Lectures on Non-Commutative Rings

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Preface.

This document is a somewhat extended record of the material covered in the Fall 2002 seminar Math 681 on non-commutative ring theory. This does not include material from the informal discussion of the representation theory of algebras that we had during the last couple of lectures. On the other hand this does include expanded versions of some items that were not covered explicitly in the lectures. The latter mostly deals with material that is prerequisite for the later topics and may very well have been covered in earlier courses.

For the most part this is simply a cleaned up version of the notes that were prepared for the class during the term. In this we have attempted to correct all of the many mathematical errors, typos, and sloppy writing that we could find or that have been pointed out to us. Experience has convinced us, though, that we have almost certainly not come close to catching all of the goofs. So we welcome any feedback from the readers on how this can be cleaned up even more.

One aspect of these notes that you should understand is that a lot of the substantive material, particularly some of the technical stuff, will be presented as exercises. Thus, to get the most from this you should probably read the statements of the exercises and at least think through what they are trying to address. And to complicate matters, some of these exercises will be the focus of later sections. For example, we may include an exercise in Section n because it fits there but that will be essential in Section $n + m$ where we may want to pause to discuss it in some detail.

We have assumed that you have more than a nodding acquaintance with rings and modules and their calculus – stuff that's pretty much standard in the local 600 algebra sequence. Still we'll review, often in the exercises and in the actual exposition, much of this material. We will also assume that you are not afraid of categories, functors, and even natural transformations. Still since your familiarity with these is probably rather varied, we will attempt to review the relevant material as we encounter it.

Fortunately, there are some excellent books that include much, if not all, of the material that we cover. In addition to the Hungerford text that you probably used in the 600 algebra I've listed in the Bibliography several appropriate ones. Not surprisingly Anderson and Fuller, [1], probably comes closest to reflecting my bias toward notation and terminology. The Stenstöm [11] and Lambek [8] books are good references for my approach to rings of quotients and the Goldie Theorems. The Goodearl and Warfield text [4] also works well, but is focused on noetherian rings. The Auslander-Reiten-Smalø text [2] is the appropriate next step for those interested in representation theory.

Contents

1 The Basics.

In this section we shall try to lay something of a foundation for the rest of the course. In the process we shall begin a review of rings and modules that will continue though most of these notes.

We shall assume that the reader has a good grasp of the basic notions and results of ring theory. In particular, we expect the reader to be comfortable with the ideas like **ring homomorphism**, **isomorphism**, (**left**/**right**) **ideal**, **factor ring**, the **isomorphism theorems**, **center**, etc. Nevertheless, often when we first encounter a concept, even though it should be familiar, we will try to pause briefly to review it. Frequently, such review will be embedded in the exercises.

For us "ring" will mean "ring with identity"; that is, an identity is part of the defining structure of the ring. Thus, if R is a ring and S is a subring of R, then not only must S have an identity, but it must be the same as the identity of R. Similarly, if R and S are rings with identities 1_R and 1_S, respectively, then for a map $\varphi : R \longrightarrow S$ to be a ring homomorphism, we must have $\varphi(1_R) = 1_S$; that is, all ring homomorphisms are "unital".

An "algebra" is a ring with some additional structure. Let K be a commutative ring, let R be a ring, and let $\gamma : K \longrightarrow \operatorname{Cen} R$ be a ring homomorphism from K into the center of R. Then the system (R, K, γ) is a K-**algebra**. Different homomorphisms γ produce different K-algebras. Still we often suppress explicit mention of γ and talk about the K-algebra R. In this case we also usually write, for each $a \in K$ and $x \in R$,

$$ax = \gamma(a)x.$$

If (R, K, γ) is a K-algebra and if S is a subring of R, then (S, K, γ) is a **subalgebra** of (R, K, γ) in case $\gamma(K) \subseteq S$. When this is the case we tend to refer to S itself as a subalgebra of R. If (R, K, γ) and (S, K, γ') are two K-algebras, then a ring homomorphism $\varphi : R \longrightarrow S$ is an **algebra homomorphism** in case $\varphi \circ \gamma = \gamma'$, or suppressing γ and γ', for all $a \in K$ and $x \in R$,

$$\varphi(ax) = a\varphi(x).$$

The kernel of such a homomorphism is an ideal of the underlying ring R. On the other hand, given an ideal I of R, there is a natural K-algebra structure on the factor ring R/I given by

$$a(x + I) = (ax) + I.$$

Thus, the kernels of K-algebra homomorphisms are precisely the kernels of ring homomorphisms of the underlying rings. So the **ideals** of the algebra (R, K, γ) are the ideals of R and the **factor algebras** are the factor rings of R with the above K-structure.

Strictly speaking, we could deal exclusively with algebras. Indeed, you will note that each ring R is uniquely an algebra over the ring $\mathbb{Z}$ of integers. However, if we drop our insistence on an identity, then we may be able to view a ring R in more than one way as an algebra over $\mathbb{Z}$. Thus, in the interest of caution we shall continue to maintain the distinction between rings and algebras.

As we have seen repeatedly in the past, it is natural to represent groups as permutations on sets. There is an analogous representation theory for rings. Thus, let M be an abelian group. Then the set

$$\mathrm{End}(M)$$

of all endomorphisms of M is a ring under the usual operations. These endomorphism rings provide a rich source of rings. Indeed, as we shall see shortly, we can realize every ring as a subring of such an endomorphism ring. But we must be alert for parity; M actually has *two* endomorphism rings

$$\mathrm{End}^{\ell}(M) \qquad \text{and} \qquad \mathrm{End}^{r}(M)$$

where in $\mathrm{End}^{\ell}(M)$ we view each endomorphism as a left operator and in $\mathrm{End}^{r}(M)$ we treat each endomorphism as a right operator. So in these two rings the operations are given by

In $\mathrm{End}^{\ell}(M)$:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (fg)(x) &= f(g(x))\end{aligned}$$

and in $\mathrm{End}^{r}(M)$:

$$\begin{aligned}(x)(f+g) &= (x)f+(x)g\\ (x)(fg) &= ((x)f)g\end{aligned}$$

Thus, it is clear that the map $f \longmapsto f$ is an anti-isomorphism of $\mathrm{End}^{\ell}(M)$ onto $\mathrm{End}^{r}(M)$, so that

$$\mathrm{End}^{r}(M) \cong (\mathrm{End}^{\ell}(M))^{op}.$$

Since the natural source for rings are the endomorphisms of abelian groups, the natural representation theory for a ring would be determined by its action as endomorphisms on some abelian group. Indeed, that is the approach that we take. Thus, let R be a ring, let M be an abelian group, and let $\lambda : R \longrightarrow \mathrm{End}^{\ell}(M)$ be a ring homomorphism. Then the pair (M, λ) is a **left** R-**representation** of R. Different choices of λ determine different representations, but we tend to suppress the mention of λ and simply write ax for $\lambda(a)(x)$. Thus, alternatively, we can characterize a left representation of R as an abelian group M together with a map $\mu : R \times M \longrightarrow M$ abbreviated

$$\mu(a, x) = ax,$$

satisfying for all $a, b \in R$ and all $x, y \in M$,

$$\begin{aligned} a(x+y) &= ax+ay \qquad & (ab)x &= a(bx)\\ (a+b)x &= ax+bx \qquad & 1x &= x.\end{aligned}$$

These, of course, are just the axioms for a **left** R-**module**. So the left representations of a ring R are simply the left R-modules. For such a module we often refer to the map μ as **scalar** **multiplication** and to the ring R as the **scalar** ring of the module. As another bit of convenient shorthand we will often write just

$$_RM$$

to indicate that M is a left R-module. This poses some small danger since a given abelian group M may admit many different left R-module structures, so we should not invoke this shorthand if there is any possibility of serious ambiguity.

A left representation $\lambda : R \longrightarrow \mathrm{End}^{\ell}(M)$, or the corresponding module $_RM$, is **faithful** if λ is injective. Equivalently $_RM$ is faithful iff for each $a \in R$

$$ax = 0 \text{ for all } x \in M \Longrightarrow a = 0.$$

The left representations of a ring R are the left R-modules. There is a dual right-hand version. A pair (M, ρ) consisting of an abelian group M and a ring homomorphism $\rho : R \longrightarrow \mathrm{End}^{r}(M)$ is a **right representation** of R Again we often downplay the homomorphism ρ and write

$$xa = (x)((a)\rho).$$

Every such a right representation can be characterized as an abelian group M together with a map $\nu : M \times R \longrightarrow M$ abbreviated

$$\nu(x, a) = xa,$$

satisfying for all $a, b \in R$ and all $x, y \in M$,

$$\begin{array}{cc} (x + y)a = xa + ya & x(ab) = (xa)b \\ x(a + b) = xa + xb & x1 = x; \end{array}$$

that is, M is a **right** R-**module** . So the right representations of R are just the right R-modules. Our shorthand for a right R-module M is

$$M_R.$$

For each ring R there are two special modules. Indeed, the maps $\mu : R \times R \longrightarrow R$ and $\nu : R \times R \longrightarrow R$ defined by

$$\mu(a, x) = ax \qquad \text{and} \qquad \nu(x, a) = xa$$

define a left R-module and a right R-module on the abelian group R. These are the **regular** R-modules. Here we allow no ambiguity. That is, the symbols

$$_RR \quad \text{and} \quad R_R$$

are reserved to represent these regular R-modules.

Note that the regular modules ${}_RR$ and R_R are both faithful. Thus, we have a "Cayley Theorem" for rings: Every ring is isomorphic to a subring of the endomorphism ring of an abelian group.

The collection of all left representations of a ring R, that is, the collection of all left R-modules, forms a very rich and interesting category. Let (M, λ) and (M', λ') be two left R-modules. A group homomorphism $f : M \longrightarrow M'$ is an R-**homomorphism** in case for each $a \in R$ the diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda(a)} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow{\lambda'(a)} & M' \end{array}$$

commutes. Thus, a map $f : M \longrightarrow M'$ is an R-homomorphism, also called an R-**linear map**, iff for all $a, b \in R$ and all $x, y \in M$,

$$f(ax + by) = af(x) + bf(y).$$

The identity map on each ${}_RM$ is an R-homomorphism and compositions of R-homomorphisms are R-homomorphisms, so that one readily checks that the collection of all left R-modules together with all R-homomorphisms forms a category. We denote this **category of left** R-**modules** by

$$R\mathbf{Mod}\,.$$

In an entirely analogous fashion the collection of all right R-modules forms a category. Of course, if M and M' are right R-modules, then an R-homomorphism $f : M \longrightarrow M'$ is characterized as a map $f : M \longrightarrow M'$ such that for all $a, b \in R$ and all $x, y \in M$,

$$f(xa + yb) = f(x)a + f(y)b.$$

We denote this **category of right** R-**modules** by

$$\mathbf{Mod}R\,.$$

In general, the categories $R\mathbf{Mod}$ and $\mathbf{Mod}R$ are quite different. Rings for which they are essentially the same (definitely a non-technical statement) often have some very strong properties. Certainly as an important special case we have that if R is commutative (a very strong property), then

$$R\mathbf{Mod} = \mathbf{Mod}R\,.$$

The two categories $R\mathbf{Mod}$ and $\mathbf{Mod}R$ have considerable structure and it is that, the representation theory of R, that will be the focus of much of this course. We will tend to concentrate on the category $R\mathbf{Mod}$ and will let you hold that up to the mirror to learn about the category $\mathbf{Mod}R$. In this Section, we will look at just one of the unusual structural features of these module categories.

Let M and N be two left R-modules. Then the collection of all R-homomorphisms from M to N is a collection of functions from M to N and is, thus, a set. We denote this set by

$$\mathrm{Hom}_R(M, N).$$

If there is no question about the ring R of scalars, we may even omit one piece of decoration and denote this set by $\mathrm{Hom}(M, N)$. Each of these hom sets comes with the structure of an abelian group. That is, if $f, g \in \mathrm{Hom}_R(M, N)$, then we define $f + g : M \longrightarrow N$ by

$$(f + g)(x) = f(x) + g(x) \qquad\qquad (x \in M).$$

Then it is easily checked that $f + g$ is R-linear and that under the operation $(f, g) \longmapsto f + g$ the set $\mathrm{Hom}_R(M, N)$ is an abelian group. Moreover, if M, M', and M'' are left R-modules, then composition

$$\circ : (g, f) \longmapsto g \circ f$$

is a bilinear map $\mathrm{Hom}(M', M'') \times \mathrm{Hom}(M, M') \longrightarrow \mathrm{Hom}(M, M'')$. That is, for all $g, \overline{g} \in \mathrm{Hom}(M', M'')$ and all $f, \overline{f} \in \mathrm{Hom}(M, M')$,

$$(g + \overline{g})f = gf + \overline{g}f \qquad \text{and} \qquad g(f + \overline{f}) = gf + g\overline{f}.$$

For the record, we will state this formally in

1.1. Theorem. *If R is a ring, then in the category R**Mod** (and in **Mod**R) each hom set $\mathrm{Hom}_R(M, N)$ is an abelian group and composition is bilinear. In other words the module categories R**Mod** and **Mod**R are additive.* ■

This has one particularly important consequence.

1.2. Corollary. *If R is a ring, then for each left (right) R-module M, the set $\mathrm{Hom}_R(M, M)$ is a ring w.r.t. the operations of pointwise addition and composition.* ■

If M is an R-module, then an R-homomorphism $f : M \longrightarrow M$ is an R-**endomorphism**. (A bijective endomorphism is an **automorphism**.) Thanks to Corollary 1.2 the set of all endomorphisms of M is actually a ring. Now each R-endomorphism of M is a group endomorphism, so can be viewed as residing in either $\mathrm{End}^{\ell}(M)$ or $\mathrm{End}^{r}(M)$. Here we adopt a very useful convention. If ${}_RM$ is a left R-module, then its R-**endomorphism ring** is the subring

$$\mathrm{End}({}_RM)$$

of $\mathrm{End}^{r}(M)$ consisting of all R-endomorphisms of ${}_RM$. Dually, if M_R is a right R-module, then its R-**endomorphism ring** is the subring

$$\mathrm{End}(M_R)$$

of $\mathrm{End}^{\ell}(M)$ consisting of all R-endomorphisms of M_R. The point is that the R-endomorphism ring of a module *operates on the side opposite the scalars.* So for a module ${}_RM$, the ring R acts on the left and the endomorphism ring $\mathrm{End}({}_RM)$ acts on the right, and they act jointly as follows: for each $r \in R$, $x \in M$, and $s \in \mathrm{End}({}_RM)$

$$(rx)s = r(xs).$$

This leads us to yet another important concept.

Suppose that R and S are two rings and that M is an abelian group that has structures ${}_RM$ and M_S as a left R and a right S module. This then gives M the structure of an (R, S)-**bimodule** in case for all $r \in R$, $x \in M$, and $s \in S$

$$(rx)s = r(xs).$$

If M is such an (R, S)-bimodule, then we abbreviate that fact by

$${}_RM_S.$$

The rub here is that M is an (R, S) bimodule iff each $s \in S$ acts as an R-endomorphism of ${}_RM$ and each $r \in R$ acts as an S-endomorphism of M_S. Maybe we should state that formally:

1.3. Proposition. *Suppose that R and S are rings, and that an abelian group M is a left R-module and a right S-module under the actions $\lambda : R \longrightarrow \mathrm{End}^{\ell}(M)$ and $\rho : R \longrightarrow \mathrm{End}^{r}(M)$. Then ${}_RM_S$ iff $\lambda(r) \in End(M_S)$ for all $r \in R$ iff $\rho(s) \in \mathrm{End}({}_RM)$ for all $s \in S$.* ■

Of course, it follows immediately that if ${}_RM$ is a left R-module and N_R is a right R-module, then there are natural bimodule structures

$${}_RM_{End({}_RM)} \qquad \text{and} \qquad {}_{\mathrm{End}(N_S)}N_S.$$

Given a left R-module ${}_RM$ there is a ring $S = \mathrm{End}({}_RM)$, the endomorphism ring of ${}_RM$. This produces a right S-module M_S. This module induces yet another ring, the endomorphism ring of M_S. The latter is the so-called **biendomorphism** **ring** of the module ${}_RM$, and is denoted by

$$\mathrm{BiEnd}({}_RM) = \mathrm{End}(M_{\mathrm{End}({}_RM)}).$$

It follows from Proposition 1.3 that M is naturally a bimodule

$${}_{\mathrm{BiEnd}({}_RM)}M_{\mathrm{End}({}_RM)}.$$

That same Proposition gives us that the R acts on M as S-endomorphisms of ${}_RM$, whence this left action $\lambda : R \longrightarrow \mathrm{BiEnd}({}_RM)$ is a ring homomorphism.

Let M and N be left R-modules. Then it is tempting to suspect that the abelian group $\mathrm{Hom}_R(M,N)$ is actually an R-module via a scalar multiplication $(a,f) \longmapsto af$ where

$$(af)(x) = a(f(x)).$$

But in general this does not work. (See Exercise 1.1.) However, as we shall now see, $\mathrm{Hom}_R(M,N)$ has a very rich module structure.

1.4. Proposition. *Suppose that R, S and T are rings, and that ${}_RM_S$ and ${}_RN_T$ are bimodules. Then*

$$Hom_R({}_RM_S, {}_RN_T)$$

is an (S,T) bimodule where for each $\varphi \in \mathrm{Hom}_R(M,N)$ and each $m \in M$

$$(s\varphi t)(m) = (\varphi(ms))t$$

for all $s \in S$ and $t \in T$. ■

The proof of this is just a matter of checking that the action as stated works. So we shall omit the details. We might mention that there are other versions, and they are all simply consequences of the behavior of the Hom functors that we shall discuss in a later lecture. For now, though, we shall use this to prove a simple little fact that will be useful in the next couple of sections.

An element e of a ring R is **idempotent** in case $e^2 = e$. If e is a non-zero idempotent of R, then the set

$$eRe = \{exe \mid x \in R\}$$

is, under the addition and multiplication of R, a ring; note, though, that unless $e = 1$, the ring eRe is not a subring of R. Also, if ${}_RM$ is an R-module, then

$$eM = \{ex \mid x \in M\}$$

is an eRe module in the obvious way. Similarly, Re is a right eRe module. So thanks to Proposition 1.4 $\mathrm{Hom}_R(Re,M)$ is a left eRe-module. Now we have the following important fact, a very special version of the Yoneda Lemma.

1.5. Proposition. *Suppose that e is a non-zero idempotent in the ring R. Then for each left R-module ${}_RM$ right multiplication*

$$\rho : eM \longrightarrow \mathrm{Hom}_R(Re,M)$$

is an eRe module isomorphism.

Proof. Here the map ρ satisfies, for all $x \in M$ and all $a \in R$

$$\rho(ex) : ae \longmapsto aex.$$

So clearly, $\rho(ex) \in \mathrm{Hom}_R(Re, M)$ and the map ρ is additive. Then for each $ere \in eRe$,

$$\begin{aligned} \rho(erex)(ae) &= aerex = (aere)(ex) \\ &= \rho(ex)(aere) = ((ere)\rho(ex))(ae), \end{aligned}$$

where the last equality is an application of Proposition 1.4. So ρ is an eRe homomorphism. Finally, one checks that ρ is surjective since for each $\varphi \in \mathrm{Hom}_R(Re, M)$

$$\varphi = \rho(e\varphi(e)),$$

and ρ is injective since if $ex \in \operatorname{Ker} \rho$, then $\rho(ex) : e \longmapsto ex = 0$. ■

As we claimed this is actually just a piece of a much more general and powerful result. The main ingredient missing here is that the isomorphism ρ is **natural**. In this case this means that if ${}_RM$ and ${}_RN$ are left R-modules and if $f : M \longrightarrow N$ is an R-homomorphism, then

$$\begin{array}{ccc} eM & \xrightarrow{f|_{eM}} & eN \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} \\ \mathrm{Hom}(Re, M) & \xrightarrow{f_*} & \mathrm{Hom}(Re, N) \end{array}$$

commutes where $f_*(\varphi) = f \circ \varphi$ for each $\varphi \in \mathrm{Hom}_R(Re, M)$. But this follows since for each $x \in M$ and $a \in R$

$$\begin{aligned} f_*(\rho_M(ex))(ae) &= f \circ \rho_M(ex)(ae) \\ &= f(aex) = aef(x) \\ &= \rho_N(ef(x))(ae). \end{aligned}$$

Exercises 1.

1.1. Here is a simple example of a ring R and left R-modules M and N for which $\mathrm{Hom}_R(M, N)$ is not a left R-module. Let R be the ring of all polynomials in x and y, non-commuting, over $\mathbb{Z}$ of the form $a + bx + cy + dyx$ subject to the relations $x^2 = y^2 = xy = 0$. [Alternatively, let $\mathbb{Z}\langle x, y\rangle$ be the ring of all polynomials in x and y, non-commuting indeterminants, let $I = \langle x^2, y^2, xy\rangle$ be the ideal generated by x^2, y^2, and xy, and let $R = \mathbb{Z}\langle x, y\rangle / I$.] Let $M = N = {}_RR$ both be the regular left R-module. Then the identity function $f : {}_RR \longrightarrow {}_RR$ is a left R-homomorphism. But show that the map $(yf) : a + bx + cy + dyx \longmapsto ay + byx$ is not R-linear. [Hint: Show that $(yf)(xa) \neq x(yf)(a)$.]

