Morita Equivalence for $C^*$-algebras.

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Abstract

Morita equivalence was adapted to $C^*$-algebras by Marc Rieffel in the 1970’s and it has since become a standard tool for analyzing group $C^*$-algebras, crossed products and representations. Roughly speaking two $C^*$-algebras $A$ and $B$ are Morita equivalent if there is a Hilbert $(A,B)$-bimodule with some compatibility conditions on the inner products.

The main goal of this talk is to show that two Morita equivalent $C^*$-algebras have equivalent categories of representations. Along the way, I will give many accessible examples. The only two perquisites for following most of the talk are to have some familiarity with Hilbert spaces and with the tensor product of modules.

1 A brief review of Hilbert Modules

Definition 1.1. Let $A$ be a $C^*$-algebra and $E$ a complex vector space which is also a right $A$-module. An $A$-valued right inner product on $E$ is a map

$$E \times E \rightarrow A$$

$$(\xi, \eta) \mapsto (\xi, \eta)_A$$

such that for any $\xi, \eta, \eta_1, \eta_2 \in E$, $a \in A$ and $\alpha \in \mathbb{C}$ we have

1. $$(\xi, \eta_1 + \alpha \eta_2)_A = (\xi, \eta_1)_A + \alpha (\xi, \eta_2)_A.$$  
2. $$(\xi, \eta a)_A = (\xi, \eta)_A a.$$  
3. $$(\xi, \eta)^*_A = (\eta, \xi)_A.$$  
4. $$(\xi, \xi)_A \geq 0$$ in $A$.  
5. $$(\xi, \xi)_A = 0 \implies \xi = 0.$$  

Definition 1.2. Let $A$ be a $C^*$-algebra. A Hilbert $A$-module is a complex vector space $E$ which is a right $A$-module with an $A$-valued right inner product and so that $E$ is complete with the norm $||\xi|| := ||(\xi, \xi)_A||^{1/2}$. We say that $E$ is full if $\langle E, E \rangle_A := \text{span}\{ (\xi, \eta)_A : \xi, \eta \in E \}$ is dense in $A$.

Example 1.3. Let $\mathcal{H}$ be a Hilbert space with the physicists convention that the inner product is linear in the second variable. Then, $\mathcal{H}$ is clearly a full Hilbert $C$-module.

Example 1.4. Any $C^*$-algebra $A$ is clearly a full Hilbert $A$-module with inner product given by $(a, b) \mapsto a^*b$. More generally, $A^n$ is also a full Hilbert $A$-module with the obvious “euclidean” inner product.

Example 1.5. The set of continuous sections of a vector bundle over a compact Hausdorff space $X$ equipped with a Riemannian metric $g$ is a Hilbert $C(X)$-module.
A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert $A$-modules has an adjoint. We will only be interested in those maps that do have an adjoint.

**Definition 1.6.** Let $E$ and $F$ be a Hilbert $A$-modules. A map $t : E \to F$ is said to be **adjointable** if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$

$$\langle t(\xi), \eta \rangle = \langle \xi, t^*(\eta) \rangle$$

The space of adjointable maps from $E$ to $F$ is denoted by $\mathcal{L}_A(E, F)$ and $\mathcal{L}_A(E) := \mathcal{L}_A(E, E)$.

It’s almost immediate that adjointable maps between Hilbert modules are linear and bounded. A standard proof shows that $\mathcal{L}_A(E)$ is a $C^*$-algebra when equipped with the operator norm. We will have special interest for a particular case of adjointable maps, those of “rank 1”:

**Definition 1.7.** Let $E$ and $F$ be a Hilbert $A$-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi, \eta} : F \to E$ by

$$\theta_{\xi, \eta}(\zeta) := \langle \eta, \zeta \rangle_A$$

One easily checks that $\theta_{\xi, \eta} \in \mathcal{L}_A(E, F)$, that $(\theta_{\xi, \eta})^* = \theta_{\eta, \xi} \in \mathcal{L}_A(F, E)$ and that $\|\theta_{\xi, \eta}\| \leq \|\xi\|\|\eta\|$. This gives an analogous of the class of rank-one operators on Hilbert spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}_A(E, F) := \text{span}\{\theta_{\xi, \eta} : \xi \in E, \eta \in F\}$$

It’s also not hard to verify that $\mathcal{K}_A(E) := \mathcal{K}_A(E, E)$ is a closed two sided ideal in $\mathcal{L}_A(E)$, whence $\mathcal{K}(E)$ is also a $C^*$-algebra.

