filter on a set $X$. We say that $\mathcal{F}$ is an ultrafilter if for any $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Example 1.4. Let $X$ be any set and $x \in X$. The filter $\mathcal{F}_x$ defined above is an ultrafilter.

Lemma 1.5. Every filter is contained in an ultrafilter and maximal filters are ultrafilters.

Proof. This is a trivial application of Zorn’s lemma.

2 Ultraproducts

Definition 2.1. Let $X$ be a set and $(E_x)_{x \in X}$ be an arbitrary collection of non-empty sets indexed by $X$. Let $\mathcal{U}$ be an ultrafilter on $X$. We define an equivalence relation $\equiv_\mathcal{U}$ on $\prod_{x \in X} E_x$ by

$$(\xi_x) \equiv_\mathcal{U} (\eta_x) \iff \{ x \in X : \xi_x = \eta_x \} \in \mathcal{U}$$

The set theoretic ultrapower of $(E_x)_{x \in X}$ with respect to $\mathcal{U}$ is $\prod_{x \in X} E_x / \equiv_\mathcal{U}$. This is sometimes denoted as $\prod_{x \in X} E_x / \mathcal{U}$ or simply by $\prod_\mathcal{U} E_x$. When all the $E_x$ are equal to a set $E$, we get the set theoretic ultrapower, usually denoted by $E^X / \mathcal{U}$.
Lemma 2.6. Moreover, the following result gives a simpler way to compute the norm on $(E_x)_{x \in X}$.

In the above construction, when each set $E_x$ is a Banach space, we do not always get a Banach space when constructing the set theoretic ultraproduct. We need to modify the construction above to always get a Banach space.

Definition 2.3. Let $X$ be a set and $E$ a topological space. If $\mathcal{U}$ is an ultrafilter on $X$, we define the ultralimit of $(\xi_x)_{x \in X}$, denoted by $\lim_{\mathcal{U}} \xi_x$, as follows

$$
\lim_{\mathcal{U}} \xi_x = \xi \in E \iff \{ x \in X : \xi_x \in A \} \in \mathcal{U}
$$

for any open neighborhood $A$ of $\xi$.

Theorem 2.4. If $E$ is a compact Hausdorff space, then for each sequence $(\xi_x)_{x \in X}$, the limit $\lim_{\mathcal{U}} \xi_x$ exists and it’s unique.

Proof. Follows from standard point-set topology arguments.

Example 2.7. If $x_0 \in X$, then $\prod_{x \neq x_0} E_x$ and $E_{x_0}$ are isomorphic as sets.

As before, when all the Banach spaces $E_x$ are equal to $E$, we get the Banach space ultrapower, which we still denote by $E^X/\mathcal{U}$. There is an isometric embedding $\Phi : E \to E^X/\mathcal{U}$ defined by

$$
\Phi(\xi) = (\xi_x)_{\mathcal{U}}
$$

where each $\xi_x := \xi$. Banach space ultrapowers are only interesting in the infinite-dimensional case. Indeed, if $E$ is finite dimensional, the map $\Phi : E \to E^X/\mathcal{U}$ is an isometric isomorphism. To check this, it suffices to show that $\Phi$ is surjective, so let’s take any $(\xi_x)_{\mathcal{U}}$ in $E^X/\mathcal{U}$. Since $M := \sup_{x \in X} \|\xi_x\| < \infty$, then $(\xi_x)$ is in $\{ \xi \in E : \|\xi\| \leq M \}$, which is a compact set because $E$ is finite dimensional. Thus, by Theorem 2.4, the limit $\lim_{\mathcal{U}} \xi_x$ exists and its unique. Call the limit $\xi$. Clearly $\lim_{\mathcal{U}} (\xi_x - \xi) = 0$ and therefore $(\xi_x)_{\mathcal{U}} = \Phi(\xi)$. 

Theorem 2.8. The following classes of Banach spaces are closed under Banach space ultraproducts:

1. Banach algebras
2. $C^*$-algebras
3. $C(K)$ spaces where $K$ is a compact Hausdorff space.
4. $L^p(\mu)$ spaces for $1 \leq p < \infty$

Proof. To prove (1) one checks that the natural multiplication

$$(\xi_x)_{U}(\eta_x)_{U} := (\xi_x\eta_x)_{U}$$

is well defined in $(E_x)_U$. Then, it follows that

$$\|((\xi_x)_{U}(\eta_x)_{U})\| = \lim_U \|\xi_x\eta_x\| \leq \lim_U \|\xi_x\\| \|\eta_x\| = \|(\xi_x)_{U}\| \|(\eta_x)_{U}\|$$

For (2), one checks that the natural involution

$$(\xi_x)_{U}^* := (\xi_x^*)_{U}$$

is well defined, and then notice that

$$\|((\xi_x)_{U}(\xi_x^*)_{U})\| = \lim_U \|\xi_x\xi_x^*\| = \lim_U \|\xi_x\|^2 = \|(\xi_x)_{U}\|^2.$$  

Now, (3) follows because $C(K)$-spaces are commutative $C^*$-algebras and therefore the ultraproduct will also be a commutative $C^*$-algebra, so Gelfand-Naimark gives that the ultraproduct is also a $C(K)$ space. Finally, (4) requires the representation theorem for $L^p$ spaces, which says that a Banach lattice with the property that $\|x + y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$ is an $L^p(\nu)$ space for some measure $\nu$.  

Having introduced the ultraproduct of Banach spaces, it makes sense to to talk about the ultraproduct of operators between Banach spaces. Let $(E_x)_{x \in X}$ and $(F_x)_{x \in X}$ be families of Banach spaces indexed by the same set $X$. Suppose that for each $x \in X$ we have $a_x \in \mathcal{L}(E_x, F_x)$ and that

$$\sup_{x \in X} \|a_x\| < \infty$$

Then, if $U$ is an ultrafilter, we define a map $(a_x)_U : (E_x)_U \to (F_x)_U$ by

$$(a_x)_U(\xi_x)_U = (a_x\xi_x)_U$$

One checks that if $\lim_U \|\xi_x\| = 0$, then $\lim_U \|a_x\xi_x\| = 0$, so the definition above is well defined. Furthermore, one has that $(a_x)_U \in \mathcal{L}((E_x)_U, (F_x)_U)$.

Corollary 2.9. For $1 \leq p < \infty$, we have that $L^p$-operator algebras are closed under Banach space ultraproducts.

Sketch of Proof. If $(L^p(\mu_x))_{x \in X}$ is a family of $L^p$ spaces indexed by $X$ and $(A_x)_{x \in X}$ is a family such that each $A_x$ is a norm-closed subalgebra of $\mathcal{L}(L^p(\mu_x))$, then one checks that for any ultrafilter $U$, the algebra $(A_x)_U$ is a norm-closed subalgebra of $\mathcal{L}(L^p(\mu_x))_U$. Having the definition of ultraproduct of operators, we can check that

$$\left(\mathcal{L}(L^p(\mu_x))\right)_U = \mathcal{L}\left(\left(\mathcal{L}(L^p(\mu_x))\right)_U\right)$$

So the result follows from Theorem 2.8, which assures that $(L^p(\mu_x))_U$ is an $L^p$ space.