1.2. Let R be a ring and M a left R-module. A subgroup N of M is a **submodule** in case it is stable under the action of R. We abbreviate the fact that N is a submodule of M by

$$N \leq M \qquad \text{or} \qquad N \leq_R M$$

to emphasize that N is an R-submodule. Then it is easy to see that the collection $\mathcal{S}(M)$ of all submodules of M is a poset (= partially ordered set) w.r.t. the relation $\leq$ with least element the zero submodule and with greatest element M. Let $\mathcal{A} = \{M_\alpha \mid \alpha \in A\}$ be some set of submodules of M. Then we let

$$\sum \mathcal{A} = \sum_A M_\alpha = \bigcup \{M_{\alpha_1} + \cdots + M_{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in A, n \in \mathbb{N}\}$$

be the set of all finite sums from the union of the set $\mathcal{A}$.

(a) Show that $\mathcal{S}(M)$ is a complete lattice where for each set $\mathcal{A}$ in $\mathcal{S}(M)$

$$\operatorname{glb} \mathcal{A} = \bigcap \mathcal{A} \qquad \text{and} \qquad \operatorname{lub} \mathcal{A} = \sum \mathcal{A}.$$

(b) Show that the lattice $\mathcal{S}(M)$ is **modular**; that is, for $H, K, L \leq M$

$$H \geq K \Longrightarrow H \cap (K + L) = K + (H \cap L).$$

(c) Show that $\mathcal{S}(M)$ need not be **distributive**; that is, it need not satisfy

$$H \cap (K + L) = H \cap K + H \cap L$$

for all $H, K, L \in \mathcal{S}(M)$.

1.3. Let M be a left R-module. If $X \subseteq M$ is any subset of M, then we set

$$RX = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in R, x_i \in X \text{ and } n \in \mathbb{N}\},$$

the set of all finite R-linear combinations of X. (Note: The element 0 is the only linear combination of the empty set.) It is elementary (if tedious) to prove that RX is a submodule of M. We say that RX is **spanned** or **generated** by X and that X is a set of **generators** or is a **spanning set** for RX. We say that a module is **finitely generated** if it has a finite spanning set. It is **cyclic** if it is spanned by a single element. A module M is **simple** if its lattice of submodules consists of just 0 and M.

(a) Prove that a module M is finitely generated iff for every set $\mathcal{A}$ of submodules of M with $\sum \mathcal{A} = M$, there is a finite subset $\mathcal{F} \subseteq \mathcal{A}$ with $\sum \mathcal{F} = M$.

(b) Prove that a left R-module M is cyclic iff it is a factor of the regular module ${}_R R$.

(c) Prove that every module M is the join (= lub) of its cyclic submodules.

(d) Prove that a module ${}_RM$ is simple iff it is the factor of ${}_RR$ modulo some maximal left ideal.

(e) Prove that if ${}_RM$ is finitely generated, then M has a maximal submodule. [Yes, folks, Professor Zorn and the Axiom of Choice are welcome participants in this course.]

1.4. A module ${}_RM$ is **Noetherian** in case the lattice $\mathcal{S}(M)$ satisfies the A.C.C. Prove that the following are equivalent for a module M:

(a) M is Noetherian;

(b) Every submodule of M is finitely generated;

(c) Every non-empty set in $\mathcal{S}(M)$ has a maximal element.

1.5. There is a perfect dual to what we found in the previous exercise. A module ${}_RM$ is **finitely cogenerated** in case for every set $\mathcal{A}$ of submodules of M with $\cap\mathcal{A} = 0$, there is some finite set $\mathcal{F} \subseteq \mathcal{A}$ with $\cap\mathcal{F} = 0$. A module is **Artinian** in case its lattice $\mathcal{S}(M)$ of submodules satisfies the D.C.C. Prove that the following are equivalent for a module M:

(a) M is Artinian;

(b) Every factor module of M is finitely cogenerated;

(c) Every non-empty set in $\mathcal{S}(M)$ has a minimal element.

1.6. A bimodule ${}_RM_S$ is (**faithfully**) **balanced** in case the ring homomorphisms $\lambda : R \longrightarrow \text{End}(M_S)$ and $\rho : S \longrightarrow \text{End}({}_RM)$ are surjective (isomorphisms). (See Proposition 1.3.)

(a) Prove that the regular bimodule ${}_RR_R$ is faithfully balanced.

(b) More generally, prove that if ${}_RM$ is a left R-module, then ${}_{\text{BiEnd}({}_RM)}M_{\text{End}({}_RM)}$ is a faithfully balanced bimodule.

1.7. A module ${}_RM$ is (**faithfully**) **balanced** in case the ring homomorphism $\lambda : R \longrightarrow \text{BiEnd}({}_RM)$ is surjective (an isomorphism). Let K be a field and let V be a finite dimensional K-vector space. Since K is commutative, we can veiw V as a bimodule ${}_KV_K$.

(a) Prove that each of the modules ${}_KV$ and V_K is faithfully balanced.

(b) Prove that the bimodule ${}_KV_K$ is not balanced.

1.8. Let R be commutative. Prove that a module ${}_RM$ is balanced (see Exercise 1.7) iff for every $s \in \text{Cen}(\text{End}({}_RM))$ three is an $\sigma(s) \in R$ with $xs = \sigma(s)x$ for every $x \in M$.

1.9. Recall that a category $\mathcal{C}$ is **additive** in case, first, that for each pair (A, B) of objects in $\mathcal{A}$ the morphism set $\mathrm{Mor}_{\mathcal{A}}(A, B)$ comes equipped with the structure of an additive abelian group, and, second, that composition in the category is bilinear w.r.t. to these additions. Thus, for each object A in such a category, the set $\mathrm{Mor}_{\mathcal{C}}(A, A)$ is a ring. This gives us an entirely different characterization of rings. Indeed, we could define a ring simply to be an additive category with a single object. (Note that leads to a sweeping generalization of the notion of a ring, namely, a small additive category.) Let's pursue this version of the notion of a ring a bit further; in particular, let's see if we can figure out what modules and homomorphisms should be in this setting. For this we introduce the category **Ab** of all abelian groups. (Of course, this is just the category $\mathbb{Z}$**Mod**.) Let R be a ring viewed as an additive category with a single object.

(a) Show that M is a left R-module iff there is a covariant additive functor $F_M : R \longrightarrow \mathbf{Ab}$ whose image lies in the full subcategory with object set $\{M\}$.

(b) Show that if M and N are left R-modules, then $f \in \mathrm{Hom}_{\mathbf{Ab}}(M, N)$ is an R-homomorphism iff f is a natural transformation of the functors F_M to F_N.

(c) Show that M is a right R-module iff there is a contravariant additive functor $F_M : R \longrightarrow \mathbf{Ab}$ whose image lies in the full subcategory with object set $\{M\}$.

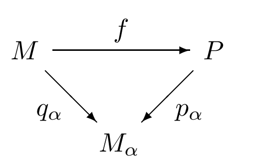
2 Splitting.

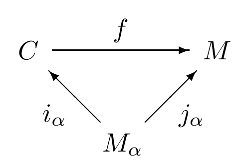
Decompositions of rings and modules play a very important role in the study of these objects, and so in this Section we shall review briefly what's involved in direct decompositions of modules. More on the topic can be found in in the exercises. Also, en route to our study of decompositions we will give a brief treatment of products and coproducts in the setting of module theory — a more general treatment can be found in the exercises.

Throughout this Section R will denote an arbitrary ring. Let $(M_\alpha)_{\alpha\in\Omega}$ be an indexed set of R-modules; we will assume here that they are left modules. Let P and C be R-modules and let $(p_\alpha)_{\alpha\in\Omega}$ and $(i_\alpha)_{\alpha\in\Omega}$ be R homomorphisms

$$p_\alpha : P \longrightarrow M_\alpha \qquad \text{and} \qquad i_\alpha : M_\alpha \longrightarrow C$$

for each $\alpha \in \Omega$. We say that the pair $(P, (p_\alpha)_{\alpha\in\Omega})$ is a **product** of $(M_\alpha)_{\alpha\in\Omega}$ if for every module M and all homomorphisms $q_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in \Omega)$, there is a unique homomorphism $f : M \longrightarrow P$ so that the left diagram below commutes. Dually, we say that the pair $(C, (i_\alpha)_{\alpha\in\Omega})$ is a **coproduct** of $(M_\alpha)_{\alpha\in\Omega}$ if for every module M and all homomorphisms $q_\alpha : M_\alpha \longrightarrow M$ $(\alpha \in \Omega)$, there is a unique homomorphism $f : C \longrightarrow M$ so that the right diagram below commutes.





Clearly, if such a product or coproduct exists, then it is unique to within isomorphism. We now construct both a product and a coproduct for the modules $(M_\alpha)_{\alpha\in\Omega}$.

First, let

$$P = \prod_\Omega M_\alpha$$

be the Cartesian product of the modules M_α equipped with coordinatewise addition and scalar multiplication. Then clearly, P is an R-module, and the coordinate projections $\pi_\alpha : P \longrightarrow M_\alpha$ are module epimorphisms. Moreover, for each $\alpha \in \Omega$ we define $\iota_\alpha : M_\alpha \longrightarrow P$ by[1]

$$\pi_\beta \iota_\alpha(x) = \begin{cases} x, & \text{if } \beta = \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Then each ι_α is an R-monomorphism with the property that

$$\pi_\alpha \iota_\alpha = 1_{M_\alpha}$$

[1] The symbol ι is the lower case Greek iota with no dot as opposed to the Latin letter i with a dot.

for each $\alpha \in \Omega$. Next, we set

$$\coprod_{\Omega} M_\alpha = \sum_{\Omega} \iota_\alpha(M_\alpha) \leq \prod_{\Omega} M_\alpha.$$

We leave the easy proof of the following Theorem to the reader.

2.1. Theorem. *The pair* $(\prod_\Omega M_\alpha, (\pi_\alpha)_{\alpha\in\Omega})$ *is a product and the pair* $(\coprod_\Omega M_\alpha, (\iota_\alpha)_{\alpha\in\Omega})$ *is a coproduct of* $(M_\alpha)_{\alpha\in\Omega}$. ■

We tend to refer to $\prod_\Omega M_\alpha$ and $\coprod_\Omega M_\alpha$ as *the* product and *the* coproduct of $(M_\alpha)_{\alpha\in\Omega}$ suppressing the use of the coordinate maps. Also, if M is a module and if $M_\alpha = M$ for all $\alpha \in \Omega$, then we may abbreviate

$$M^\Omega = \prod_{\Omega} M_\alpha \qquad \text{and} \qquad M^{(\Omega)} = \coprod_{\Omega} M_\alpha.$$

You should note that M^Ω is the R module of all functions from Ω to M with coordinatewise operations, and that $M^{(\Omega)}$ the submodule of M^Ω of all functions $\Omega \longrightarrow M$ with finite support.

Suppose that M is a module and that there are homomorphisms

$$f_\alpha : M \longrightarrow M_\alpha \qquad \text{and} \qquad g_\alpha : M_\alpha \longrightarrow M.$$

Then there exits unique homomorphisms $f : M \longrightarrow \prod M_\alpha$ and $g : \coprod M_\alpha \longrightarrow M$ such that

$$\pi_\alpha \circ f = f_\alpha \qquad \text{and} \qquad g \circ \iota_\alpha = g_\alpha$$

for all $\alpha \in \Omega$. We call the map f the **product** of the maps (f_α) and the map g the **coproduct** of the maps (g_α). Moreover, these will be denoted by

$$f = \prod_{\Omega} f_\alpha \qquad \text{and} \qquad g = \coprod_{\Omega} g_\alpha.$$

2.2. Corollary.

$$\operatorname{Ker} \prod_{\Omega} f_\alpha = \bigcap_{\Omega} \operatorname{Ker} f_\alpha \qquad \textit{and} \qquad \operatorname{Im} \coprod_{\Omega} f_\alpha = \sum_{\Omega} \operatorname{Im} f_\alpha.$$ ■

Now for each $x \in \coprod_\Omega M_\alpha$ we have $\pi_\alpha(x) = 0$ for almost all values of α. Thus,

$$\sum_{\Omega} \iota_\alpha \pi_\alpha : \coprod_{\Omega} M_\alpha \longrightarrow \coprod_{\Omega} M_\alpha$$

is an endomorphism of $\coprod_\Omega M_\alpha$. In fact, we have

2.3. Proposition. *The coordinate maps of the coproduct* $\coprod_\Omega M_\alpha$ *satisfy for all* $\alpha, \beta \in \Omega$

$$\pi_\alpha \iota_\beta = \begin{cases} 1_{M_\alpha}, & \textit{if } \beta = \alpha; \\ 0, & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad \sum_{\Omega} \iota_\alpha \pi_\alpha = 1_{\coprod_\Omega M_\alpha}.$$ ■

A particularly important case occurs with the coproducts of the regular modules ${}_RR$ and R_R. These are the **free** R-modules. For example, the left R-module $R^{(\Omega)}$ is the free left R-module on Ω. These free modules generate all modules in the following sense. Let ${}_RM$ have generating set X, so that $M = RX$. For each $x \in X$ right multiplication $\rho_x : R \longrightarrow M$ defined by $\rho_x : a \longmapsto ax$ is an R-homomorphism with $\operatorname{Im}\rho_x = Rx$. Thus, the coproduct

$$\coprod_X \rho_x : R^{(X)} \longrightarrow M$$

is an R-homomorphism from the free module $R^{(X)}$ to M with image $\sum_X Rx = M$. Thus, we have proved

2.4. Corollary. *If M is an R-module, then M is a factor of a free module.* ■

Next, suppose that each M_α is a submodule of some module M. Then for each α let $i_\alpha : M_\alpha \longrightarrow M$ be the inclusion map. Then the coproduct of the inclusion maps is an R-homomorphism

$$\coprod_\Omega i_\alpha : \coprod_\Omega M_\alpha \longrightarrow M.$$

Note that an element of the coproduct $\coprod M_\alpha$ is just a vector $\mathbf{x} = (x_\alpha)_{\alpha\in\Omega}$ with $x_\alpha \in M_\alpha$ equal to zero for almost all α. So the coproduct of the inclusion maps is characterized by

$$\coprod i_\alpha : \mathbf{x} \longmapsto \sum_\Omega x_\alpha.$$

Our immediate goal is to determine conditions under which we can guarantee that $(M, (i_\alpha)_{\alpha\in\Omega})$ is a coproduct of the submodules $(M_\alpha)_{\alpha\in\Omega}$. The solution is easy and, we assume, well known.

2.5. Theorem. *For submodules $M_\alpha \leq M$ with inclusion maps $i_\alpha : M_\alpha \longrightarrow M$ for all $\alpha \in \Omega$, the following statements are equivalent:*

(a) *$(M, (i_\alpha)_{\alpha\in\Omega})$ is a coproduct of the submodules $(M_\alpha)_{\alpha\in\Omega}$;*

(b) *The coproduct $\coprod_\Omega i_\alpha$ of the inclusion maps is an isomorphism;*

(c) *For each $x \in M$ there exist unique $x_\alpha \in M_\alpha$ with $x_\alpha = 0$ for almost all $\alpha \in \Omega$ such that $x = \sum_\Omega x_\alpha$;*

(d) *$M = \sum_\Omega M_\alpha$ and for each $\alpha_0, \alpha_1, \ldots, \alpha_n$ distinct in Ω*

$$M_{\alpha_0} \cap (M_{\alpha_1} + \cdots + M_{\alpha_n}) = 0.$$ ■

We say that the submodules $(M_\alpha)_{\alpha\in\Omega}$ of M are **independent** if they satisfy the second condition of part (d) in Theorem 2.5. If the submodules $(M_\alpha)_{\alpha\in\Omega}$ of M satisfy any (and hence all) of the conditions of Theorem 2.5, then we say that M is the **direct sum** of its submodules $(M_\alpha)_{\alpha\in\Omega}$, and we denote this by

$$\bigoplus_\Omega M_\alpha,$$

and in the finite case if $\Omega = \{1, 2, \ldots, n\}$, then by

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

It is this finite case that will be of greatest interest to us in this course. Using Proposition 2.3 and Theorem 2.5 it is easy to obtain a valuable characterization of such decompositions in terms of idempotent endomorphisms.

Recall that an element $e \in R$ is **idempotent** in case $e^2 = e$. A set $\{e_\alpha \mid \alpha \in \Omega\}$ of idempotents is (**pairwise**) **orthogonal** in case $e_\alpha e_\beta = 0$ for all $\alpha \neq \beta$ in Ω.

2.6. Theorem. *Let $M_1, \ldots, M_n$ be submodules of the module ${}_RM$. Then*

$$M = M_1 \oplus \cdots \oplus M_n$$

iff there exist pairwise orthogonal idempotents $e_1, \ldots, e_n \in \operatorname{End}({}_RM)$ such that $M_i = Me_i$ for all $i = 1, \ldots, n$ and

$$e_1 + \cdots + e_n = 1.$$

Proof. ($\Longrightarrow$) Let $i_1, \ldots, i_n$ be the inclusion maps for $M_1, \ldots, M_n$ in M, and let

$$f = \coprod_{k=1}^{n} i_k : \coprod_{k=1}^{n} M_k \longrightarrow M.$$

By Theorem 2.5, f is an isomorphism. Then by Proposition 2.3 for each $k = 1, \ldots, n$ the map

$$e_k = f\iota_k\pi_k f^{-1} : M \longrightarrow M$$

satisfies for all j, $\operatorname{Im} e_k = M_k$, $e_k \circ e_j = \delta_{jk}e_k$ and $e_1 + \cdots + e_n = 1_M$. So letting each e_k operate on the right, the $e_1, \ldots, e_n$ are idempotent endomorphisms satisfying the conditions of the Theorem.

($\Longleftarrow$) Since $e_1 + \cdots + e_n = 1_M$ we have that $M = M_1 + \cdots + M_n$. Since the e_k are orthogonal, we have that $Me_je_k = \delta_{jk}M_k$, whence $M_1, \ldots, M_n$ are independent. Thus, an appeal to Theorem 2.5 completes the proof. ■

2.7. Corollary. *Let $I_1, \ldots, I_n$ be left ideals of R. Then*

$${}_RR = I_1 \oplus \cdots \oplus I_n$$

iff there exist pairwise orthogonal idempotents $e_1, \ldots, e_n \in R$ with $e_1 + \cdots + e_n = 1$ and $I_k = Re_k$ for $k = 1, \ldots, n$.

Proof. This follows from Theorem 2.6 using the fact that right multiplication is a ring isomorphism $\rho : R \longrightarrow \mathrm{End}({}_RR)$. ■

Another particularly useful corollary is

2.8. Corollary. *Let R and S be rings and let $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ be an additive functor. If ${}_RM$ is a module and $e_1, \dots, e_n \in \mathrm{End}({}_RM)$ are orthogonal idempotents with $e_1 + \cdots + e_n = 1_M$, then $F(e_1), \dots, F(e_n) \in \mathrm{End}({}_SF(M))$ are orthogonal idempotents with $F(e_1) + \cdots + F(e_n) = 1_{F(M)}$. In particular,*

$$F(M) = F(M)F(e_1) \oplus \cdots \oplus F(M)F(e_n).$$ ■

Let M be a module and let $N \leq M$ be a submodule. Then N is a **direct summand** of M in case there exists some submodule $K \leq M$ with

$$M = N \oplus K.$$

If such a submodule K exists, it is called a **direct complement** of N in M.

If N is a direct summand of M with direct complement K, then there exists an epimorphism $f : M \longrightarrow N$ with kernel K and a homomorphism $g : N \longrightarrow M$ with $fg = 1_N$. This latter condition alone characterizes direct summands.

2.9. Lemma. *Let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be homomorphisms with*

$$fg = 1_N.$$

Then f is an epimorphism, g is a monomorphism, and

$$M = \mathrm{Ker}\, f \oplus \mathrm{Im}\, g.$$

Proof. Trivially f is epic and g is monic. Finally, gf is idempotent and $M = (gf)M \oplus (1 - gf)M = \mathrm{Im}\, g \oplus \mathrm{Ker}\, f$. ■

If, as in Lemma 2.9, $fg = 1_N$, then we say that f is a **split epimorphism** and g is a **split monomorphism**. More generally, a short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is **split** or **split exact** in case f is a split mono and g is a split epi.

2.10. Lemma. *For a short exact sequence*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

of R-modules, the following are equivalent:

(a) *The sequence is split;*

(b) *The monomorphism f is split;*

(c) *The epimorphism g is split;*

(d) $\operatorname{Im} f = \operatorname{Ker} g$ *is a direct summand of M.*

Proof. We'll prove that (d) implies (c), and leave the rest of the easy proof as an exercise. So assume (d), say $M = \operatorname{Im} f \oplus H$. If both $m = f(k) + h$ and $m' = f(k') + h'$ satisfy $g(m) = g(m')$, then $h - h' \in \operatorname{Im} f \cap H$, so that $h = h'$. Thus, there is an R-isomorphism $g' : N \longrightarrow H$ with $g'g = 1_N$. ■

A direct summand of a module M is determined by an idempotent endomorphism of M. That same idempotent determines the endomorphisms of the direct summand from the endomorphisms of M.

2.11. Proposition. *Let ${}_RM$ be a module and let $e \in \operatorname{End}({}_RM)$ is an idempotent. Then $\operatorname{End}({}_RMe) \cong e \operatorname{End}({}_RM)e$.*

Proof. For each $\varphi \in \operatorname{End}(Me)$ let $\overline{\varphi} : x \longmapsto xe\varphi$. Then $\overline{\varphi} \in \operatorname{End}(M)$ and $\overline{\varphi} = e\overline{\varphi}e$. Then $\varphi \longmapsto \overline{\varphi}$ is an injective ring homomorphism. Finally, if $\psi \in \operatorname{End}(M)$, then $\overline{e\psi e} = e\psi e$. ■

2.12. Corollary. *Let $e \in R$ be a nonzero idempotent. Then right multiplication*

$$\rho : eRe \longrightarrow \operatorname{End}({}_RRe)$$

defines a ring isomorphism $eRe \cong \operatorname{End}({}_RRe)$. ■

Exercises 2.