## 2 Morita Equivalence

Given a Hilbert $A$-module $E$, there is a close connection between the $C^*$-algebras $A$ and $\mathcal{K}(E)$. Observe that $E$ is a left $\mathcal{K}(E)$-module when equipped with the obvious left action $v \cdot \xi := v(\xi)$. Further, there is a $\mathcal{K}(E)$-valued left inner product on $E$ defined by

$$\langle \xi, \eta \rangle := \theta_{\xi, \eta}$$

for any $\xi, \eta \in E$. Indeed:

- $\langle (\xi_1 + \alpha \xi_2), \eta \rangle = \theta_{\xi_1 + \alpha \xi_2, \eta} = \theta_{\xi_1, \eta} + \alpha \theta_{\xi_2, \eta}$.
- $(v \xi, \eta) = \theta_{v \xi, \eta} = v \theta_{\xi, \eta} = v(\xi, \eta)$.
- $(\xi, \eta)^* = \theta_{\eta, \xi}^* = \theta_{\eta, \xi} = \langle \eta, \xi \rangle$.
- $\langle \xi, \eta \rangle \geq 0$, whence $(\xi, \xi) \geq 0$.
- If $(\xi, \xi) = 0$, then $(\xi, \xi) = 0$ and therefore $\xi = 0$.
- $\|\langle \xi, \eta \rangle\| \leq \|\xi\|\|\eta\|$ (≤ is immediate and ≥ requires some play with functional calculus). Form this, it follows that $E$ is complete with the norm induced by $(\cdot, \cdot)$.

Hence $E$ is also a left Hilbert $\mathcal{K}(E)$-module. Even better, the right action of $A$ on $E$ is compatible with the left action of $\mathcal{K}(E)$ on $E$. Indeed, for $v \in \mathcal{K}(E)$, $\xi \in E$ and $a \in A$

$$(v \cdot \xi)a = v(\xi)a = v(\xi a) = v \cdot (\xi a)$$

The correct terminology is to say that $E$ is a Hilbert $(\mathcal{K}(E), A)$-bimodule.
Definition 2.1. Two $C^*$-algebras $A$ and $B$ are said to be Morita equivalent if there is a Hilbert $(A, B)$-bimodule $E$ (we use $A(\cdot, \cdot)$ for $A$-valued inner product and $(\cdot, \cdot)_B$ for the $B$-valued one) such that

1. $E$ is a full left Hilbert $A$-module, $E$ is a full right Hilbert $B$-module.
2. For all $\xi, \eta, \zeta \in E$, $a \in A$ and $b \in B$
   \begin{align}
   \langle a\xi, \eta \rangle_B &= \langle \xi, a^*\eta \rangle_B. \\
   A(\xi b, \eta) &= (A(\xi, \eta b^*)). \\
   A(\xi, \eta) \cdot \zeta &= \xi \cdot (\langle \eta, \zeta \rangle_B).
   \end{align}

If $A$ and $B$ are Morita equivalent $C^*$-algebras, then the module $E$ implementing the equivalence is called an $A$-$B$ imprimitivity bimodule.

Example 2.2. We already saw that any full Hilbert $A$-module implements a Morita equivalence between the $C^*$-algebras $A$ and $\mathcal{K}_A(E)$. In particular, if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\mathbb{C}$ and $\mathcal{K}(\mathcal{H})$ are Morita equivalent $C^*$-algebras via the $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$ imprimitivity bimodule $\mathcal{H}$.

Example 2.3. Morita equivalence is weaker than isomorphism. Indeed, given $\varphi : A \to B$, an isomorphism of $C^*$-algebras, we can construct an imprimitive bimodule whose underlying space is $B$, right action of $A$ is $a \cdot b := \varphi(a)b$, left action is left multiplication on $B$, and inner products are given by

\[ A(b_1, b_2) := \varphi^{-1}(b_1b_2^*), \quad \langle b_1, b_2 \rangle_B := b_1^*b_2 \]

Example 2.4. Let $X$ be a locally compact Hausdorff space and $\mathcal{H}$ a Hilbert space. The $C^*$-algebras $A := C_0(X, \mathcal{K}(\mathcal{H}))$ and $B := C_0(X)$ are Morita equivalent. To see this we construct an $(A, B)$ imprimitivity bimodule whose underlying space is $C_0(X, \mathcal{H})$ and operations as follows

- Left action $A \acts C_0(X, \mathcal{H})$ is $(a \cdot f) \in C_0(X, \mathcal{H})$ given by
  \[ (a \cdot f)(x) := a(x)(f(x)) \]
  for any $a \in C_0(X, \mathcal{K}(\mathcal{H}))$ and $f \in C_0(X, \mathcal{H})$.