2.1. An abelian group M is **divisible** in case for each $a \neq 0$ in $\mathbb{Z}$, we have $aM = M$. Let $(M_\alpha)_{\alpha \in \Omega}$ be an indexed set of abelian groups.

(a) Prove that the product $\prod_\Omega M_\alpha$ is divisble (torsionfree) iff each M_α is divisible (torsionfree).

(b) Prove that the coproduct $\coprod_\Omega M_\alpha$ is torsion iff each M_α is torsion.

(c) Show that even though each M_α is torsion, then product $\prod_\Omega M_\alpha$ need not be.

2.2. Prove that the abelian group $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ has a non-zero element divisible by 2^n for every $n \in \mathbb{N}$. Deduce that the inclusion monomorphism $0 \longrightarrow \mathbb{Z}^{(\mathbb{N})} \longrightarrow \mathbb{Z}^{\mathbb{N}}$ does not split.

2.3. Let $e, f \in R$ be idempotent.

(a) Prove that as (eRe, fRf) bimodules, $\mathrm{Hom}_R(Re, Rf) \cong eRf$.

(b) Deduce that $Re \cong Rf$ iff there exists some $a \in eRf$ and $b \in fRe$ with $ab = e$ and $ba = f$.

(c) Show that $Re \cong Rf$ iff $eR \cong fR$.

(d) Suppose that $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are idempotents in R with $1 = e_1 + \cdots + e_n = f_1 + \cdots + f_n$ and $Re_i \cong Rf_i$ for each $i = 1, \ldots, n$. Prove that there is an inner automorphism φ of the ring R with $\varphi(e_i) = f_i$ for all $i = 1, \ldots, n$.

2.4. Let $F \cong R^{(\Omega)}$ be a free right module on Ω. Then there exist elements $(x_\alpha)_{\alpha \in \Omega}$ in F with the property that for each $x \in F$ there exists a unique element $(a_\alpha)_{\alpha \in \Omega}$ in $R^{(\Omega)}$ such that $x = \sum_\Omega x_\alpha a_\alpha$. Such a set $(x_\alpha)_{\alpha \in \Omega}$ is a **free basis** for F. Let $\varphi \in \mathrm{End}(F_R)$. Then for each $\beta \in \Omega$ there exists an element $(\varphi_{\alpha\beta})_{\beta \in \Omega} \in R^{(\Omega)}$ with

$$\varphi(x_\beta) = \sum_\Omega \varphi_{\alpha\beta} x_\alpha.$$

(a) Show that the matrix $[\![\varphi_{\alpha\beta}]\!]$ is an $\Omega \times \Omega$ column finite matrix over R.

(b) Show that the map $\varphi \longmapsto [\![\varphi_{\alpha\beta}]\!]$ is a ring isomorphism from $\mathrm{End}(F_R)$ onto the ring $\mathbb{CFM}_\Omega(R)$ of all $\Omega \times \Omega$ column finite matrices over R.

2.5. A free module F is said to have **rank** $\operatorname{card}\Omega$ in case $F \cong R^{(\Omega)}$. If F is free of rank $\operatorname{card}\Omega$, prove that

(a) If Ω is infinite and $F \cong R^{(\Lambda)}$, then $\operatorname{card}\Lambda = \operatorname{card}\Omega$.

(b) If F has a finite generating set, then Ω is finite.

2.6. A ring R is said to be left SBN in case every non-zero free left module of finite rank is isomorphic to ${}_RR$.

(a) Prove that for a ring R the following statements are equivalent:

i. R is SBN;

ii. ${}_RR^{(2)} \cong {}_RR$;

iii. There exist $p, p', i, i' \in R$ with $ip + i'p' = pi = p'i' = 1$ and $p'i = pi' = 0$;

(b) Prove that if Ω is infinite, then the ring $\mathbb{CFM}_\Omega(R)$ of all $\Omega \times \Omega$ column finite matrices over R is SBN.

(c) Prove that no commutative ring can be SBN.

2.7. Let ${}_RT$ be a non-zero module. For every module ${}_RM$ the **trace of** T **in** M and the **co-trace of** T **in** M are the submodules $\text{Tr}_M(T)$ and $\text{coTr}_M(T)$ defined by

$$\text{Tr}_M(T) = \sum_{f\in\text{Hom}(T,M)} \text{Im}\, f, \qquad \text{and} \qquad \text{coTr}_M(T) = \bigcap_{f\in\text{Hom}_R(M,T)} \text{Ker}\, f.$$

We say that T **generates** the left R-module M in case $\text{Tr}_M(T) = M$. Dually, we say that T **co-generates** M in case $\text{coTr}_M(T) = 0$.

(a) Prove that if $f : M \longrightarrow N$ is an R-homomorphism, then

$$f(\text{Tr}_M(T)) \leq \text{Tr}_N(T), \qquad \text{and} \qquad f(\text{coTr}_M(T)) \leq \text{coTr}_N(T).$$

(b) Prove that both $\text{Tr}_R(T)$ and $\text{coTr}_R(T)$ are ideals of R.

(c) For a module ${}_RM$ prove that the following are equivalent:

i. T generates M;

ii. M is isomorphic to a factor of some coproduct $T^{(X)}$ of T;

iii. For each $f : M \longrightarrow N$, if $fh = 0$ for all $h \in \text{Hom}(T, M)$, then $f = 0$.

(d) For a module ${}_RM$ prove that the following are equivalent:

i. T cogenerates M;

ii. M is isomorphic to a submodule of some product T^X of T;

iii. For each $f : N \longrightarrow M$, if $hf = 0$ for all $h \in \text{Hom}(M.T)$, then $f = 0$.

(e) In the category of all finitely generated abelian groups describe the class of groups that are

i. Generated by $\mathbb{Z}$; cogenerated by $\mathbb{Z}$;

ii. Generated by $\mathbb{Q}$; cogenerated by $\mathbb{Q}$;

iii. Generated by $\mathbb{Z}_2$; cogenerated by $\mathbb{Z}_2$.

2.8. Let $R_1, \ldots, R_n$ be a set of rings and let $R = R_1 \times \cdots \times R_n$ be their cartesian product. For each $i = 1, \ldots, n$, let $\iota_i : R_i \longrightarrow R$ be the i^{th} coordinate map. Then notice that $\iota_i(R_i)$ is an ideal of R, that $u_i = \iota_i(1_i)$ is a central idempotent of R, and that ι_i is a ring isomorphism $\iota_i : R_i \longrightarrow \iota_i(R_i)$. Finally, observe that the central idempotents $u_1, \ldots, u_n$ are orthogonal and $1 = u_1 + \cdots + u_n$. Next, let S be an arbitrary ring and let $v_1, \ldots, v_n$ be central idempotents of S. Then for each i, $Sv_i = v_iS = v_iSv_i$ is a ring with identity v_i and is an ideal of S. We say that S is the **ring direct sum** of the ideals $Sv_i, \ldots, Sv_n$ and we write

$$S = Sv_1 \dot{+} \cdots \dot{+} Sv_n$$

if the central idempotents $v_1, \ldots, v_n$ are pairwise orthogonal and $1 = v_1 + \cdots + v_n$. In particular, $R = \iota_1(R_1) \dot{+} \cdots \dot{+} \iota_n(R_n)$.

(a) Let R be a ring and let $I_1, \dots, I_n$ be left ideals with $R = I_1 \oplus \cdots \oplus I_n$. Prove that each I_i is a ring and $R = I_1 \dot{+} \cdots \dot{+} I_n$ iff each I_i is an ideal of R.

(b) Show that it is possible to have an ideal I of a ring with ${}_RR = I \oplus K$ for some left ideal K of R yet $R \neq I \dot{+} K$. [Hint: Consider a ring of 2×2 upper triangular matrices.]

(c) Let I be an ideal of a ring R. Show that $R = I \dot{+} K$ for some ideal K of R iff there is an idempotent $e \in R$ with $I = Re = eR$.

2.9. Recall that a **poset** (or **partially ordered set**) is a pair $(X, \leq)$ consisting of a set X and a partial order $\leq$ on X where a **partial order** is a reflexive, transitive, and anti-symmetric relation. Usually, for a poset $(X, \leq)$ we ignore the relation and call X itself the poset. Each poset X can be completely characterized by means of a category $\mathcal{X}(X) = \mathcal{X}$. The objects of this category $\mathcal{X}$ are the elements of X and the morphisms are the pairs (x, y) with $x \leq y$. For each $x, y \in X$ there is a unique morphism, namely (x, y), from x to y iff $x \leq y$ and no morphism from x to y otherwise. Finally, composition is given by

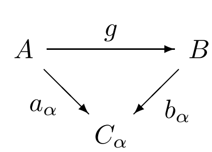
$$(x, y)(y, z) = (x, z).$$

(a) Prove that if X is a poset, then $\mathcal{X}(X)$ is a category with the property that if x, y are objects of $\mathcal{X}(X)$, then $\mathrm{Mor}_{\mathcal{X}}(x, y)$ has at most one element.

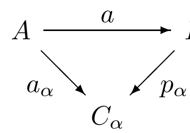
(b) Let X and Y be two posets. Prove that $X \cong Y$ iff $\mathcal{X}(X) \cong \mathcal{X}(Y)$. Thanks to this we usually identify each poset X with its category $\mathcal{X}(X)$.

(c) Let $\mathcal{C}$ be a small category (*i.e,* its object class is a set). Prove that $\mathcal{C}$ is isomorphic to the category $\mathcal{X}(X)$ of a poset X iff for each pair x, y of objects of $\mathcal{C}$, the set $\mathrm{Mor}_{\mathcal{C}}(x, y) \cup \mathrm{Mor}_{\mathcal{C}}(y, x)$ of morphisms between x and y has at most one element.

2.10. In this Exercise we define products and coproducts in an arbitrary category. Let $\mathcal{C}$ be a category and let $\mathbf{C} = (C_\alpha)_{\alpha \in \Omega}$ be an indexed set of objects. (That is, $\mathbf{C} = (C_\alpha)_{\alpha \in \Omega}$ is a function on the set Ω to the object class of $\mathcal{C}$.) We build a category $\mathrm{Mor}(-, \mathbf{C})$ whose objects are all pairs $(A, (a_\alpha)_{\alpha \in \Omega})$ with A an object of $\mathcal{C}$, and $a_\alpha : A \longrightarrow C_\alpha$ a morphism in $\mathcal{C}$ for each $\alpha \in \Omega$. A morphism $g : (A, (a_\alpha)_{\alpha \in \Omega}) \longrightarrow (B, (b_\alpha)_{\alpha \in \Omega})$ is a morphism $g : A :\longrightarrow B$ in $\mathcal{C}$ such that for all $\alpha \in \Omega$ we have that $b_\alpha \circ g = a_\alpha$ or that the diagram



commutes. A terminal object $(P, (p_\alpha)_{\alpha \in \Omega})$ of $\mathrm{Mor}(-, \mathbf{C})$, if it exists, is a **product** in $\mathcal{C}$ of $\mathbf{C} = (C_\alpha)_{\alpha \in \Omega}$. So if a product of $\mathbf{C}$ exists in $\mathcal{C}$, then it is unique to within isomorphism. Note that if $(P, (p_\alpha)_{\alpha \in \Omega})$ is a product for $\mathbf{C}$, then for each $(A, (a_\alpha)_{\alpha \in \Omega})$ in $\mathrm{Mor}(-, \mathbf{C})$ there is a unique morphism $a : A \longrightarrow P$ in $\mathcal{C}$ such that

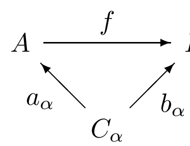


commutes for all $\alpha \in \Omega$. If a product exists, choose one $(P, (p_\alpha)_{\alpha\in\Omega})$. The morphism $a : A \longrightarrow P$ is the **product** of the morphisms $(a_\alpha)_{\alpha\in\Omega}$. We often write

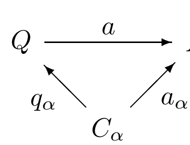
$$P = \prod_{\alpha\in\Omega} C_\alpha$$

and refer to this object as the product. This is slightly dangerous because the morphisms $(p_\alpha)_{\alpha\in\Omega}$, called the **coordinate projections**, are definitely part of the package.

Next, we define the dual notion of a "coproduct". Although we could define this very efficiently as a product in the opposite category $\mathcal{C}^{op}$, we'll start it from scratch. So let $\mathcal{C}$ be a category and let $\mathbf{C} = (C_\alpha)_{\alpha\in\Omega}$ be an indexed set of objects. This time we build a category $\mathrm{Mor}(\mathbf{C}, -)$ whose objects are all pairs $(A, (a_\alpha)_{\alpha\in\Omega})$ with A an object of $\mathcal{C}$, and $a_\alpha : C_\alpha \longrightarrow A$ a morphism in $\mathcal{C}$ for each $\alpha \in \Omega$. A morphism $g : (A, (a_\alpha)_{\alpha\in\Omega}) \longrightarrow (B, (b_\alpha)_{\alpha\in\Omega})$ is a morphism $g : A \longrightarrow B$ in $\mathcal{C}$ such that for all $\alpha \in \Omega$ we have that $f \circ a_\alpha = b_\alpha$ or that the diagram



commutes. Now an initial object $(Q, (q_\alpha)_{\alpha\in\Omega})$ of $\mathrm{Mor}(\mathbf{C}, -)$, if it exists, is a **coproduct** in $\mathcal{C}$ of $\mathbf{C} = (C_\alpha)_{\alpha\in\Omega}$. So if a coproduct of $\mathbf{C}$ exists in $\mathcal{C}$, then it is unique to within isomorphism. Here if $(Q, (q_\alpha)_{\alpha\in\Omega})$ is a coproduct for $\mathbf{C}$, then for each $(A, (a_\alpha)_{\alpha\in\Omega})$ in $\mathrm{Mor}(\mathbf{C}, -)$ there is a unique morphism $a : Q \longrightarrow A$ in $\mathcal{C}$ such that



commutes for all $\alpha \in \Omega$. The morphism $a : Q \longrightarrow A$ is the **coproduct** of the morphisms $(a_\alpha)_{\alpha\in\Omega}$. If a product exists, choose one $(Q, (q_\alpha)_{\alpha\in\Omega})$. We often write

$$Q = \coprod_{\alpha\in\Omega} C_\alpha$$

and refer to this object as the coproduct. Again, this is slightly dangerous because the morphisms $(q_\alpha)_{\alpha\in\Omega}$, also called the **coordinate morphisms**, are part of the coproduct.

(a) Let $\mathcal{C}$ be a poset. (See the previous exercise.) Prove that if $\mathbf{C} = (C_\alpha)_{\alpha\in\Omega}$ has a product $(P, (p_\alpha)_{\alpha\in\Omega})$, then P is the meet of the set $\{C_\alpha\}_{\alpha\in\Omega}$ and hence is unique.

(b) Let $\mathcal{C}$ be a poset. Prove that if $\mathbf{C} = (C_\alpha)_{\alpha\in\Omega}$ has a coproduct $(Q, (q_\alpha)_{\alpha\in\Omega})$, then Q is the join of the set $\{C_\alpha\}_{\alpha\in\Omega}$ and hence is unique.

(c) Again let $\mathcal{C}$ be a poset. Deduce that $\mathcal{C}$ is a lattice iff every finite set of objects has a product and coproduct and that $\mathcal{C}$ is complete iff every set has such a product and coproduct.

(d) Let G be a non-trivial group; so G is a category with a single object and for which every morphism is an automorphism. Show that it fails to have products and coproducts even for finite sets.

3 The Hom Functors — Projectivity and Injectivity.

Our immediate goal is to study the phenomenon of category equivalence, and that we shall do in the next Section. First, however, we have to be in control of the so-called Hom functors and projective modules. Later in the term, the duals of projective modules, the injective modules, will play a crucial role. So in this Section we will treat the basics of these two types of modules.

We begin with a pair of rings R and S and an (R, S)-bimodule ${}_RU_S$. This bimodule determines two Hom functors

$$\mathrm{Hom}_R({}_RU_S, \_) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \text{and} \qquad \mathrm{Hom}_R(\_, {}_RU_S) : R\mathbf{Mod} \longrightarrow \mathbf{Mod}S$$

defined by

$$\mathrm{Hom}_R({}_RU_S, \_) : {}_RM \longmapsto \mathrm{Hom}_R({}_RU_S, {}_RM) \qquad \text{and} \qquad \mathrm{Hom}_R(\_, {}_RU_S) : {}_RM \longmapsto \mathrm{Hom}_R({}_RM, {}_RU_S)$$

for each ${}_RM$ in $R\mathbf{Mod}$, and

$$\mathrm{Hom}_R({}_RU_S, f) : \varphi \longmapsto f \circ \varphi \qquad \text{and} \qquad \mathrm{Hom}_R(f, {}_RU_S) : \psi \longmapsto \psi \circ f$$

for all $f : M \longrightarrow N$ in $R\mathbf{Mod}$ and all $\varphi \in \mathrm{Hom}_R(U, M)$ and all $\psi \in \mathrm{Hom}_R(N, U)$. One readily checks that both of these are additive, that $\mathrm{Hom}_R({}_RU_S, \_)$ is covariant and $\mathrm{Hom}_R(\_, {}_RU_S)$ is contravariant. In general, these functors are not exact, but each is **left exact**. That is, for each short exact sequence

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $R\mathbf{Mod}$ the sequence

$$0 \longrightarrow \mathrm{Hom}(U, K) \overset{\mathrm{Hom}(U,f)}{\longrightarrow} \mathrm{Hom}(U, M) \overset{\mathrm{Hom}(U,g)}{\longrightarrow} \mathrm{Hom}(U, N)$$

is exact in $S\mathbf{Mod}$, and

$$0 \longrightarrow \mathrm{Hom}(N, U) \overset{\mathrm{Hom}(g,U)}{\longrightarrow} \mathrm{Hom}(M, U) \overset{\mathrm{Hom}(f,U)}{\longrightarrow} \mathrm{Hom}(K, U)$$

is exact in $\mathbf{Mod}S$. We do not want to take the time to prove these facts here. Although exactness at M is a bit of a challenge, the proof is pretty strightforward diagram chasing. Proofs are available in standard references; see, for example, [1] and [6]. In the interest of completeness, we'll record all of this formally.

3.1. Theorem. *If ${}_RU_S$ is a bimodule, then the functors*

$$\mathrm{Hom}_R({}_RU_S, \_) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \textit{and} \qquad \mathrm{Hom}_R(\_, {}_RU_S) : R\mathbf{Mod} \longrightarrow \mathbf{Mod}S$$

are both additive and left exact with $\mathrm{Hom}_R({}_RU_S, \_)$ *covariant and* $\mathrm{Hom}_R(\_, {}_RU_S)$ *contravariant.* ∎

Since the notation for these functors is more than a little cumbersome, it is fairly common, when the bimodule U is fixed, to abbreviate

$$M_* = \mathrm{Hom}(U, M), f_* = \mathrm{Hom}(U, f) \qquad \text{and} \qquad M^* = \mathrm{Hom}(M, U), f^* = \mathrm{Hom}(f, U).$$

These Hom functors need not be exact, but as we shall see the modules U for which they are exact play a very important role in our study. Let R be a ring. A left module ${}_RP$ is **projective** in case the covariant evaluation functor $\mathrm{Hom}_R(P, -) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is exact. Dually, a left module ${}_RQ$ is **injective** in case the contravariant duality functor $\mathrm{Hom}_R(-, Q) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is exact. We defined projective and injective left modules — the definition for right modules should be clear. Also, note that although we defined these in terms of functors into $\mathbf{Ab}$, exactness holds more generally. For one example, if S is another ring, then ${}_RP_S$ is projective as a left R-module iff $\mathrm{Hom}_R(P, -) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ is exact. You can easily check out this and the other cases.

The first thing we want to determine is whether any of these modules exist. And, once we learn that there are plenty of them, we'd like to find out just how nice, or not, they may be. Although these concepts are homologically dual, we cannot automatically take every true statement about projective modules, turn it around, and have a true statement about injective modules. So to some extent we have to treat them separately. We'll begin with projectives.

But, first, let $0 \neq e \in R$ be an idempotent, so eRe is a ring with identity e. Define a function $\Lambda_e : R\mathbf{Mod} \longrightarrow eRe\mathbf{Mod}$ by $\Lambda_e : M \longmapsto eM$ for each ${}_RM$ and $\Lambda_e(f) : ex \longmapsto ef(x)$ for each $f : M \longrightarrow N$ in $R\mathbf{Mod}$. Then one easily checks that Λ_e is an additive, exact, covariant functor. The following Lemma then describes a natural isomorphism of the functors Λ_e and $\mathrm{Hom}_R(Re, \_)$

3.2. Lemma. *Let $e \in R$ be a non-zero idempotent. Then for each ${}_RM$ there is a natural eRe-isomorphism*

$$\rho_M : eM \longrightarrow \mathrm{Hom}_R(Re, M),$$

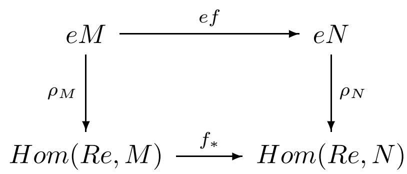
defined by $\rho_M(ex) : ae \longmapsto aex$ for all $ex \in eM$ and $ae \in Re$.

Proof. Clearly, each $\rho_M(ex)$ is an R-linear map from Re to M and the map $\rho_M : ex \longmapsto \rho_M(ex)$ is additive. So let $ere \in eRe$, $ex \in eM$, and $ae \in Re$. Then

$$\begin{aligned} \rho_M(erex)(ae) &= aerex = (aere)(ex) \\ &= \rho_M(ex)(aere) \\ &= [ere\rho_M(ex)](ae), \end{aligned}$$

(where the last equality comes from the way eRe acts on $\mathrm{Hom}_R(Re, M)$), so that ρ_M is R-linear. For each $\varphi \in \mathrm{Hom}_R(Re, M)$ we see that $\varphi = \rho_M(e\varphi(e))$, so ρ_M is epic, and if $ex \in \mathrm{Ker}\,\rho_M$, then

$ex = \rho_M(ex)(e) = 0$, so ρ_M is monic. Finally, let $f : M \longrightarrow N$ in R**Mod** and consider the following diagram:



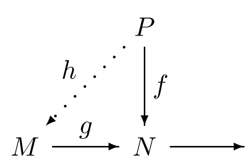
Then for each $ex \in eM$ and $ae \in Re$

$$\begin{aligned} f_*(\rho_M(ex))(ae) &= f(aex) = aef(ex) \\ &= \rho_N(ef(ex))(ae), \end{aligned}$$

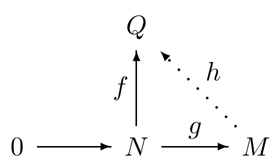
so the diagram commutes and ρ is natural. ■

Since the functor Λ_e is exact, this means that $\mathrm{Hom}_R(Re, \_)$ is exact and Re is projective. In particular, the regular module ${}_RR$ is projective.

Now a module ${}_RP$ is projective precisely when the functor $\mathrm{Hom}_R(P, -)$ is exact. Equivalently, ${}_RP$ is projective iff for every R-epimorphism $M \xrightarrow{g} N \to 0$, every R-homomorphism $f : P \longrightarrow N$ factors through g; that is, there exists a homomorphism $h : P \longrightarrow M$ with $f = g \circ h$.



Dually, a module ${}_RQ$ is injective provided the functor $\mathrm{Hom}_R(-, Q)$ is exact. So, equivalently, ${}_RQ$ is injective iff for every R-monomorphism $0 \to N \xrightarrow{g} M$, every R-homomorphism $f : N \longrightarrow Q$ factors through g; that is, there exists a homomorphism $h : M \longrightarrow Q$ with $f = h \circ g$.



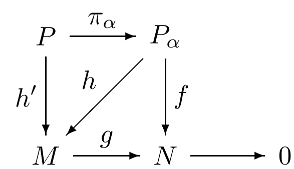
3.3. Proposition. *In R***Mod***,*

(1) *A coproduct $(P, (i_\alpha)_\Omega)$ of left R–modules $(P_\alpha)_\Omega$ is projective iff each P_α is projective;*

(2) *A product $(Q, (p_\alpha)_\Omega)$ of left R–modules $(Q_\alpha)_\Omega$ is injective iff each Q_α is injective.*

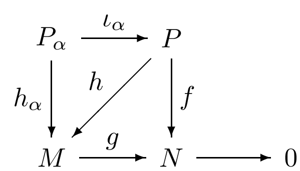
Proof. We'll prove (1) and leave the dual proof of (2) to you. Moreover, it will suffice to assume that P is the coproduct $P = \coprod_\Omega P_\alpha$ with coordinate monomorphisms $\iota_\alpha : P_\alpha \longrightarrow P$. Then also let π_α be the coordinate projection $\pi_\alpha : P \longrightarrow P_\alpha$.

($\Longrightarrow$) Consider the diagram



with g an epimorphism and f a homomorphism. Since P is projective and $f\pi_\alpha : P \longrightarrow N$, there is a homomorphism $h' : P \longrightarrow M$ for which $f\pi_\alpha = gh'$. Define h by $h = h'\iota_\alpha : P_\alpha \longrightarrow M$. Then $gh = gh'\iota_\alpha = f\pi_\alpha\iota_\alpha = f$ and so P_α is projective.

($\Longleftarrow$) Consider the diagram



where g is an epimorphism and $f : P \longrightarrow N$ is a homomorphism. Since P_α is projective and $f\iota_\alpha : P_\alpha \longrightarrow N$, there exists an $h_\alpha : P_\alpha \longrightarrow M$ with $gh_\alpha = f\iota_\alpha$. Then $h = \coprod_\Omega h_\alpha : P \longrightarrow M$ and $(gh - f)\iota_\alpha = gh\iota_\alpha - f\iota_\alpha = gh_\alpha - f\iota_\alpha = 0$ for each $\alpha \in \Omega$ so $gh = f$ and P is projective. ■

Now by Lemma 3.2, the regular module ${}_RR$ is projective, so it follows from Proposition 3.3

3.4. Corollary. *Every free module is projective.*

Proof. By Lemma 3.2, the regular modules ${}_RR$ and R_R are projective. ■

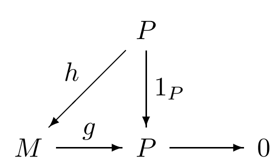
3.5. Proposition. *For a module ${}_RP$ the following are equivalent:*

(a) *${}_RP$ is projective;*

(b) *Every epimorphism $M \xrightarrow{g} P \longrightarrow 0$ splits;*

(c) *${}_RP$ is isomorphic to a direct summand of a free module.*

Proof. (a) $\Longrightarrow$ (b). Consider the diagram



with exact row. If it commutes, then $gh = 1_P$ and the epimorphism g is split.

(b) $\Longrightarrow$ (c). There is some free module ${}_RF$ and some epimorphism $F \longrightarrow P \longrightarrow 0$. Apply (b).

(c) $\Longrightarrow$ (a). By Corollary 3.4 every free module is projective and by Proposition 3.3 every direct summand of a projective module is projective. ■

Since every module is a factor of a free module, we have trivially the following important

3.6. Corollary. *For every left R–module ${}_RM$ there is a projective module ${}_RP$ and an R-epimorphism $P \longrightarrow M \longrightarrow 0$.* ■

The above results for projective modules all have duals for injective modules. There is one small hitch. The notion dual to that of a free module is not quite so obvious. So we will postpone discussion of these "injective cogenerators". Fortunately, though, we can fairly easily find a dual to Corollary 3.6. We begin with a really nice test for injectivity.

3.7. Lemma. [The Injective Test Lemma.] *A module ${}_RQ$ is injective iff for every left ideal I of R and every R–homomorphism $f : I \longrightarrow Q$ there is a homomorphism $g : R \longrightarrow Q$ with $g|_I = f$.*

Proof. The direction ($\Longrightarrow$) is trivial. For the other direction, ($\Longleftarrow$), asume that Q satisfies the stated condition. As a first step we prove the claim

Claim. *If N is a submodule of a cyclic module Rx, and if $f : N \longrightarrow Q$ is an R–homomorphism, then there is a homomorphism $g : Rx \longrightarrow Q$ with $g|N = f$.*

To prove the claim, first note that right multiplication by x, $\rho = \rho_x : R \longrightarrow Rx$ is an epimorphism and $\rho^{\leftarrow}(N) = I$ is a left ideal of R. Now consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{inc} & R \\ & & {\scriptstyle \rho|_I}\downarrow & & \downarrow{\scriptstyle \rho} \\ 0 & \longrightarrow & N & \xrightarrow{inc} & Rx \\ & & {\scriptstyle f}\downarrow & & \\ & & Q & & \end{array}$$

By hypothesis there is a homomorphism $h : R \longrightarrow Q$ such that $h|_I = f\rho|_I$. Since $\operatorname{Ker}\rho \subseteq I$, we have $\operatorname{Ker}\rho \subseteq \operatorname{Ker}h$, so that h lifts to some $g : Rx \longrightarrow Q$ with $h = g\rho$. But then for all $a \in I$, we have $g\rho(a) = h(a) = f\rho(a)$, so $g|_N = f$, as claimed.

Now consider the diagram

$$\begin{array}{ccccc} & & Q & & \\ & & {\scriptstyle f}\uparrow & & \\ 0 & \longrightarrow & N & \xrightarrow{inc} & M \end{array}$$

Let $\mathcal{F}$ be the set of all R–homomorphisms g such that ${}_RN \leq D_g = \operatorname{Dom} g \leq_R M$ and $g : D_g \longrightarrow Q$ with $g|_N = f$. Then $\mathcal{F}$ is a poset w.r.t. the relation $g \leq g'$ iff both $D_g \leq D_{g'}$ and $g'|D_g = g$. By Zorn there exists a maximal $g \in \mathcal{F}$. Let $D = D_g$. We claim $D = M$. Indeed, suppose $x \in M \backslash D$. Consisder the cyclic submodule Rx. Let $K = Rx \cap D \leq Rx$. By the Claim there is an $h : Rx \longrightarrow Q$ with $h|_K = g|_K$. Let $y \in D$ and $a \in R$ with $ax + y = 0$, so $ax = -y \in K$. Then $h(ax) + g(y) = g(ax) - g(y) = 0$. That is, there is an R–homomorphism $\overline{g} : Rx + D \longrightarrow Q$ defined by $\overline{g}(ax + y) = h(ax) + g(y)$, contrary to the maximality of g. ∎

Now we can begin to prove that injectives exist; we begin by finding some injective abelian groups. More generally, let R be a PID. A module ${}_RQ$ is **<u>divisible</u>** in case for each $0 \neq d \in R$, $dQ = Q$. For example, the $\mathbb{Z}$–modules (= abelian groups) $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ are divisible.

3.8. Lemma. *A module ${}_RQ$ over a PID R is injective iff it is divisible.*

Proof. Let $d \in R$ be non-zero. For each $q \in Q$ there is an R-homomorphism $f : Rd \longrightarrow Q$ with $f(d) = q$. Then f has an extension $g : R \longrightarrow Q$ iff $q = f(d) = g(d) = dg(1)$ iff $q \in dQ$. So by the Injective Test Lemma 3.7 ${}_RQ$ is injective iff $Q = dQ$ for each non-zero $d \in R$. ∎

So there do exist some injective abelian groups. Now we prove that there exist a lot of them, enough so that every abelian group lies in at least one injecive.

3.9. Corollary. *If M is an abelian group then there is some divisible group D and a monomorphism $M \longrightarrow D$.*

Proof. There is an epimorphism from the free abelian group $\mathbb{Z}^{(M)}$ onto M, so that for some $K \leq \mathbb{Z}^{(M)}$,

$$M \cong \mathbb{Z}^{(M)}/K \leq \mathbb{Q}^{(M)}/K = D$$

a divisible group. ∎

But for an arbitrary ring R each of these divisible groups produces an injective R-module, so the injectives are starting to pile up.

3.10. Lemma. *Let R be a ring. Then for every divisible abelian group D, the R-module $\operatorname{Hom}_{\mathbb{Z}}(R_R, D)$ is injective.*

Proof. The proof uses one version of "Adjoint Associativity"; namely, if R and S are rings and if there are modules ${}_RM, {}_SN_R, {}_SK$, then there is an isomorphism natural in each variable

$$\operatorname{Hom}_R(M, \operatorname{Hom}_S(N, K)) \cong \operatorname{Hom}_S(N \otimes_R M, K).$$

Now R_R is projective, so the functor $R \otimes_R (\_)$ is exact. Also ${}_{\mathbb{Z}}D$ is injective, so the functor $\operatorname{Hom}_{\mathbb{Z}}(\_, D)$

is exact. Thus, the functor $\mathrm{Hom}_{\mathbb{Z}}(R \otimes_R (\_), D)$ is exact. Then by Adjoint Associativity, the functor $\mathrm{Hom}_R(\_, \mathrm{Hom}_{\mathbb{Z}}(R, D))$ is exact. Therefore, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective. ■

Now, at last we have the important dual to Corollary 3.6.

3.11. Corollary. *For every module ${}_RM$ there exists some injective ${}_RQ$ and a monomorphism $M \longrightarrow Q$.*

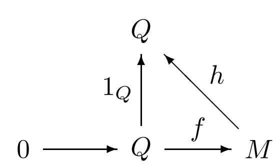
Proof. By Corollary 3.9 the abelian group M can be embedded in some divisible group D. But then by Lemma 3.10, $\mathrm{Hom}(R_R, D)$ is injective, and clearly,

$${}_RM \cong \mathrm{Hom}_R(R, M) \leq \mathrm{Hom}_{\mathbb{Z}}(R, M) \leq \mathrm{Hom}_{\mathbb{Z}}(R, D).$$ ■

And with that we can obtain a (partial) dual to the important Proposition 3.5.

3.12. Proposition. *A module ${}_RQ$ is injective iff every monomorphism $0 \longrightarrow Q \longrightarrow M$ splits.*

Proof. $(\Rightarrow)$ Consider the diagram



If it commutes, then $hf = 1_Q$, and the monomorphism f is split.

$(\Leftarrow)$ By Corollary 3.11, there is an injective module E with $Q \leq E$. So if the inclusion map $Q \hookrightarrow E$ splits, then $E = Q \oplus Q'$ and by Proposition 3.3, Q is injective. ■

We conclude this Section with a brief return to projective modules. The point is to discuss an important characteristic property of finitely generated projectives. We begin with an arbitrary left R-module ${}_RM$. Then its R-**dual**

$$M^* = \mathrm{Hom}_R(M, R)$$

is a right module. The elements of this R-dual M^* are often called **linear functionals**. And, here, we will write them as right operators. Thus, for each $x \in M$, $r \in R$, and $f \in M^*$ we have

$$(rx)f = r(xf) \in R.$$

A pair of sequences $(x_1, \ldots, x_n)$ in M and $(f_1, \ldots, f_n)$ in M^* is called a **dual basis** for M in case for each $x \in M$,

$$x = \sum_{i=1}^{n} (xf_i)x_i.$$

As an important example of such a dual basis, let ${}_RF$ be a free module with free basis $(x_1, \ldots, x_n)$. (See Exercise 2.4.) Then this free basis determines a unique dual basis for F. Indeed, for each $x \in F$ there exists a unique element $(a_1, \ldots, a_n) \in R^{(n)}$ with

$$x = \sum_{i=1}^{n} a_i x_i.$$

Then for each $i = 1, \ldots, n$ the map $p_i : F \longrightarrow R$ defined by $p_i : x \longmapsto a_i$ is a linear functional and the pair $(x_1, \ldots, x_n)$ and $(p_1, \ldots, p_n)$ forms a dual basis for F. And now the important

3.13. Lemma. [Dual Basis Lemma.] *An R-module ${}_RP$ is finitely generated projective iff it has a finite dual basis.*

Proof. ($\Longrightarrow$) Since P is finitely generated there is a free module F with free basis $(x_1, \ldots, x_n)$ and an epimorphism $\varphi : F \longrightarrow P$. Since P is also projective, this map splits and there is a monomorphism $\psi : P \longrightarrow F$ with (we'll write these functions **on the right**) $\psi\varphi = 1_P$. By the above example, there is a dual basis $(x_1, \ldots, x_n)$, $(p_1, \ldots, p_n)$ for F. But then clearly, $(x_1\varphi, \ldots, x_n\varphi)$, $(\psi p_1, \ldots, \psi p_n)$ is a dual basis for P.

($\Longleftarrow$) Let $(y_1, \ldots, y_n)$, $(f_1, \ldots, f_n)$ be a dual basis for P. Then $y_1, \ldots, y_n$ spans P, and there is a free module F with free basis $(x_1, \ldots, x_n)$ and an epimorpihsm $\varphi : F \longrightarrow P$ such that $\varphi : x_i \longmapsto y_i$ for $i = 1, \ldots, n$. But there is a homomorphosm $\psi : P \longrightarrow F$ via

$$\psi : x = \sum_{i=1}^{n} (xf_i)x_i \longmapsto \sum_{i=1}^{n} (xf_i)y_i.$$

Then $\psi\varphi = 1_P$ and φ splits; thus, P is projective. ■

Finally, as a useful application we have the following result whose straightforward proof we will leave for the reader.

3.14. Lemma. *Let ${}_RP$ be a finitely generated projective module with endomorphism ring $S =$ $\mathrm{End}({}_RP)$ and dual $P^* = \mathrm{Hom}_R(P, R)$. Then there is a natural isomorphism*

$$\eta : P^* \otimes_R (\_) \longrightarrow \mathrm{Hom}_R(P, \_),$$

where for each ${}_RM$

$$\eta_M(f \otimes m)(x) = f(x)m$$

for all $f \in P^*$, $m \in M$, and $x \in P$. ■

Exercises 3.

3.1. Let $e \in R$ be a non-zero idempotent and consider the functor $\Lambda_e : R\mathbf{Mod} \longrightarrow eRe\mathbf{Mod}$.

(a) Show that Λ_e is exact.

(b) Show that there is a natural isomorphism $\eta : eR \otimes_R (\_) \longrightarrow \Lambda_e$.

(c) Deduce that every projective module is flat. [A module M_R is **flat** in case the functor $M \otimes_R (\_) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is exact.]

3.2. Prove that if $e, f \in R$ are non-zero idempotents, then

$$\mathrm{Hom}_R(Re, Rf) \cong_{eRe} eRf_{fRf} \cong \mathrm{Hom}_R(fR, eR).$$

3.3. Let ${}_RQ$ be a divisible module over a PID, R.

(a) Prove that every factor module Q/K of Q is divisible.

(b) Must every submodule of Q be divisible? Explain.

(c) Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence over R. If both A and C are divisible, must B be divisible? Explain.

3.4. Products of projective modules need not be projective. Indeed, prove that the abelian group $\mathbb{Z}^{\mathbb{N}}$ is not free. [Hint: Consider the subgroup M of $\mathbb{Z}^{\mathbb{N}}$ consisting of all the sequences $(x_1, x_2, \ldots) \in \mathbb{Z}^{\mathbb{N}}$ such that each power of 2 divides all but finitely many of the terms. If $\mathbb{Z}^{\mathbb{N}}$ is free, then so is M. But then consider $2M$.]

3.5. Prove that every coproduct of injective left R-modules is injective iff R is left noetherian. [Hint: Use the Injective Test Lemma. ($\Leftarrow$) Every left ideal I of R is finitely generated, so if $f : I \longrightarrow \coprod Q_\alpha$, then the image is in a finite sum of the Q_α's. ($\Rightarrow$) Suppose there is an infinite ascending chain $I_1 < I_2 < \cdots$ of left ideals. For each n, let Q_n be an injective module containing R/I_n. Let$I = \cup_{n=1}^{\infty} I_n$ and let $f : I \longrightarrow Q = \coprod_{n=1}^{\infty} Q_n$ be defined by $\pi_n f(a) = a + I_n$. Show that this cannot be extended to R.]

3.6. A ring R is (**left**) **self-injective** if the regular module ${}_RR$ is injective.

(a) Let R be a PID and let I be a non-trivial ideal of R. Prove that the ring R/I is self-injective.

(b) More generally, let R be a commutative ring whose lattice of ideals is a finite chain, say $0 < I_1 < \cdots < I_n = R$. Prove that R is self-injective.

3.7. Prove Lemma 3.14.

3.8. There is a version of the Dual Basis Lemma more general than that given in Lemma 3.13. Indeed, let ${}_RM$ be an R-module. A pair of indexed sets $(x_\alpha)_\alpha \in \Omega$ in M and $(f_\alpha)_\alpha \in \Omega$ in M^* is a **dual basis** for M in case for each $x \in M$, $xf_\alpha = 0$ for almost all $\alpha \in \Omega$ and

$$x = \sum_{\alpha \in \Omega} (xf_\alpha) f_\alpha.$$

Prove that a module ${}_RP$ is projective iff it has a dual basis.

4 Equivalent Categories.

Suppose that R and S are two rings. We want to determine what it would mean for them to have the same representation theories — or for these theories to be equivalent in some meaningful way. What we want should be pretty clear — the two rings have equivalent representation theories when their module (= representation) categories $R\mathbf{Mod}$ and $S\mathbf{Mod}$ are "the same". But that just trades one problem for another — what does it mean for two categories "to be the same"? This is a bit dicey. We'd probably like to say that they are the same if they are isomorphic. But that's too strong. For example, the category of all finite sets, an enormous sucker, has exactly the same categorical structure as the category of all finite subsets of $\mathbb{N}$, but there is no possible way to build a bijective function between these two categories. As we shall see, natural transformations will come to the rescue, and we will be able to determine an appropriate notion of equivalent categories and hence of equivalent representation theories for rings.

Let $\mathcal{C}$ and $\mathcal{D}$ be two (additive) categories. A covariant (additive) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an **equivalence** in case there exists a covariant (additive) functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that GF and FG are naturally isomorphic, respectively, to the identity functors $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$. If such an equivalence exists, then we say that the categories are **equivalent**. This relation is symmetric. Indeed, the two categories are equivalent iff there exist (additive) covariant functors

$$F : \mathcal{C} \longrightarrow \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \longrightarrow \mathcal{C}$$

and natural isomorphisms

$$\eta : GF \longrightarrow 1_{\mathcal{C}} \quad \text{and} \quad \mu : FG \longrightarrow 1_{\mathcal{D}}.$$

This means that η assigns to each object C in $\mathcal{C}$ an isomorphism $\eta_C : GF(C) \longrightarrow C$ in $\mathcal{C}$ and μ assigns to each object D in $\mathcal{D}$ an isomorphism $\mu_D : FG(D) \longrightarrow D$ in $\mathcal{D}$ such that for all $\mathcal{C}$-morphisms $f : C \longrightarrow C'$, and all $\mathcal{D}$-morphisms $g : D \longrightarrow D'$, the diagrams

$$\begin{array}{ccc} GF(C) & \xrightarrow{GF(f)} & GF(C') \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_{C'}} \\ C & \xrightarrow{\quad f \quad} & C' \end{array} \qquad\qquad \begin{array}{ccc} FG(D) & \xrightarrow{FG(g)} & FG(D') \\ {\scriptstyle \mu_D}\downarrow & & \downarrow{\scriptstyle \mu_{D'}} \\ D & \xrightarrow{\quad g \quad} & D' \end{array}$$

commute. When this happens, we call F and G **inverse equivalences** for $\mathcal{C}$ and $\mathcal{D}$.