- Right action $C_0(X, \mathcal{H}) \acts B$ is $(f \cdot b) \in C_0(X, \mathcal{H})$ given by
  \[ (f \cdot b)(x) := f(x)b(x) \]
  for any $b \in C_0(X)$ and $f \in C_0(X, \mathcal{H})$.

- Left $A$-valued inner product is $A(f, g) \in C_0(X, \mathcal{K}(\mathcal{H}))$ given by
  \[ A(f, g)(x) := \theta_{f(x), g(x)} \]
  for $f, g \in C_0(X, \mathcal{H})$.

- Right $B$-valued inner product is $(f, g)_B \in C_0(X, \mathcal{K}(\mathcal{H}))$ given by
  \[ (f, g)_B(x) := (f(x), g(x))_\mathbb{C} \]
  for $f, g \in C_0(X, \mathcal{H})$.

That $C_0(X, \mathcal{H})$ is indeed a $(A, B)$-bimodule follows working pointwise and using that $\mathcal{H}$ is a $(\mathcal{K}(\mathcal{H}), \mathbb{C})$-bimodule. Some analysis is needed to actually check the fullness of the modules but we omit this.

If $A$ and $B$ are Morita equivalent, there is an equivalence between the categories of representations of $A$ and representations of $B$. To see this, we need to discuss first inner tensor products of Hilbert modules.
3 Inner Tensor product and the Induced representation

Let $A$ and $B$ be $C^*$-algebras. Suppose $E$ is a Hilbert $B$-module, that $F$ is a Hilbert $A$-module and that there is a $*$-homomorphism $\varphi : B \to \mathcal{L}(F)$. This naturally makes $F$ a left $B$-module with the action induced by $\varphi$. We can then form the algebraic tensor product of $E$ and $F$ over $B$, denoted by $E \odot_B F$. To do so, we start with the algebraic tensor product $E \otimes F$ and take the quotient by the subspace generated by

$$\{\xi b \otimes \eta - \xi \otimes \varphi(b)\eta : \xi \in E, \eta \in F, b \in B\}$$

This quotient is $E \odot_B F$. We abuse notation and call the image of $\xi \otimes \eta$ in $E \odot_B F$ also by $\xi \otimes \eta$. Then, $E \odot_B F$ is a right $A$-module with an action defined by

$$(\xi \otimes \eta)a = \xi \otimes (\eta a)$$

We now define an $A$-valued inner product on $E \odot_B F$. First we put

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \eta, \varphi(\langle \xi, \xi' \rangle)\eta' \rangle$$

for any $\xi, \xi' \in E$ and $\eta, \eta' \in F$. One checks that this is indeed a well defined $A$-valued inner product on $E \odot_B F$, so to get a Hilbert $A$-module we complete $E \odot_B F$ with respect to the norm induced by this inner product. We denote the completion $E \otimes_\varphi F$ and we call it the interior tensor product of $E$ and $F$ by $\varphi$.

**Theorem 3.1.** If $A$ and $B$ are Morita equivalent $C^*$-algebras, then the category of representations of $A$ is equivalent to the one on $B$.

**Sketch of Proof.** Let $E$ be the $A$-$B$ imprimitivity bimodule implementing the equivalence and $\pi : B \to \mathcal{L}(\mathcal{H}_\pi)$ be a representation of $B$. Write $\langle \cdot, \cdot \rangle_B$ for the $B$-valued right inner product on $E$. Then, regarding $\mathcal{H}_\pi$ as a right $\mathbb{C}$-module, we can form the Hilbert space $E \otimes_\pi \mathcal{H}_\pi$ whose inner product on elementary tensors looks like

$$\langle \xi_1 \otimes v_1, \xi_2 \otimes v_2 \rangle = \langle v_1, \pi(\langle \xi_1, \xi_2 \rangle_B)v_2 \rangle$$

for $\xi_k \in E$ and $v_k \in \mathcal{H}_B$. We define Ind$\pi : A \to \mathcal{L}(E \otimes_\pi \mathcal{H}_\pi)$ by first letting

$$[\text{Ind}\pi(a)](\xi \otimes v) = (a\xi) \otimes v$$

and then extending to all $E \otimes_\pi \mathcal{H}_\pi$. Using that $A$ is Morita equivalent to $B$, this gives a $*$-homomorphism and therefore Ind$\pi$ is a representation of $A$. One checks that $\pi$ is irreducible if and only if Ind$\pi$ is irreducible and every irreducible representation of $A$ is of this form. The Functor Ind from the category of representations of $A$ to the one of representations of $B$ is the one implementing the equivalence. "□"