When such inverse equivalences exist, it means that all "categorical" information available in one of the categories carries over unchanged to the other. In other words, using only categorical machinery we have no way to distinguish equivalent categories. So now let's see how to characterize category equivalences.

A covariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is **faithful** in case it is one-to-one on the class of morphisms of $\mathcal{C}$. The functor F is **full** in case for each pair C, C' in $\mathcal{C}$ the map $F : \mathrm{Mor}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$ is surjective. Finally, the functor F is **dense** in case for each object D in $\mathcal{D}$ there is an object C in $\mathcal{C}$ with $F(C) \cong D$. It turns out that these three properties characterize category equivalences.

4.1. Theorem. *Let $\mathcal{C}$ and $\mathcal{D}$ be (additive) categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant (additive) functor. Then F is an equivalence iff F is full, faithful, and dense.*

Proof. ($\Longrightarrow$) Suppose that F is an equivalence and that G, η, and μ are as above. Then the map $f \longmapsto GF(f)$ is a bijection (isomorphism) from $\mathrm{Mor}_{\mathcal{C}}(C, C')$ onto $\mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$. Thus, the map $f \longmapsto F(f)$ is injective (monic) from $\mathrm{Mor}_{\mathcal{C}}(C, C')$ to $\mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$, and also the map $G : F(f) \longmapsto GF(f)$ from $\mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$ to $\mathrm{Mor}_{\mathcal{C}}(GF(C), GF(C'))$ is surjective. Using the natural isomorphism μ gives that $f \longmapsto F(f)$ is surjective (epic). Thus, F is full and faithful. That it is dense is trivial.

($\Longleftarrow$) This time suppose that F is full, faithful, and dense. We must build a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\varphi : 1_{\mathcal{D}} \longrightarrow FG$ and $\psi : 1_{\mathcal{C}} \longrightarrow GF$.

Since F is dense, for each object D in $\mathcal{D}$ we can choose an object C in $\mathcal{C}$ and an isomorphism $\varphi_D : D \longrightarrow F(C)$. Set $G(D) = C$. Then φ_D is an isomorphism $D \longrightarrow FG(D)$. Now for each morphism $g : D \longrightarrow D'$ in $\mathcal{D}$, since F is full and faithful, there is a unique morphism $f : C = G(D) \longrightarrow C' = G(D')$ for which the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ g\ } & D' \\ {\scriptstyle \varphi_D}\downarrow & & \downarrow{\scriptstyle \varphi_{D'}} \\ FG(D) & \xrightarrow{F(f)} & FG(D') \end{array}$$

commutes. Set $G(g) = f$. By uniqueness it is clear that for each D we have $G(1_D) = 1_{G(D)}$ and that for every pair of morphisms g and g' in $\mathcal{D}$, if $g \circ g'$ is defined, then $G(g \circ g') = G(g) \circ G(g')$. Thus, $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a covariant functor, and $\varphi : 1_{\mathcal{D}} \longrightarrow FG$ is a natural isomorphism. In the additive case, if $g, g' \in Mor_{\mathcal{D}}(D, D')$, then since $F(G(g) + G(g')) = FG(g) + FG(g')$, uniqueness again gives that $G(g + g') = G(g) + G(g')$, so that G is additive.

Finally, we claim that there is a natural isomorphism $\psi : 1_{\mathcal{C}} \longrightarrow GF$. Let C be an object of $\mathcal{C}$. Then there is an isomorphism $\varphi_{F(C)} : F(C) \longrightarrow FGF(C)$. But F is full and faithful, so there is a unique morphism $\psi_C : C \longrightarrow GF(C)$ with $F(\psi_C) = \varphi_{F(C)}$. Since $\varphi_{F(C)}$ is an isomorphism, there exists a morphism $g : FGF(C) \longrightarrow F(C)$ with

$$g\varphi_{F(C)} = 1_{F(C)} \quad \text{and} \quad \varphi_{F(C)}g = 1_{FGF(C)}.$$

Again, since F is full and faithful there is a unique morphism $h : GF(C) \longrightarrow C$ with $F(h) = g$. Thus, $F(h\psi_C) = 1_{F(C)} = F(1_C)$ and $F(\psi_C h) = 1_{FGF(C)} = F(1_{GF(C)}$. Since F is faithful, this means that

$h\psi_C = 1_C$ and $\psi_C h = 1_{GF(C)}$ so that $\psi_C : C \longrightarrow GF(C)$ is an isomorphism. We finish by showing that it is natural. Let $f : C \longrightarrow C'$ be a morphism and consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ {\scriptstyle \psi_C}\downarrow & & \downarrow{\scriptstyle \psi_{C'}} \\ GF(C) & \xrightarrow{GF(f)} & GF(C') \end{array}$$

Hit it with F and it commutes. But F is faithful, so the diagram itself commutes. ∎

Although an equivalence of two categories need not be an isomorphism, under which virtually eveything of mathematical interest is preserved, it does preserve the really important categorical stuff. For example,

4.2. Corollary. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a categorical equivalence. Then a morphism $f : C \longrightarrow C'$ in $\mathcal{C}$ is a monomorphism, epimorphism, or isomorphism, respectively, iff $F(f) : F(C) \longrightarrow F(C')$ is a monomorphism, epimorphism, or isomorphism in $\mathcal{D}$. If $\mathcal{C}$, $\mathcal{D}$, and F are additive, then for each pair C, C' of objects in $\mathcal{C}$,*

$$F : \mathrm{Mor}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C), F(C'))$$

is a group isomorphism, and for each object C in $\mathcal{C}$

$$F : \mathrm{Mor}_{\mathcal{C}}(C, C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C), F(C))$$

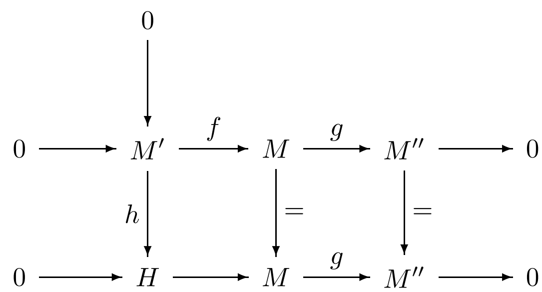
is a ring isomorphism.

Proof. Suppose that $f : C \longrightarrow C'$ is a monomorphism. Let $g, g' : D \longrightarrow F(C)$ be morphisms in $\mathcal{D}$ with $F(f)g = F(f)g'$. Since F is dense, there is some C'' and some isomorphism $g'' : F(C'') \longrightarrow D$. So gg' and $g'g''$ are morphisms $F(C'') \longrightarrow F(C)$. Since F is full, there exist morphisms h, h' in $\mathcal{C}$ with $F(h) = gg''$ and $F(h') = g'g''$. So $F(fh) = F(f)F(h) = F(f)gg'' = F(f)g'g'' = F(f)F(h') = F(fh')$. But F is faithful, so $fh = fh'$ and since f is monic, $h = h'$. So $gg'' = g'g''$. But g'' is an isomorphism, so $g = g'$. Thus, $F(f)$ is monic. We'll leave the rest of this to the reader. ∎

We want to apply this to the representation theory of rings. Two rings R and S are (**Morita**) **equivalent** in case the additive categories R**Mod** and S**Mod** are equivalent. Not at all surprisingly, if $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ is an additive equivalence, then F is an exact functor.

4.3. Theorem. *If R and S are equivalent rings with $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ an additive equivalence, then F is exact.*

Proof. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be exact over R. Then by Corollary 4.2, $F(f)$ is monic and $F(g)$ is epic. Also, $F(g) \circ F(f) = F(g \circ f) = F(0) = 0$, and thus, $\operatorname{Im} F(f) \leq \operatorname{Ker} F(g)$. So, it will suffice to show that $\operatorname{Ker} F(g) \leq \operatorname{Im} F(f)$. Let $K = \operatorname{Ker} F(g)$. Since F is dense, there is an H in $R\mathbf{Mod}$ with $F(H) \cong K$. Since F is also full and faithful, there exist R-homomorphisms for which the diagram



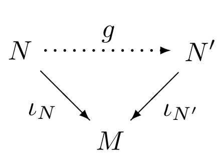
commutes and has exact rows and columns. Thus, h is an isomorphism, so that $F(h) : F(M') \longrightarrow K$ is an isomorphism, and $\operatorname{Ker} F(g) = \operatorname{Im} F(f)$. ■

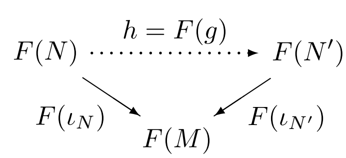
A property of modules is said to be a **Morita invariant** if it is preserved by additive equivalences. Our next result is the key to identifying a whole lot of important Morita invariants.

4.4. Theorem. *If R and S are equivalent rings with $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ an additive equivalence, then for each module ${}_RM$ the lattice of submodules of ${}_RM$ and the lattice of submodules of ${}_SF(M)$ are isomorphic.*

Proof. Let ${}_RM$ be a left R-module. For each $N \leq M$, let $\iota_N : N \hookrightarrow M$ be the inclusion monomorphism, and let $\Phi(N) = \operatorname{Im} F(\iota_N) \leq F(M)$. We claim that the map $\Phi : N \longmapsto \Phi(N)$ from the submodule lattice of M to that of $F(M)$ is a lattice isomorphism.

To begin with Φ is injective. For if $\Phi(N) = \Phi(N')$, then there is an S-isomorphism $h : F(N) \longrightarrow F(N')$ with $h \circ F(\iota_{N'}) = F(\iota_N)$. But since F is full, there must be an R-homomorphism, necessarily an isomorphism by Corollary 4.2, $g : N \longrightarrow N'$ with $F(g) = h$. Then since F is faithful, $g \circ \iota_{N'} = \iota_N$, and so $N = N'$.





Next, Φ is surjective. Let K be a submodule of $F(M)$ with, say, $\iota_K : K \hookrightarrow F(M)$ the inclusion mono. Since F is dense, there is some ${}_RH$ and some S-isomorphism $f : F(H) \longrightarrow K$. Since F is full, there is an R-monomorphism $g : H \longrightarrow M$ with $F(g) = \iota_K \circ f : F(H) \longrightarrow F(M)$. Thus, $\Phi(\operatorname{Im} g) = K$.

Finally, for submodules N, N' of M we have $N \leq N'$ iff there is some monomorphism $j : N \longrightarrow N'$ with $\iota_N = \iota_{N'} \circ j$ iff there is some monomorphism $g : F(N) \longrightarrow F(N')$ with $F(\iota_N) = F(\iota_{N'}) \circ g$ iff $\Phi(N) \leq \Phi(N')$. ■

4.5. Corollary. *Each of the following properties of a module is a Morita invariant. That is, if F is an equivalence from the category R**Mod** of left R-modules to the category S**Mod** of left S-modules, then one of the following applies to $_RM$ iff it applies to $_SF(M)$:*

(a)	*simple;*	**(d)**	*artinian;*	**(g)**	*finitely generated;*
(b)	*semisimple;*	**(e)**	*noetherian;*	**(h)**	*projective;*
(c)	*indecomposable*	**(f)**	*composition length c;*	**(i)**	*injective.*

Proof. By Corollary 4.2, epimorphisms and monomorphisms are preserved, and splitting is functorial, so projective and injective are Morita invariants. All the rest of these are characterized in the lattice of submodules. ∎

So what isn't a Morita invariant? By and large properties that deal with the elements or with the arithmetic of the ring are not invariants. Once we have found another characterization of equivalence, some of these will become quite clear. But some properties that are not invariants are commutativity, division ring, existence of nilpotent elements — all dealing with the arithmetic of the ring. A possibly surprising non-invariant is that of being a free module. Projectivity is, but freeness is not.

Exercises 4.

4.1. Finish the proof of Corollary 4.2 by proving that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence and if $f : C \longrightarrow C'$ is a surjective, then so is $F(f) : F(C) \longrightarrow F(C')$.

4.2. Prove that the category **sets** of all finite sets is equivalent to its full subcategory $\mathcal{C}$ with object class the collection of all finite initial subsets of $\mathbb{N}$.

4.3. Let $\mathcal{C}$ be a category. Choose an object from each isomorphism class of objects of $\mathcal{C}$. (Yes, we are using a monster version of the Axiom of Choice here.) For each object A denote the representative of its class by $\hat{A}$. So

$$\hat{A} = \hat{B} \iff A \cong B.$$

Let $\hat{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ with object class the representatives $\hat{A}$ of the objects of $\mathcal{C}$. We call this the **skeleton** of $\mathcal{C}$. Using the word "the" is premature, but in Exercise 4.4 we will prove that there is essentially one skeleton.

(a) Find a skeleton for **sets** whose objects are all subsets of $\mathbb{N}$;

(b) Find a skeleton for **ab** the category of finitely generated abelian groups.

(c) Find a skeleton for the category of finite abelian 2-groups.

4.4. Two categories $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** in case there exist covariant functors, $S : \mathcal{C} \longrightarrow \mathcal{D}$ and $T : \mathcal{D} \longrightarrow \mathcal{C}$ such that $TS = 1_{\mathcal{C}}$ and $ST = 1_{\mathcal{D}}$.

(a) Show that there exists a category with two objects that is not isomorphic to any of its skeletons.

(b) Show that if $\hat{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are two skeletons of the category $\mathcal{C}$, then they are isomorphic.

4.5. Prove that two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent iff they have isomorphic skeletons.

4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be two (additive) categories. A contravariant (additive) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a **duality** in case there exists a contravariant (additive) functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that GF and FG are naturally isomorphic, respectively, to the identity functors $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$. Prove that for a pair $\mathcal{C}$ and $\mathcal{D}$ of categories the following statements are equivalent:

(a) There is a duality $F : \mathcal{C} \longrightarrow \mathcal{D}$;

(b) There is an equivalence $H : \mathcal{C}^{op} \longrightarrow \mathcal{D}$;

(c) There is a full, faithful, and dense contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$.

4.7. Let F : **Top** $\longrightarrow$ **Sets** be the forgetful functor from the category of topological spaces to the category of sets. On the other hand, let T : **Sets** $\longrightarrow$ **Top** be the functor that assigns to each set S the discrete space $T(S)$ on S, and to each $f : S \longrightarrow S'$ in **Sets**, the continuous function $T(f) = f : T(S) \longrightarrow T(S')$ in **Top**.

(a) Show that the functor F : **Top** $\longrightarrow$ **Sets** is faithful and dense but not full.

(b) Show that the functor T : **Sets** $\longrightarrow$ **Top** is faithful and full but not dense.

4.8. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be two functors. These two functors then determine two bi-functors

$$\mathrm{Mor}_{\mathcal{D}}(F(-),-) : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathbf{Sets} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-,G(-)) : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathbf{Sets}.$$

We say that (F, G) is an **adjoint pair** in case there is a natural isomorphism

$$\eta : \mathrm{Mor}_{\mathcal{C}}(F(-),-) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(-,G(-)).$$

Alternatively, we say that F is a **left adjoint** of G and that G is a **right adjoint** of F. Of course, this means that for each $f : C' \longrightarrow C$ in $\mathcal{C}$ and each $g : D \longrightarrow D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathcal{D}}(F(C),D) & \xrightarrow{\ \mathrm{Mor}(F(f),g)\ } & \mathrm{Mor}_{\mathcal{D}}(F(C'),D') \\
\Big\downarrow \eta_{(C,D)} & & \Big\downarrow \eta_{(C',D')} \\
\mathrm{Mor}_{\mathcal{C}}(C,G(D)) & \xrightarrow{\ \mathrm{Mor}(f,G(g))\ } & \mathrm{Mor}_{\mathcal{C}}(C',G(D'))
\end{array}$$

commutes. That is, for each $\varphi \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$,

$$G(g) \circ \eta_{(C,D)}(\varphi) \circ f = \eta_{(C',D')}(g \circ \varphi \circ F(f)).$$

(a) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be inverse equivalences. Prove that both (F, G) and (G, F) are adjoint pairs.

(b) Let K be a field and let V be a K-vector space. Consider the additive functors

$$F = \mathrm{Hom}_K(V, -) : K\mathbf{Mod} \longrightarrow K\mathbf{Mod} \quad \text{and} \quad G = (-) \otimes_K V : K\mathbf{Mod} \longrightarrow K\mathbf{Mod}\,.$$

Show that (F, G) is an adjoint pair.

(c) Show that if F and T are the functors of Exercise 4.7, then (T, F) is an adjoint pair.

5 An Informative Example.

Our primary concern with equivalence is what it involves for module categories. Once we have the Morita characterization of equivalences of module categories, we'll be able to see the incredible power of that property. To set the stage for stating and proving the Morita results, in this Section we shall discuss some necessary conditions and, based on those, investigate one particularly revealing example. That simple example will provide a template for the complete result.

So let's begin by trying to find some necessary conditions. Thus, let R and S be two rings and suppose that there are additive functors

$$F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \text{and} \qquad G : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}$$

that are inverse equivalences. When this happens, we say that the rings R and S are (**Morita**) **equivalent**.

Suppose, then, that F and G determine a Morita equivalence. Set

$${}_SQ = F({}_RR) \qquad \text{and} \qquad {}_RP = G({}_SS).$$

Then by Corollary 4.5, ${}_SQ$ and ${}_RP$ are both finitely generated projective. By Corollary 4.2 under F and G we have ring isomorphisms

$$\operatorname{End}({}_SQ) \cong \operatorname{End}({}_RR) \cong R \qquad \text{and} \qquad \operatorname{End}({}_RP) \cong \operatorname{End}({}_SS) \cong S.$$

In particular,

$${}_SQ_R \qquad \text{and} \qquad {}_RP_S$$

are bimodules. But there is more. For if we compute the R-dual of P, we find, using Corollary 4.2,

$${}_SP^*_R = \operatorname{Hom}_R({}_RP_S, R) \cong \operatorname{Hom}_R(G(S), G(Q)) \cong \operatorname{Hom}_S(S, {}_SQ_R) \cong {}_SQ_R,$$

with all isomorphisms natural. Since a similar thing holds for the dual of Q, we see that there are natural isomorphisms

$${}_SP^*_R = \operatorname{Hom}_R(P, R) \cong {}_SQ_R \quad \text{and} \quad {}_RQ^*_S = \operatorname{Hom}_S(Q, S) \cong {}_RP_S.$$

Now we can actually compute the functors F and G. Indeed, if ${}_RM$, then by Lemma 3.2 and Corollary 4.2 we have natural isomorphisms

$$\begin{aligned} F(M) &\cong \operatorname{Hom}_S(S, F(M)) \\ &\cong \operatorname{Hom}_S(FG(S), F(M)) \\ &\cong \operatorname{Hom}_R(G(S), M) \\ &= \operatorname{Hom}_R(P, M). \end{aligned}$$

A perfectly symmetric argument shows that for each ${}_SN$ there is a natural isomorphism

$$G(N) \cong \mathrm{Hom}_S(Q, N).$$

Remarkable! If R and S are equivalent, then each of them is the endomorphism ring of a finitely generated projective module over the other one. And those projectives characterize the functors that produce the equivalence. That's not quite the whole story, but at least let's record what we have so far.

5.1. Lemma. *Let R and S be equivalent rings via the inverse equivalences*

$$F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \textit{and} \qquad G : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}\,.$$

Then there exist finitely generated projective modules ${}_RP$ and ${}_SQ$ such that

$$R \cong \mathrm{End}({}_SQ) \qquad \textit{and} \qquad S \cong \mathrm{End}({}_RP),$$

and there exist natural isomorphisms

$$F \cong \mathrm{Hom}_R(P, \_\_) \qquad \textit{and} \qquad G \cong \mathrm{Hom}_S(Q, \_\_).$$

Morevover, there are bimodule isomorphisms

$${}_SP^*_R = \mathrm{Hom}_R(P, R) \cong {}_SQ_R \quad \textit{and} \quad {}_RQ^*_S = \mathrm{Hom}_S(Q, S) \cong {}_RP_S. \qquad \blacksquare$$

For the rest of this Section we want to consider a simple exmple of rings and projective modules that satisfy the conditions of Lemma 5.1 and see what it would take in this example to deduce that the two rings are actually equivalent. Once we have worked our way through that example, the general result characterizing equivalent rings should be quite transparent. So let's start with a ring R and choose some, particularly manageble, but not trivial, finitely generated projective. The first thing that comes to mind would be a cyclic projective, but such a projective is just some direct summand of ${}_RR$. So let $e \in R$ be a non-zero idempotent and set

$$P = Re.$$

Now for that projective module we have (see Corollary 2.12 and Lemma 3.2)

$$\mathrm{End}({}_RRe) \cong eRe \qquad \text{and} \qquad P^* = \mathrm{Hom}_R(Re, R) \cong eR.$$

So for S and Q we should take

$$S = eRe \qquad \text{and} \qquad {}_SQ_R = {}_{eRe}eR_R.$$

So our task is to determine whether or under what conditions the additive functors

$$\mathrm{Hom}_R(Re, \_\_) : R\mathbf{Mod} \longrightarrow eRe\mathbf{Mod} \quad \text{and} \quad \mathrm{Hom}_{eRe}(eR, \_\_) : eRe\mathbf{Mod} \longrightarrow R\mathbf{Mod}$$

define inverse equivalences. But by Lemma 3.2 and Exercise 3.1 we have that the functors $\mathrm{Hom}_R(Re, \_)$ and $eR \otimes_R (\_)$ are naturally isomorphic. Thus, instead, we consider the equivalent task of determining whether the functors

$$F : eR \otimes_R (\_) : R\mathbf{Mod} \longrightarrow eRe\mathbf{Mod} \quad \text{and} \quad G : Re \otimes_{eRe} (\_) : eRe\mathbf{Mod} \longrightarrow R\mathbf{Mod}$$

are inverse equivalences. Now the job has been reduced to quite manageable dimensions, and the first step is easy. Indeed, we have

5.2. Lemma. *There is a natural isomorphism $\mu : FG = eR \otimes_R Re \otimes_{eRe} \otimes (\_) \longrightarrow 1_{eRe\mathbf{Mod}}$ where for each ${}_{eRe}N$, each $n \in N$, each $ea \in eR$, and each $be \in Re$,*

$$\mu_N : ea \otimes be \otimes n \longmapsto eaben.$$

Proof. That μ is a natural surjective homomorphism is trivial. But if ${}_{eRe}N$ and if

$$x = \sum_{i=1}^{n} ea_i \otimes be \otimes n_i \in \mathrm{Ker}\, \mu_N,$$

then

$$x = \sum_{i=1}^{n} ea_i \otimes be \otimes n_i = e \otimes e \otimes \mu_N(x) = 0,$$

so μ is also monic. ∎

Thus, we have one of the natural isomorphisms we want. The issue, then, is coming back the other way: can we find a natural isomorphism

$$\nu : GF = Re \otimes_{eRe} \otimes_R eR \otimes (\_) \longrightarrow 1_{R\mathbf{Mod}}$$

There is certainly an obvious candidate, the one given by, for each ${}_RM$

$$\nu_M : ae \otimes eb \otimes m \longmapsto aebm.$$

It's pretty clear that this gives us a natural homomorphism. But, in general, it just misses being an isomorphism. The problem is that to be an isomorphism it must be surjective on $GF(R)$; that is, it must satisfy (at least)

$$\nu_R(Re \otimes eR \otimes R) = R.$$

But the image of ν_R on $Re \otimes eR \otimes R$ is just the ideal ReR. So for ν to be the natural isomorphism that we want, we must have at least that $ReR = R$. And, happily, that turns out to be enough.

5.3. Lemma. *There is a natural homorphism* $\nu : GF = Re \otimes_{eRe} eR \otimes_R \otimes(\_) \longrightarrow 1_{R\mathbf{Mod}}$ *where for each* ${}_RM$*, each* $m \in M$*, each* $ae \in Re$*, and each* $eb \in eR$*,*

$$\nu_M : ae \otimes eb \otimes m \longmapsto aebm.$$

In addition, ν *is a natural isomorphism iff* $ReR = R$*.*

Proof. The first statement is easy. And certainly if ν is an isomorphism, then $ReR = R$. So conversely, assume that $ReR = R$. Let ${}_RM$. Then

$$\nu_M(Re \otimes eR \otimes M) = ReRM = RM = M,$$

so that ν_M is a natural epimorphism. Now since $ReR = R$, we have $1 \in ReR$, so there exist $a_i, b_i \in R$ with

$$1 = \sum_{i=1}^{n} a_i e b_i.$$

So if

$$x = \sum_{j=1}^{k} (s_j e \otimes e t_j \otimes m_j) \in \operatorname{Ker} \nu_M,$$

then

$$\begin{aligned} x = 1x &= \sum_{j=1}^{k} \left[\left(\sum_{i=1}^{n} a_i e b_i \right) s_j e \otimes e t_j \otimes m_j \right] \\ &= \sum_{i=1}^{n} \left[a_i e \sum_{j=1}^{k} (e b_i s_j e \otimes e t_j \otimes m_j) \right] \\ &= \sum_{i=1}^{n} [a_i e \otimes e b_i \otimes \nu_M(x)] = 0. \end{aligned}$$

Thus, ν is a natural isomorphism. ∎

Putting this all together, it provides us with the really slick example of equivalent rings.

5.4. Theorem. *Let* $e \in R$ *be a non-zero idempotent. The functors*

$$F : eR \otimes_R (\_) : R\mathbf{Mod} \longrightarrow eRe\mathbf{Mod} \quad \text{and} \quad G : Re \otimes_{eRe} (\_) : eRe\mathbf{Mod} \longrightarrow R\mathbf{Mod}$$

are inverse equivalences via the natural homomorphisms

$$\mu : FG = eR \otimes_R Re \otimes_{eRe} \otimes(\_) \longrightarrow 1_{eRe\mathbf{Mod}} \quad \text{and} \quad \nu : GF = Re \otimes_{eRe} \otimes_R eR \otimes (\_) \longrightarrow 1_{R\mathbf{Mod}}$$

iff $ReR = R$*.* ∎

As we mentioned before, this does not tell the whole story. But as we shall see in the next Section, it comes amazingly close. Even with this partial result, however, we can draw some very interesting corollaries. Right now we'll mention just a few. First, though, let's introduce some notation. If R and S are rings, then let's write

$$R \approx S$$

in case R and S are Morita equivalent rings. Of course, it is clear that the relation $\approx$ is an equivalence relation on the class of all rings.

5.5. Corollary. *For every ring R and every natural number $n \in \mathbb{N}$, we have $R \approx \mathbb{M}_n(R)$.*

Proof. If $S = \mathbb{M}_n(R)$ and if $e = e_{11}$ is the $(1,1)$ matrix unit, then $SeS = S$, so that by Theorem 5.4, $S \approx eSe \cong R$. ■

Even more generally, we have that

5.6. Corollary. *Let R be a ring and $n \in \mathbb{N}$ be a natural number. If $E \in \mathbb{M}_n(R)$ is an idempotent matrix with $\mathbb{M}_n(R) = \mathbb{M}_n(R)E\mathbb{M}_n(R)$, then $R \approx E\mathbb{M}_n(R)E$.* ■

Of course, these corollaries make it quite clear that ring properties such as commutativity or being a division ring are not Morita invariants. But here is a final Corollary for this Section; it puts the Wedderburn Theorem about simple Artinian rings in the right context.

5.7. Corollary. *If D is a division ring, then for every finite dimensional D-vector space ${}_DV$, D is equivalent to* $\mathrm{End}({}_DV)$. ■

Exercises 5.

5.1. Let ${}_RM_S$ be a bimodule with $R = \mathrm{End}(M_S)$ and $S = \mathrm{End}({}_RM)$. (In this case we say that the bimodule is **faithfully** **balanced**.)

(a) For each $r \in \mathrm{Cen}(R)$ define $\rho_r : M \longrightarrow M$ as the right operator $\rho_r : x \longmapsto x\rho_r = rx$. Prove that $\rho_r \in S$.

(b) Prove that $\rho : r \longmapsto \rho_r$ is a ring isomorphism $\mathrm{Cen}(R) \cong \mathrm{Cen}(S)$.

(c) Deduce that if $e \in R$ is an idempotent in a ring R and if $ReR = R$, then $\mathrm{Cen}(eRe) \cong \mathrm{Cen}(R)$.

5.2. Let $e \in R$ be a non-zero idempotent such that $ReR = R$. Thus $R \approx eRe$. For each (two-sided) ideal I of R set $\Phi(I) = eIe$.

(a) Prove that Φ defines a lattice isomorphism between the lattices of ideals of R and eRe.

(b) Deduce that R is simple iff eRe is simple.

(c) Prove that for all ideals I, J of R, $\Phi(IJ) = \Phi(I)\Phi(J)$.

(d) Prove that an ideal I of R is prime iff $\Phi(I)$ is prime in eRe. [Hint: Recall that an ideal I of R is prime in case for every pair J, K of ideals $JK \subseteq I$ implies J or K is in I.]

(e) Deduce that R is a prime ring iff eRe is prime.

5.3. Let $e \in R$ be a non-zero idempotent such that $ReR = R$. Thus $R \approx S = eRe$. Suppose that $R_1, \ldots, R_n$ are rings and that

$$R \cong R_1 \times \cdots \times R_n.$$

Prove that there exist rings $S_1, \ldots, S_n$ with

$$S \cong S_1 \times \cdots \times S_n$$

and $S_i \approx R_i$ for $i = 1, \ldots, n$. [Hint: See Exercise 2.8.]

6 The Morita Characterization of Equivalence.

In the last Section we found some necessary conditions for two rings to be equivalent. But, as our example in that Section showed, we needed a little bit more to have a set of sufficient conditions. The additional requirement in that example was that for the idempotent $e \in R$, we needed $ReR = R$. In this Section we will see that a version of this added to the necessary conditions from the last Section give us a characterization of duality. In addition we will find a good form for the inverse equivalences involved. This is a somewhat streamlined version, due largely to Hyman Bass, of the brilliant work of Morita in which these characterizations were first presented.

We begin by reviewing the notion of generating treated in Exercise 2.7. Consider left R-modules ${}_RG$ and ${}_RM$. The **trace of** G **in** M is the submodule

$$\mathrm{Tr}_M(G) = \sum_{f \in \mathrm{Hom}_R(G,M)} \mathrm{Im}\, f$$

of M. We say that G **generates** M in case $M = \mathrm{Tr}_M(G)$. If $\mathcal{S}$ is some collection of R-modules, then we say that G **generates** $\mathcal{S}$ if it generates every module in the class $\mathcal{S}$. In particular, it generates R**Mod** if it generates every R-module and we'll say then that G is a **generator** for R**Mod** or simply that G is a **generator**. In Exercise 2.7 we also saw that

6.1. Lemma. *A module ${}_RG$ generates a module ${}_RM$ iff there exists a set A and an epimorphism $g : G^{(A)} \longrightarrow M$.* ∎

6.2. Proposition. *For a left R-module ${}_RG$ the following are equivalent:*

(a) *G generates R**Mod**;*

(b) *G generates ${}_RR$;*

(c) *For every non-zero R-homomorphism $f : M \longrightarrow N$, there is an R-homomorphism $g : G \longrightarrow M$ with $f \circ g \neq 0$.*

Proof. (a) ⇒ (c). Since $f(\mathrm{Tr}_M(G)) = f(M) \neq 0$, and $\mathrm{Tr}_M(G)$ is the sum of all $g(G)$ with $g : G \longrightarrow M$, there is some $g : G \longrightarrow M$ with $f \circ g \neq 0$.

(c) ⇒ (b). If $T = \mathrm{Tr}_R(G) \neq R$, then the factor map $p : R \longrightarrow R/T$ is not zero. So by (c) there is a homomorphism $g : G \longrightarrow R$ with $p \circ g \neq 0$. But $\mathrm{Im}\, g \leq \mathrm{Tr}_R(G) = T$, so $p \circ g = 0$.

(b) ⇒ (a). Let ${}_RM$ and let $x \in M$. Then right multiplication by x, $\rho_x : R \longrightarrow M$ is an R-homomorphism with $x \in \mathrm{Im}\, \rho_x$. So $\mathrm{Tr}_M(R) = M$ and R generates M. But by (b), $\mathrm{Tr}_R(G) = R$, so clearly, $\mathrm{Tr}_M(G) \geq \mathrm{Tr}_M(R) = M$. ∎

Since condition (c) of Proposition 6.2 is categorical, that is, characterized solely in terms of the algebra of morphisms in the category, we clearly have that being a generator is a Morita invariant. That is,

6.3. Corollary. *If F is an additive equivalence from R**Mod** to S**Mod**, then an R-module ${}_RG$ is a generator for ${}_RR$ iff $F(G)$ is a generator for ${}_SS$.* ∎

We have now come to the ingredient missing in the last Section to describe equivalent rings. A module ${}_RP$ is a **progenerator** (for R**Mod**) in case it is a finitely generated projective generator (for R**Mod**). Trivially, every free module is generated by ${}_RR$ and every module is generated by some free (see Lemma 6.1), so ${}_RR$ is a generator for R**Mod**. But certainly, ${}_RR$ is a finitely generated projective, so ${}_RR$ is a progenerator. Here is an important chacterization of progenerators whose proof we'll leave to you.

6.4. Lemma. *A module ${}_RP$ is a progenerator (for R**Mod**) iff there are modules P' and R' and natural numbers $m, n \in \mathbb{N}$ such that*

$$P^m \cong R \oplus R' \qquad \textit{and} \qquad R^n \cong P \oplus P'.$$ ∎

One curious fact we can read from this is that if ${}_RP$ is a progenerator, then there is some $n \in \mathbb{N}$ and some idempotent endomorphism $e \in \mathrm{End}(R^n)$ of the free module R^n with $P \cong R^n e$. So

$$\mathrm{End}({}_RP) \cong e\,\mathrm{End}(R^n)e \cong E\mathbb{M}_n(R)E,$$

for some idempotent matrix $E \in \mathbb{M}_n(R)$.

Now let's return to our study of equivalent rings. Indeed, suppose that $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ and $G : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ are inverse additive equivalences. Let

$${}_SQ = F({}_RR) \qquad \text{and} \qquad {}_RP = G({}_SS).$$

Then by Corollary 6.3, not only are these modules finitely generated projective, but they are progenerators. Then as we saw in the last Section, we have ring isomorphisms

$$\mathrm{End}({}_SQ) \cong \mathrm{End}({}_RR) \cong R \qquad \text{and} \qquad \mathrm{End}({}_RP) \cong \mathrm{End}({}_SS) \cong S,$$

so that

$${}_SQ_R \qquad \text{and} \qquad {}_RP_S$$

are bimodules. Moroever, we saw that

$${}_SP^*_R = \mathrm{Hom}_R(P, R) \cong {}_SQ_R \quad \text{and} \quad {}_RQ^*_S = \mathrm{Hom}_S(Q, S) \cong {}_RP_S$$

with all homomorphisms natural. Finally, using Lemma 3.14 and what we observed about F and G before, we have natural isomorphisms

$$F \cong \mathrm{Hom}_R(P, \_) \cong P^* \otimes_R (\_) \cong Q \otimes_R (\_) \quad \text{and} \quad G \cong \mathrm{Hom}_S(Q, \_) \cong Q^* \otimes_S (\_) \cong P \otimes_S (\_).$$

To seek a characterization of equivalence, we can distill these conditions down by observing that we have the two rings R and S and two bimodules ${}_RP_S$ and ${}_SQ_R$. (And it so happens that P and Q are duals if that could be of any help.) These induce additive functors

$$Q \otimes_R (\_) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \text{and} \qquad P \otimes_S (\_) : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}\,.$$

Then the issue is simply whether there are natural isomorphisms

$$P \otimes_S Q \otimes_R (\_) \cong 1_{R\mathbf{Mod}} \quad \text{and} \quad Q \otimes_R P \otimes_S (\_) \cong 1_{S\mathbf{Mod}}.$$

We begin our attack on this problem by looking at what ought to work for a module ${}_RP_S$ and its R-dual ${}_SP^*_R$. And we will be guided by what we learned in our study of the special case with $P = Re$ and $P^* = eR$.

Thus, let ${}_RP$ be an arbitrary module, let S be its endomorphism ring and P^* its R-dual:

$$S = \mathrm{End}({}_RP) \qquad \text{and} \qquad P^* = \mathrm{Hom}_R(P, R).$$

Then we have bimodules

$${}_RP_S \qquad \text{and} \qquad {}_SP^*_R.$$

Now one easily checks that there are bimodule homomorphisms

$$\varphi_P : P \otimes_S P^* \longrightarrow {}_RR_R \quad \text{and} \quad \theta_P : P^* \otimes_S P \longrightarrow {}_SS_S$$

such that for all $x \in P$ and $f \in P^*$,

$$\varphi_P : x \otimes f \longmapsto xf \qquad \text{and} \qquad \theta_P : f \otimes x \longmapsto ((-)f)x.$$

For θ_P note that $\theta_P(f \otimes x)$ is the endomorphism of ${}_RP$ that assigns to each $y \in P$ the element $((y)f)x$. Another easy computation shows that for all $x, y \in P$ and $f, g \in P^*$

$$\varphi_P(x \otimes f)y = x\theta_P(f \otimes y) \qquad \text{and} \qquad \theta_P(f \otimes x)g = f\varphi_P(x \otimes g).$$

6.5. Lemma. *For a module ${}_RP$ with $S = \mathrm{End}({}_RP)$,*

(a) $\varphi_P : P \otimes_S P^* \longrightarrow {}_RR_R$ *is surjective iff ${}_RP$ is a generator;*

(b) $\theta_P : P^* \otimes_R P \longrightarrow {}_SS_S$ *is surjective iff ${}_RP$ is finitely generated projective.*

Proof. (a) Clearly, $\operatorname{Im}\varphi_P = \operatorname{Tr}_R(P)$.

(b) Note that θ_P is surjective iff $1_S \in \operatorname{Im}\theta_P$ iff there exist $f_1, \ldots, f_n \in P^*$ and $x_1, \ldots, x_n \in P$ with

$$\theta_P(\sum_{i=1}^{n}(f_i \otimes x_i) = \sum_{i=1}^{n}(-)f_i x_i = 1_S.$$

But this last simply means that for each $x \in P$

$$x = (x)1_S = \sum_{i=1}^{n}(x)f_i x_i,$$

or that there exists a dual basis for ${}_RP$. Now just apply the Dual Basis Lemma (Lemma 13.3). ■

Now we come to the machine that will build equivalences. Let R and S be rings. Consider a pair ${}_RP_S$ and ${}_SQ_R$ of bimodules. A pair (φ, θ) of bimodule homomorphisms

$$\varphi : P \otimes_S Q \longrightarrow R \qquad \text{And} \qquad \theta : Q \otimes_r P \longrightarrow S$$

is a **Morita pair** for (P, Q) in case

(M.1) For all $x, y \in P$ and all $f, g \in Q$

$$\varphi(x \otimes f)y = x\theta(f \otimes y) \qquad \text{and} \qquad \theta(f \otimes x)g = f\varphi(x \otimes g).$$

(M.2) Both φ and θ are surjective.

Thus, for example, by Lemma 6.5, if ${}_RP$ is a progenerator, and $S = \operatorname{End}({}_RP)$, then (φ_P, θ_P) is a Morita pair for the bimodules $({}_RP_S, {}_SP^*_R)$. Now the key step.

6.6. Lemma. *Let (φ, θ) be a Morita pair for the bimodules $({}_RP_S, {}_SQ_R)$. Then for each ${}_RM$ and ${}_SN$ there exist natural isomorphisms*

$$\eta_M : P \otimes_S Q \otimes_R M \longrightarrow M \qquad \textit{and} \qquad \mu_N : Q \otimes_R P \otimes_S N \longrightarrow N$$

given by

$$\eta_M : x \otimes f \otimes m \longmapsto \varphi(x \otimes f)m \qquad \textit{and} \qquad \mu_N : f \otimes x \otimes n \longmapsto \theta(f \otimes x)n.$$

Proof. By the complete symmetry of the notion of Morita pair, it will suffice to prove just one of the assertions. Now using standard tensor arguments, for each ${}_RM$, there certainly does exist an R-homomorphism $\eta : P \otimes_S Q \otimes_R M \longrightarrow M$ with $\eta : x \otimes f \otimes m \longmapsto \varphi(x \otimes f)m$. That this is also natural in M is an easy exercise that we will leave to the reader. So we are left with proving that η is an isomorphism.

Since by (M.2), φ is surjective, there exist $x_1, \ldots, x_n \in P$ and $f_1, \ldots, f_n \in Q$ with

$$\varphi(\sum_{i=1}^{n} x_i \otimes f_i) = 1.$$

Thus, η is clearly surjective, since for each $m \in M$

$$\eta\big(\sum_{i=1}^{n}(x_i \otimes f_i \otimes m)\big) = (\varphi(\sum_{i=1}^{n} x_i \otimes f_i))m = m.$$

Finally, η is injective. Indeed, let $\sum_{j=1}^{m}(y_j \otimes g_j \otimes m_j) \in \operatorname{Ker}\eta_M$. Then

$$\begin{aligned}
\sum_j (y_j \otimes g_j \otimes m_j) &= \sum_j \big((\sum_i \varphi(x_i \otimes f_i)y_j) \otimes g_j \otimes m_j\big) \\
&= \sum_j \sum_i \big(x_i\theta(f_i \otimes y_j) \otimes g_j \otimes m_j\big) \\
&= \sum_j \sum_i \big(x_i \otimes \theta(f_i \otimes y_j)g_j \otimes m_j\big) \\
&= \sum_j \sum_i \big(x_i \otimes f_i\varphi(y_j \otimes g_j) \otimes m_j\big) \\
&= \sum_i \big(x_i \otimes f_i \otimes \sum_j \varphi(y_j \otimes g_j)m_j\big) \\
&= \sum_i (x_i \otimes f_i \otimes 0) = 0.
\end{aligned}$$

Thus, η is monic and hence an isomorphism. ■

At last all the pieces are in place and we can state the Morita Theorem characterizing equivalent rings and spelling out the inverse equivalences themselves.

6.7. Theorem. **[Morita]** *Let R and S be rings and let ${}_RP_S$ and ${}_SQ_R$ be bimodules. Consider the additive functors*

$$T_Q = Q \otimes_R (-) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \quad \textit{and} \quad T_P = P \otimes_S (-) : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}\,.$$

Then the following are equivalent:

- **(a)** *T_P and T_Q are inverse equivalences;*
- **(b)** *There exists a Morita pair (φ, θ) for (P, Q);*
- **(c)** *${}_RP$ is a progenerator, $S \cong \operatorname{End}({}_RP)$, and ${}_SQ_R \cong P^* = \operatorname{Hom}_R(P, R)$;*
- **(d)** *${}_SQ$ is a progenerator, $R \cong \operatorname{End}({}_SQ)$, and ${}_RP_Q \cong Q^* = \operatorname{Hom}_S(Q, S)$.*

Proof. (a) ⇒ (c) This is immediate from our discussion following Lemma 6.4.

(c) ⇒ (b) By Lemma 6.5, the maps φ_P and θ_P form a Morita pair for the bimodules ${}_RP_S$ and ${}_SP^*_R$.

(b) $\Rightarrow$ (a) This follows at once from Lemma 6.6

(d) $\iff$ (a) This follows from the symmetry of (a) and the statements (c) and (d). ∎

And with that we have a variety of simple answers to the question of when two rings are Morita equivalent.

6.8. Corollary. *For two rings R and S the following are equivalent:*

(a) $R \approx S$;

(b) *There is a natural equivalence from R**Mod** to S**Mod**;*

(c) $S \cong \mathrm{End}({}_RP)$ *for some progenerator* ${}_RP$*;*

(d) $R \cong \mathrm{End}({}_SQ)$ *for some progenerator* ${}_SQ$*;*

(e) *There is a natural equivalence from* **Mod**R *to* **Mod**S*;*

(f) $S \cong \mathrm{End}(Q_R)$ *for some progenerator* Q_R*;*

(g) $R \cong \mathrm{End}(P_S)$ *for some progenerator* P_S*.*

Proof. The only real issue here is the implication (b) $\Longrightarrow$ (e). But by Theorem 6.7, (b) implies the existence of bimodules ${}_RP_S$ and ${}_SQ_R$ and a Morita pair (φ, θ) for them. Then the right hand version of Lemma 6.6 gives that there are natural isomorphisms

$$\eta : (\_) \otimes_R P \otimes_S Q \otimes_R \longrightarrow 1_{\mathbf{Mod}R} \qquad \text{and} \qquad \mu : (\_) \otimes_S Q \otimes_R P :\longrightarrow 1_{\mathbf{Mod}S},$$

and and we have (e). ∎

If you think about what's happened, you'll recognize that there would be no change in anything we've done in the last two sections if we had been working exclusively in the categories R**mod** and S**mod** of finitely generated modules. Thus, we have the answer to some obvious Qual question.

6.9. Corollary. *For two rings R and S, the categories R**Mod** and S**Mod** are additively equivalent iff the categories R**mod** and S**mod** are additively equivalent.* ∎

There is sometimes a temptation to read too much into having inverse equivalences F and G between the two categories R**Mod** and S**Mod**. The usual slip-up occurs since the image $F(R)$ of R is usually not S and the image $G(S)$ of S is not usually R. So the lattices of left ideals of R and S need not be isomorphic. Nevertheless, there is something almost as good.

6.10. Corollary. *If R and S are equivalent, then there exist $m, n \in \mathbb{N}$ and idempotent matrices U and V with*

$$S \cong U\mathbb{M}_n(R)U \qquad \text{and} \qquad R \cong V\mathbb{M}_m(S)V.$$ ∎

And from this, for example, we read at once

6.11. Corollary. *If R and S are equivalent rings, then R is left (right) artinian or noetherian iff S is left (right) artinian or noetherian.* ■

None of this, however, gives us any information about the two-sided ideal structure of R and S. But, fortunately, this information is built into the Morita pairs. Indeed, let R and S be two rings, and let (φ, θ) be a Morita pair for bimodules ${}_RP_S$ and ${}_SQ_R$.

6.12. Theorem. *If R and S are equivalent rings, then the maps*

$$\Theta : I \longmapsto \theta(QI \otimes P) \qquad \textit{and} \qquad \Phi : J \longmapsto \varphi(PJ \otimes Q)$$

define inverse lattice isomorphisms between the lattices of ideals of R and S. Moreover, these maps are multiplicative in the sense that for each pair I, I' of ideals of R and J, J' of ideals of S,

$$\Theta(II') = \Theta(I)\Theta(I') \qquad \textit{and} \qquad \Phi(JJ') = \Phi(J)\Phi(J').$$

Proof. We'll leave the proof of this as an exercise. ■

We conclude with the following important Corollary whose easy proof we leave to you.

6.13. Corollary. *Let R and S be equivalent rings and let Θ and Φ be the lattice isomorphisms of Theorem 6.12. Then*

(1.) R is simple iff S is simple;

(2.) An ideal I of R is prime iff $\Theta(I)$ is a prime ideal of S;

(3.) R is the ring direct sum (see Exercise 2.8) of ideals $I_1, \dots, I_n$ iff S is the ring direct sum of the ideals $\Theta(I_1), \dots, \Theta(I_n)$. ■

Exercises 6.

6.1. Let R and S be equivalent rings where (φ, θ) is a Morita pair for bimodules ${}_RP_S$ and ${}_SQ_R$.

(a) Prove Theorem 6.12.

(b) Prove Corollary 6.13.

6.2. Let $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ be an additive equivalence.

(a) Prove that if ${}_RM$ is flat, then ${}_SF(M)$ is flat.

(b) Prove that for each left R-module ${}_RM$, $\mathrm{Pd}({}_\mathrm{S}\mathrm{F}(\mathrm{M})) = \mathrm{Pd}({}_\mathrm{R}\mathrm{M})$.

(c) Deduce that $\mathrm{lgldim(S)} = \mathrm{lgldim(R)}$.

6.3. A ring R is (**<u>von</u> <u>Neumann</u>**) **<u>regular</u>** if for each $a \in R$, $a \in aRa$. (In the language of linear algebra every element of R has a psuedoinverse.)

(a) For a ring R prove that the following are equivalent: (a) R is von Neumann regular; (b) every principal left ideal Ra is a direct summand of ${}_RR$; (c) every principal right ideal aR is a direct summand of R_R.

(b) Prove that a ring R is von Neumann regular iff every finitely generated left ideal is a direct summand of ${}_RR$. [Hint: Suppose that $a \in R$ and that $e, f \in R$ are idempotents. If $Ra(1-e) = Rf$, then $e + f - ef$ is idempotent and $Re + Ra = Re + Rf = R(e + f - ef)$.]

(c) Prove that if $R \approx S$, then R is von Neumann regular iff S is.

(d) Prove that a commutative ring R is von Neumann regular iff $I^2 = I$ for every ideal I.

6.4. A ring R is **<u>left</u> <u>self</u>-<u>injective</u>** if the regular module ${}_RR$ is injective. (See Exercise 3.6.)

(a) Prove that if R is left self-injective and has zero Jacobson radical, then R is von Neumann regular.

(b) Prove that if $R \approx S$, then R is left self-injective iff S is.

6.5. Let K be a ring, and consider two right modules M_K and N_K. These modules are **<u>similar</u>** if for some natual numbers m and n there exist split monomorphisms

$$M \longrightarrow N^m \qquad \text{and} \qquad N \longrightarrow M^n.$$

Consider the rings $R = \mathrm{End}(M_K)$ and $S = \mathrm{End}(N_K)$ and the bimodules

$${}_RP_S = \mathrm{Hom}_K(N, M) \qquad \text{and} \qquad {}_SQ_R = \mathrm{Hom}_K(M, N).$$

Then there are bimodule homomorphisms

$$\varphi : P \otimes_S Q \longrightarrow R \qquad \text{and} \qquad \theta : Q \otimes_R P \longrightarrow S$$

with

$$\varphi(p \otimes q) = pq \qquad \text{and} \qquad \theta(q \otimes p) = qp.$$

(a) Prove that (φ, θ) is a Morita pair for (P, Q) iff M and N are similar.

(b) Deduce that if M and N are similar, then $R \approx S$.

6.6. Let R be a ring and let ${}_RM$ be a left R-module with $S = \mathrm{End}({}_RM)$. Let $e \in S$ be an idempotent and suppose that M and its direct summand Me are similar.

(a) Prove that $SeS = S$. [Hint: This is easy to do directly or you can use Exercise 6.5.]

(b) Prove that $\mathrm{End}(M_S) \cong \mathrm{End}(Me_{eSe})$. [Hint: For each $a \in \mathrm{End}(M_S)$ let $\hat{a}$ be the restriction to Me. Then the map $a \longmapsto \hat{a}$ is an injective ring homomorphism to $\mathrm{End}(Me_{eSe})$. To see that it is also surjective, let $a \in \mathrm{End}(Me_{eSe})$. By part (a) we have $MeS = M$. Also, using part (a) you can show that if $x_i \in Me$ and $s_i \in S$ with $\sum x_i s_i = 0$, then $\sum (ax_i)s_i = 0$, so there is an S-endomorphism $\overline{a} : MeS = M \longrightarrow M$ by $\overline{a} : \sum x_i s_s \longmapsto \sum (ax_i)s_i$ that restricts to a.]

(c) Deduce that if ${}_RM$ is a left R-module with $S = \mathrm{End}({}_RM)$, and if $n \in \mathbb{N}$, then

$$\mathrm{End}(M_S) \cong \mathrm{End}(M^n_{\mathbb{M}_n(S)}).$$

6.7. Let R be a ring and let ${}_RP$ be a left R-module with $S = \mathrm{End}({}_RP)$.

(a) Prove that P is a progenerator iff P is similar to ${}_RR$.

(b) Prove that if ${}_RP$ is a progenerator, then $R \cong \mathrm{End}(P_S)$. [Hint: Exercise 6.6.]

(c) Deduce that $\mathrm{Cen}(R) \cong \mathrm{Cen}(S)$. [Hint; See Exercise 5.1.]

7 Rings with Semisimple Generators.

It is now quite easy to use Morita to obtain the classical Wedderburn and Artin-Wedderburn characterizations of simple Artinian and semisimple rings. We begin by reminding you of a few elementary facts about semisimple modules.

Recall that a module ${}_RM$ is **simple** in case it is a non-zero module with no non-trivial submodules.

7.1. Lemma. **[Schur's Lemma]** *If M is a simple module and N is a module, then*

7.1. *Every non-zero homomorphism $M \longrightarrow N$ is a monomorphism;*

7.2. *Every non-zero homomorphism $N \longrightarrow M$ is an epimorphism;*

7.3. *If N is simple, then every non-zero homomorphism $M \longrightarrow N$ is an isomorphism;*

In particular, $\mathrm{End}({}_RM)$ *is a division ring.* ■

A module ${}_RM$ is **semisimple** if it is the sum of its simple submodules. Then an easy, but important, characterization of semisimple modules is given in [1] (see Theorem 9.6):

7.2. Proposition. *For left R-module ${}_RM$ the following are equivalent:*

(a) *M is semisimple;*

(b) *M is a direct sum of simple submodules;*

(c) *Every submodule of M is a direct summand.* ■

Let's begin with a particularly nice example. Indeed, let D be a division ring. Then the regular module ${}_DD$ is a simple module, and hence, a simple progenerator. In fact, every finitely generated module over D is just a finite dimensional D-vector space, and hence, free. That is, the progenerators for D are simply the non-zero finite dimensional D-vector spaces, and so the rings equivalent to D are nothing more than the $n \times n$ matrix rings over D. That leads us to investigate what can be said about an arbitrary ring that has a simple generator.

Well, suppose that R is a ring and that it has a simple generator ${}_RT$. But then T generates ${}_RR$, and so R is a sum of (by Lemma 7.1) simple left ideals all isomorphic to T. Since $1 \in R$, this means that there is a finite set of these simple left ideals that sum to R. Thus, there is a minimal such set, say, $I_1, \ldots, I_n$. By minimality, these left ideals must be independent, and so this sum must be direct. That is,

$$R = I_1 \oplus \cdots \oplus I_n$$

with $I_k \cong T$ for each $k = 1, \dots, n$. That, in turn, implies that ${}_RR$ has finite composition length, and so R is both left Artinian and left Northerian. It also implies that T is projective and hence actually a progenerator for R. But

$$\mathrm{End}({}_RT) = D$$

is a division ring (see Lemma 7.1). So (see Corollary 6.8), R is equivalent to D, and hence, R is equivalent to S iff D is equivalent to S. But as we saw above, this means that R is equivalent to S iff $S \cong \mathbb{M}_n(D)$ for some $n \in \mathbb{N}$. Finally, we note that D is a simple ring, and so (see Corollary 6.13) R is a simple ring. This gives us most of the following characterization of simple Artinian rings.

7.3. Theorem. *For a ring R the following are equivalent:*

(a) *R is simple left Artinian;*

(b) *R has a simple left generator;*

(c) *R is left equivalent to a division ring;*

(d) *$R \cong \mathbb{M}_n(D)$ for some division ring D and some $n \in \mathbb{N}$;*

(e) *R is right equivalent to a division ring;*

(f) *R has a simple right generator;*

(g) *R is simple right Artinian.*

Proof. As we saw above (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a). For (a) $\Longrightarrow$ (b), since R is left Artinian, it has a minimal left ideal I. By Lemma 7.1 for each $a \in R$, the left ideal Ia is either zero or isomorphic to the simple module I. But since R is a simple ring,

$$R = IR = \sum\{Ia \ : \ a \in R\}$$

is a sum of simple modules isomorphic to I. So I is a simple generator. Finally, the equivalence of the last four statements follows from the first four and the left-right symmetry of (d). ∎

Next, let's try to extend this and look at rings with a semisimple generator. We begin with a particularly nice example. Indeed, let $D_1, \dots, D_n$ be division rings (with possible repetitions) and let

$$R = D_1 \times \cdots \times D_n$$

be their ring product. If $\iota_j : D_j \longrightarrow R$ is the coordinate injection ($j = 1, \dots, , n$), then each $\iota_j(D_j)$ is a simple left ideal (actually an ideal) of R, and so

$$R = \iota_1(D_1) \oplus \cdots \oplus \iota_n(D_n)$$

so ${}_RR$ is semisimple. Then, of course, R is a semisimple left generator. Also, (see Corollary 6.13) R is equivalent to S iff $S \cong S_1 \times \cdots \times S_n$ with each S_i equivalent to D_i. But D_i is a division ring, and so (see Theorem 7.3) it is equivalent to S_i iff $S_i \cong \mathbb{M}_{n_i}(D_i)$ for some $n_i \in \mathbb{N}$. So for this example we conclude that R is equivalent to S iff there exist $n_1, \ldots, n_n \in \mathbb{N}$ with $S \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_n}(D_n)$.

Let's now look at the general case. So suppose that R has a semisimple left generator, say ${}_RG$. Then G generates ${}_RR$ and G is a sum of simple submodules, so R is a sum of simple submodules. This means that ${}_RR$ is semisimple (such a ring is **left semisimple**), so by Proposition 7.2 ${}_RR$ is a direct sum of simple left ideals. But ${}_RR$ is finitely generated, so there is a finite set $I_1, \ldots, I_m$ of simple left ideals with

$$ {}_RR = I_1 \oplus \cdots \oplus I_m. $$

This tells us, among other things, that ${}_RR$ has finite composition length and hence is both left Artinian and left Noetherian, and that each of the simple left ideals I_k is projective. Let $T_1, \ldots, T_n$ be a complete set of representatives of the simple modules $I_1, \ldots, I_k$. (That is, each I_j is isomorphic to one and only one of the T_i.) Let T be the coproduct of $T_1, \ldots, T_n$. Then T is certainly finitely generated projective and it generates each I_j, so it generates ${}_RR$. That is, T is a progenerator for R. Now if $i \neq j$, then (see Lemma 7.1) $\mathrm{Hom}_R(T_i, T_j) = 0$, so

$$ \mathrm{End}({}_RT) \cong \mathrm{End}({}_RT_1) \times \cdots \times \mathrm{End}({}_RT_n). $$

But each T_i is simple, so each $\mathrm{End}({}_RT_i) = D_i$ is a division ring. Thus, R is equivalent to S iff S is equivalent to $D_1 \times \cdots \times D_n$, and we're back to our above simple example. This gives us the following characterizations of semisimple rings.

7.4. Theorem. *For a ring R the following are equivalent:*

(a) *R is left semisimple;*

(b) *R has a semisimple left generator;*

(c) *R is left equivalent to a finite product of division rings;*

(d) *$R \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_n}(D_n)$ for $D_1, \ldots, D_n$ division rings and $n_1, \ldots, n_n \in \mathbb{N}$;*

(e) *R is isomorphic to a product of a finite number of simple Artinian rings;*

(f) *R is right equivalent to a finite product of division rings;*

(g) *R has a semisimple right generator;*

(h) *R is right semisimple.* ■

One consequence of this is that we usually drop the left-right adjectives when talking about semisimple rings. There is another slick bonus to this. The ring R is semisimiple iff it is equivalent to a finite product S of division rings, so the categories R**Mod** and S**Mod** are equivalent. But the category S**Mod** is just a product of vector space categories. So we have yet another way to characterize semisimple rings.

7.5. Corollary. *For a ring R the following are equivalent:*

(a) *R is semisimple;*

(b) *Every module in R**Mod** is semisimple;*

(c) *Every short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ in R**Mod** splits;*

Proof. The implications (a) $\Longrightarrow$ (b), (c) follow from Theorem 7.4 since conditions (b) and (c) hole when R is a division ring. The implications (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a) follow from Proposition 7.2. ■

Exercises 7.

7.1. If R is a semisimple ring, then it is isomorphic to a finite product of simple Artinian rings. There is a version of this that holds for any semisimple module. Thus, let ${}_RM$ be a semisimple module over a ring R (not itself necessarily semisimple), and let $S = \text{End}({}_RM)$ acting, as usual, on the right. Let $(T_\alpha)_{\alpha\in\Omega}$ be a complete set of representatives of the simple submodules of ${}_RM$. (That is, each simple submodule ${}_RK$ of M is isomorphic to exactly one T_α.) For each $\alpha \in \Omega$ let

$$H_\alpha = \sum\{K \leq_R M \ : \ K \cong T_\alpha\}.$$

So H_α is the trace of T_α in M; it is called the T_α-**homogeneous component** of M. Prove that

(a) Each H_α is an (R, S) sub bimodule of ${}_RM_S$.

(b) There is a *central* idempotent $e_\alpha \in S$ with $H_\alpha = Me_\alpha$ for each $\alpha \in \Omega$.

(c) $\text{End}((H_\alpha)_{e_\alpha S e_\alpha}) \cong \text{End}({}_RT_\alpha)$ is a division ring.

(d) ${}_{D_\alpha}(H_\alpha)_{e_\alpha S e_\alpha}$ is faithfully balanced where $D_\alpha = \text{End}({}_RT_\alpha)$. (In particular, each $e_\alpha S e_\alpha$ is the endomorphism ring of some vector space over the division ring D_α.)

(e) S is isomorphic to the product $\prod_{\alpha\in\Omega} e_\alpha S a_\alpha$.

7.2. The converse of Schur's Lemma 7.1 fails. Indeed,

(a) Prove that for every $n \in \mathbb{N}$ there exists a module ${}_RM$ of length n with $\operatorname{End}{}_RM$ a division ring. [Hint: Try upper triangular matrix rings.]

(b) On the other hand suppose that ${}_RI$ is a left ideal of R with $\operatorname{End}({}_RI)$ a division ring. Prove that if $x^2 = 0$ implies that $x = 0$ for all $x \in I$, then ${}_RI$ is simple.

7.3. Here is an important characterization of semisimple rings. It begins in (a) with a simple lemma.

(a) Prove that a minimal left ideal I of a ring R is a direct summand of R iff $I^2 \neq 0$. [Hint: Recall that I is a direct summand iff $I = Re$ for some idempotent $e \in R$.]

(b) Let R be a left Artinian ring. Prove that the following are equivalent:

i. R is semisimple;

ii. R has no non-zero nilpotent left ideals;

iii. For all $x \in R$, if $xRx = 0$, then $x = 0$.

7.4. Here we consider the dual of those rings with a simple generator. A ring R has a simple left **cogenerator** ${}_RC$ if C is a simple module that cogenerates every left R-module. (See Exercise 2.7.) Say that a ring R is **co-simple** in case it has a simple left cogenerator.

(a) Prove that every cosimple ring is simple.

(b) For a left Artinian ring R prove that the following are equivalent:

i. R is simple Artinian;

ii. R is cosimple;

iii. Every non-zero R module is a generator;

iv. Every non-zero R module is a cogenerator.

8 The Socle and Radical.

Simple and semisimple modules are clearly the main building blocks in much of ring theory. Of coure, not every module can be built from semisimple modules, but for many modules its semisimple submodules and semisimple factor modules play important roles in understanding the module. Here we look briefly at two submodules of a module M that provide us with important preliminary information about the extent to which semisimple modules determine its structure.

Let ${}_RM$ be a module. Then its **socle** is the submodule

$$\operatorname{Soc} M = \sum \{N \leq M \ : \ N \text{ is simple}\}.$$

So the socle of M is the largest submodule of M generated by simple modules, or equivalently, it is the largest semisimple submodule of M. Dually, the **radical** of M is the submodule

$$\operatorname{Rad} M = \bigcap \{N \leq M \ : \ N \text{ is maximal in } M\}.$$

So the radical of M is the smallest submodule of M modulo which M is cogenerated by simples, or equivalently, it is the smallest submodule modulo which M can be embedded in a product of simples.

8.1. Lemma. *Let $f : M \longrightarrow N$ be an R-homomorphism. Then*

$$f(\operatorname{Soc} M) \leq \operatorname{Soc} N \qquad \textit{and} \qquad f(\operatorname{Rad} M) \leq \operatorname{Rad} N.$$

Thus, if ${}_RM_S$ is a bimodule, then both $\operatorname{Soc}({}_RM)$ and $\operatorname{Rad}({}_RM)$ are (R, S) sub bimodules of M.

Proof. The first assertion is a trivial consequence of Schur's Lemma (Lemma 7.1). For the second, observe that for every simple ${}_RT$ and every R-homomorphism $g : N \longrightarrow T$, $g \circ f : M \longrightarrow T$, so we have $\operatorname{Rad} M \leq \operatorname{Ker}(g \circ f)$, and $f(\operatorname{Rad} M) \leq \operatorname{Ker} g$. ■

8.2. Corollary. *For a module ${}_RM$, both $\operatorname{Soc} M$ and $\operatorname{Rad} M$ are stable under $\operatorname{End}({}_RM)$. In particular, $\operatorname{Soc}{}_RR$ and $\operatorname{Rad}{}_RR$ are ideals of R.* ■

Of course, we have a right socle and a right radical for R. It is easy to see that the left and the right socles of R, $\operatorname{Soc}{}_RR$ and $\operatorname{Soc} R_R$, need not be the same. In fact, consider the ring

$$R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$$

of 2×2 upper triangular matrices over a field. Then

$$\operatorname{Soc}{}_RR = Ke_{11} + Ke_{12} \quad \text{and} \quad \operatorname{Soc} R_R = Ke_{12} + Ke_{22}.$$

On the other hand, the left and right radicals of R are the same. (See, for example, [1], Theorem 15.2.) Notice that a module is semisimple iff it is generated by the class of simple modules. Dually,h we say that a module ${}_RM$ is **cosemisimple** in case it is cogenerated by the class of simple modules. Thus, and this is just words,

8.3. Corollary. *For a left R-module M*

$$M \text{ is semisimple} \iff \operatorname{Soc} M = M \qquad \text{and} \qquad M \text{ is cosemisimple} \iff \operatorname{Rad} M = 0.$$

In particular,

$$\operatorname{Soc}(\operatorname{Soc} M) = \operatorname{Soc} M \qquad \text{and} \qquad \operatorname{Rad}(M/\operatorname{Rad} M) = 0. \qquad \blacksquare.$$

Here is another dual pair of results about the socle and radical that sharpen Lemma 8.1 in certain settings. The proofs are easy and so we'll leave them as exercises for the reader.

8.4. Corollary. *Let K be a submodule of a module M. Then $\operatorname{Soc} K = K \cap \operatorname{Soc} M$, and if $K \leq \operatorname{Rad} M$, then $\operatorname{Rad}(M/K) = (\operatorname{Rad} M)/K$.* $\blacksquare$

Both the socle and the radical functions commute with coproducts. The details of the proof of this fact, stated in the next Corollary, are again left to the reader.

8.5. Corollary. *If $M = \oplus_{\alpha\in A} M_\alpha$ is the direct sum of submodules $(M_\alpha)_{\alpha\in A}$, then*

$$\operatorname{Soc} M = \oplus_A \operatorname{Soc} M_\alpha \qquad \text{and} \qquad \operatorname{Rad} M = \oplus_A \operatorname{Rad} M_\alpha. \qquad \blacksquare$$

In at least one case the near perfect duality that we've seen so far does break down. Every semisimple module is cosemisimple, but there are cosemisimple modules that are not semisimple. Fortunately, in one important case they are the same.

8.6. Corollary. *For every module ${}_RM$, $\operatorname{Rad}(\operatorname{Soc} M) = 0$. That is, every semisimple module has zero radical. If M is artinian, then it is cosemisimple iff it is semisimple.*

Proof. The first assertion is immediate since a coproduct of simples can certainly be embedded in a product of simples. For the final assertion let M be artinian and cosemisimple. It will suffice to show that M is semisimple. Let $\mathcal{J}$ be the set of all submodules of M that are each a finite intersection of maximal submodules of M. Since M satisfies the minimum condition, $\mathcal{J}$ contains a minimal element, say $N = N_1 \cap \cdots \cap N_k$. We claim that $N = 0$. If not, then since the intersection of all maximal submodules of M is zero, there must be some maximal submodule K of M with $K \cap N < N$, a clear contradiction to the minimality of N. Thus, M is isomorphic to a submodule of the product $M/N_1 \times \cdots \times M/N_k$ of simple modules. But a finite product is also a coproduct, so M is isomorphic to a submodule of a semisimple module and hence, is semisimple. $\blacksquare$

For an example of a cosemisimple module that is not semisimple, let k be a field and let R be the product $R = k^{\mathbb{N}}$. So R is a commutative ring and ${}_RR$ is decidedly not semisimple. But for each $n \in \mathbb{N}$, let M_n be the kernel of the projection of R onto the n^{th} coordinate. Then M_n is a maximal (left) ideal and $\cap_{\mathbb{N}} M_n = 0$, so ${}_RR$ is cosemisimple.

The important notions of socle and radical are actually categorical. Both are defined in terms of the lattice of submodules of a module — the sum of minimal submodules and the intersection of maximal submodules. Thus, they are both Morita invariants in the following sense.

8.7. Theorem. *If $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ is an equivalence of categories, then for each left R-module ${}_RM$,*

$$\mathrm{Im}(F(\mathrm{Soc}\, M \hookrightarrow M)) = \mathrm{Soc}\, F(M) \quad \textit{and} \quad \mathrm{Im}(F(\mathrm{Rad}\, M \hookrightarrow M)) = \mathrm{Rad}\, F(M). \qquad \blacksquare$$

Next, we want to find a couple of alternate characterizations of the socle and radical. This will involve a few new concepts. Let M be a module and let N and N' be submodules. If $M = N \oplus N'$, then we say that N' is a **complement** of N (in M). This means, of course, that

$$N + N' = M \qquad \text{and} \qquad N \cap N' = 0.$$

In general, if a complement exists, then it is not unique. On the other hand, there are two ways we might view the antithesis of being complemented in M. That is, a submodule N of M is **essential** (or **large**) in M, abbreviated

$$N \trianglelefteq M$$

in case for every submodule $K \leq M$

$$K \cap N = 0 \Longrightarrow K = 0.$$

The submodule $N \leq M$ is **superfluous** (or **small**) in M, abbreviated

$$N \ll M$$

in case for every submodule $K \leq M$,

$$K + N = M \Longrightarrow K = M.$$

So a submodule is essential when it is so broad or ubiquitous that it bumps into every non-trivial submodule, whereas a submodule is superfluous if it is so weak that it contributes nothing toward generating the overmodule. Good topological analogs for these are the notions of "dense" and "nowhere dense" subspaces.

It should be noted that a submodule can be both essential and superfluous. For example, if the submodule lattice is a chain (e.g., the abelian group $\mathbb{Z}_{p^n}$ where p is prime and $n \in \mathbb{N}$), then every non-trivial submodule is both essential and superfluous in the module.

We can now characterize the socle and radical of a module in terms of its essential and superfluous submodules.

8.8. Proposition. *For a non-zero left R-module M, its socle is the intersection of all essential submodules, and its radical is the sum of all superfluous submodules. That is,*

$$\operatorname{Soc} M = \bigcap\{N : N \trianglelefteq M\} \qquad \textit{and} \qquad \operatorname{Rad} M = \sum\{N : N \ll M\}.$$

Proof. Let $L = \cap\{N : N \trianglelefteq M\}$. Let $T \leq M$ be simple. If $N \trianglelefteq M$, then $T \cap N \neq 0$, so that $T \cap N = T$ and $T \leq N$. Thus, $T \leq L$ and so $\operatorname{Soc} M \leq L$. The other inquality will follow if L is semisimple. But let $K \leq L$ and let $K' \leq M$ be maximal w.r.t. $K \cap K' = 0$. Then $K + K' = K \oplus K' \trianglelefteq M$, and so $K \leq L \leq K + K'$. Thus, using modularity

$$L = L \cap (K \oplus K') = K \oplus (L \cap K').$$

That is, every submodule of L is a direct summand of L, and L is semisimple.

For the other assertion, let $L = \sum\{N : N \ll M\}$. Let K be a maximal submodule of M. If $N \ll M$, then $N \leq K$, for otherwise $K + N = M$ and $K = M$, a contradiction. Thus, $L \leq \operatorname{Rad} M$. On the other hand, suppose that $x \notin L$. Then Rx is not superfluous in M so there exists a proper submodule $H < M$ with $Rx + H = M$. But then there is a maximal submodule $H \leq K < M$ with $Rx + K = M$. That is, $x \notin K$, so x is not in $\operatorname{Rad} M$. ∎

Although the socle of a module M is the intersection of all the essential submodules of M, the socle itself need not be essential. (Think about the abelian group $\mathbb{Z}$.) Similarly, the radical of M is the sum of superfluous submodules of M, but the radical need not be superfluous. (Think about the abelian group $\mathbb{Z}_{p^\infty}$.) There are important cases when the socle is essential and the radical is superfluous.

First, though, recall that a module M is finitely generated iff for every family $\mathcal{A}$ of submodules of M,

$$\sum \mathcal{A} = M \Longrightarrow \sum \mathcal{F} = M$$

for some finite set $\mathcal{F}$ of $\mathcal{A}$. (See Exercise 1.3.) Dually, recall that a module M is finitely cogenerated in case for every family $\mathcal{A}$ of submodules of M

$$\bigcap \mathcal{A} = 0 \Longrightarrow \bigcap \mathcal{F} = 0$$

for some finite set $\mathcal{F}$ in $\mathcal{A}$. (See Exercise 1.5.) Concerning the socle and the radical, we now have

8.9. Proposition. *For a module ${}_RM$,*

$$M \textit{ finitely cogenerated} \Longrightarrow \operatorname{Soc} M \trianglelefteq M;$$

$$M \textit{ finitely generated} \Longrightarrow \operatorname{Rad} M \ll M.$$

Proof. We'll prove the second statement and leave the first to you. (Hint: Theorem 10.4 of [1].) So let $H \leq M$ with $\operatorname{Rad} M + H = M$. If $H \neq M$, then since M is finitely generated, there is a maximal proper submodule $K < M$ with $H \leq K$. But then $\operatorname{Rad} M + K = K < M$. Thus, $H = M$ and so $\operatorname{Rad} M \ll M$. ∎

Exercises 8.

8.1. Here is a little bit of the "calculus of essential and superfluous submodules." So let M be a module and let $K \leq N \leq M$ and let $H \leq M$. Then prove that

(a) $K \trianglelefteq M$ iff $K \trianglelefteq N$ and $N \trianglelefteq M$;

(b) $N \ll M$ iff $K \ll M$ and $N/K \ll M/K$;

(c) $H \cap K \trianglelefteq M$ iff $H \trianglelefteq M$ and $K \trianglelefteq M$;

(d) $H + K \ll M$ iff $H \ll M$ and $K \ll M$.

8.2. Some more elementary properties of essential and superfluous submodules. Let $M_1, \ldots, M_n \leq M$.

(a) Prove that $M_1 \cap \cdots \cap M_n \trianglelefteq M$ iff $M_i \trianglelefteq M$ for all $i = 1, \ldots, n$.

(b) Prove that $M_1 + \cdots + M_n \ll M$ iff $M_i \ll M$ for all $i = 1, \ldots, n$.

(c) On the other hand show that it is possible for an infinite intersection of essential submodules to be non-essential and that it is also possible for an infinite sum of superfluous submodules to be non-superfluous.

8.3. Let $M = M_1 \oplus M_2$ and let $N_i \leq M_i$ for $i = 1, 2$. Set $N = N_1 \oplus N_2$.

(a) Prove that $N \ll M$ iff $N_i \ll M_i$ for $i = 1, 2$.

(b) Prove that $N \trianglelefteq M$ iff $N_i \trianglelefteq M_i$ for $i = 1, 2$.

8.4. Let ${}_RM$ be a module and let $N \leq M$.

(a) Show that there exists a submodule $N' \leq M$ maximal w.r.t. the property $N \cap N' = 0$. [Such a submodule N' is an M-**complement** of N.]

(b) Show that if $N' \leq M$ is an M-complement of N, then $N \oplus N' \trianglelefteq M$;

(c) Show that if $N' \leq M$ is an M-complement of N, then $(N \oplus N')/N' \trianglelefteq M/N'$.

8.5. Compute both the socle and the radical of each of the following abelian groups:

$$\mathbb{Z}; \qquad \mathbb{Z}_{p^\infty}; \qquad \mathbb{Z}_{360}; \qquad \mathbb{Q}.$$

8.6. Now let's see how the socle and radical provide potentially useful machines for horizontally slicing a module. Given a module ${}_RM$, its **socle series** and **radical series**

$$0 \le \operatorname{Soc}_1 M \le \operatorname{Soc}_2 M \le \cdots \qquad \text{and} \qquad M \ge \operatorname{Rad}_1 M \ge \operatorname{Rad}_2 M \ge \cdots,$$

are defined recursively by

$$\operatorname{Soc}_1 M = \operatorname{Soc} M, \quad \text{and} \quad \operatorname{Soc}_{n+1} M/\operatorname{Soc}_n M = \operatorname{Soc}(M/\operatorname{Soc}_n M) \quad \text{for } n \ge 1,$$

and

$$\operatorname{Rad}_1 M = \operatorname{Rad} M, \quad \text{and} \quad \operatorname{Rad}_{n+1} = \operatorname{Rad}(\operatorname{Rad}_n M) \quad \text{for } n \ge 1.$$

In general, these sequences need not terminate; they may even be stable from the beginning. (See the previous exercise.) But if $\operatorname{Soc}_n M < \operatorname{Soc}_{n+1} M$, then the factor or "slice" $\operatorname{Soc}_{n+1} M/\operatorname{Soc}_n M$ is semisimple, and if $\operatorname{Rad}_n M > \operatorname{Rad}_{n+1} M$, then the slice $\operatorname{Rad}_n M/\operatorname{Rad}_{n+1} M$ is cosemisimple.

(a) Show that if M is artinian, then its socle series stabilizes iff for some n, $\operatorname{Soc}_n M = M$. [Hint: Every factor module of an artinian module is artinian.]

(b) Show that if M is noetherian, then its radical series stabilizes iff for some n, $\operatorname{Rad}_n M = 0$.

(c) Show that if M is both artinian and noetherian, then every factor $\operatorname{Soc}_{n+1} M/\operatorname{Soc}_n M$ and $\operatorname{Rad}_n M/\operatorname{Rad}_{n+1} M$ is semisimple. [These are the **socle factors** and **radical factors**, respectively.]

8.7. Recall from the previous exercise that the socle and radical factors of a module M are the factors of its socle and radical series.

(a) Find the socle and radical factors for the abelian group $\mathbb{Z}_{720}$;

(b) Find the socle and radical factors for the $\mathbb{Q}[x]$-module $\mathbb{Q}[x]/(x^3)$;

(c) Find the socle and radical factors for the $\mathbb{Q}[x,y]$-module $\mathbb{Q}[x,y]/(x^2,y^2)$.

9 The Injective Envelope.

In Corollaries 3.6 and 3.11 we saw that every module ${}_RM$ is a factor of a projective and the submodule of an injective. So we would like to learn whether there exist such projective and injective modules that are in some sense minimal and universal for M, sort of closures for M among the projective and injective modules. It turns out, strangely, that here we discover that our usual duality for projectives and injectives breaks down; there exist minimal injective extensions for all modules, but not minimal projectives for all modules. Although we shall discuss the projective case briefly, in this section we focus on the important injective case. The first big issue is to decide what we mean by minimal projective and injective modules for M. Fortunately, the notions of of essential and superfluous submodules will serve nicely as criteria.

Let R be a ring — initially we place no further restriction.

A(n) $\begin{cases} \text{epimorphism} \\ \text{monomorphism} \end{cases}$ $f : M \longrightarrow N$ is $\begin{cases} \underline{\textbf{superfluous}} \\ \underline{\textbf{essential}} \end{cases}$ in case $\begin{cases} \operatorname{Ker} f \ll M \\ \operatorname{Im} f \trianglelefteq N \end{cases}$

So for example, if M is a module of finite length, then the usual map $M \longrightarrow M/\operatorname{Rad} M$ is a superfluous epimorphism and the inclusion map $\operatorname{Soc} M \longrightarrow M$ is an essential monomorphism. (See Proposition 8.9.)

9.1. Lemma. **(1)** *An epimorphism $f : M \longrightarrow N$ is superfluous iff for every homomorphism $g : K \longrightarrow M$, if $f \circ g$ is epic, then g is epic.*
(2) *A monomorphism $f : M \longrightarrow N$ is essential iff for every homomorphism $g : N \longrightarrow K$, if $g \circ f$ is monic, then g is monic.*

Proof. (1) Note that fg is epic iff $\operatorname{Ker} f + \operatorname{Im} g = M$. So if $\operatorname{Ker} f \ll M$ and fg is epic, then $\operatorname{Im} g = M$. Conversely, let $K \leq M$ and let $g : K \hookrightarrow M$. If $\operatorname{Ker} f + K = M$, then fg is epic, so by hypothesis, $K = M$. The proof of (2) is dual. ■

Let ${}_RM$ be a left R-module. A superfluous epimorphism $p : P \longrightarrow M$ with P projective is a **projective** **cover** of M. Dually, an essential monomorphism $q : M \longrightarrow Q$ with Q injective is an **injective** **envelope** of M.

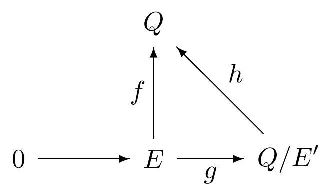
For example, if R is a ring then the epimorphism $R \longrightarrow R/J$ is a projective cover. If R is a P.I.D. with field of fractions Q, then the inclusion ${}_RR \hookrightarrow {}_RQ$ is an injective envelope.

As we shall see, every module has an injective envelope. For once our duality takes a vacation; not every module has a projective cover. Indeed, as we shall see in a moment, if the $\mathbb{Z}$-module $\mathbb{Z}_2$ had a projective cover, the usual epimorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ would have to be one. But clearly, this epimorphism is not superfluous. So $\mathbb{Z}_2$ does not have a projective cover over $\mathbb{Z}$.

We now prove that injective envelopes exist.

9.2. Theorem. *If $_RM$ is a module over a ring R, then M has an injective envelope over R.*

Proof. First, by Corollary 3.11, we may assume that $M \leq Q$ for some injective module $_RQ$. Then there exists a submodule $E \leq Q$ maximal w.r.t. $M \trianglelefteq E \leq Q$. Next, there is a submodule $E' \leq Q$ maximal w.r.t. $E \cap E' = 0$. So the obvious map $g : E \longrightarrow Q/E'$ is an essential monomorphism. If $f : E \hookrightarrow Q$ is the inclusion map, then by the injectivity of Q, there is a homomorphism $h : Q/E' \longrightarrow Q$ such that



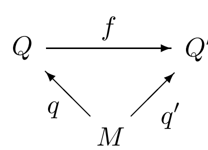
commutes. Since f is monic and g is essential, h is monic. (See Lemma 9.1.) But then $E = \operatorname{Im} f \trianglelefteq \operatorname{Im} h$ so by the maximality of E, $\operatorname{Im} h = E$. Hence, g is actually an isomorphism, so $Q = E \oplus E'$ and E is injective. That is, $M \hookrightarrow E$ is an injective envelope of M. ■

The following characterization of injective envelopes will serve to prove their essential uniqueness.

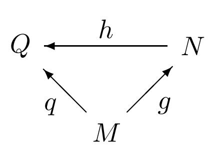
9.3. Theorem. *Let $_RM$ and $_RQ$ be left R-modules. Then for every monomorphism $q : M \longrightarrow Q$, the following are equivalent:*

(a) *$q : M \longrightarrow Q$ is an injective envelope;*

(b) *Q is injective and for every monomorphism $q' : M \longrightarrow Q'$ with $_RQ'$ injective, there exists a, necessarily split, monomorphism $f : Q \longrightarrow Q'$ with $q' = fq$:*



(c) *$q : M \longrightarrow Q$ is essential and for every essential monomorphism $g : M \longrightarrow N$ there exists a monomorphism $h : N \longrightarrow Q$ with $hg = q$*



■

Proof. (a) $\Longrightarrow$ (b) Since q is a monomorphism and Q' is injective, f exists with $q' = fq$. But q' is monic and q is essential, so by Lemma 9.1, f is monic and so it splits.

(b) $\Longrightarrow$ (c) Since g is monic and Q is injective, $h : N \longrightarrow Q$ exists with $q = gh$. But g and q are essential monomorphisms, so by Lemma 9.1, h is a monomorphism.

(c) $\Longrightarrow$ (a) Let $g : M \longrightarrow N$ be an injective envelope. Then by (c) there is a monomorphism $h : N \longrightarrow Q$ with $q = hg$. Since N is injective, h splits. But $\operatorname{Im} hg \leq \operatorname{Im} h$ and $\operatorname{Im} hg = \operatorname{Im} q \trianglelefteq Q$, so $\operatorname{Im} h = Q$. That is, Q is injective. ■

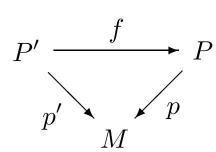
9.4. Corollary. *Let $q : M \longrightarrow Q$ and $q' : M \longrightarrow Q'$ be injective envelopes of M. Then there is an isomorphism $f : Q \longrightarrow Q'$ such that $q' = fq$. In fact, if $h : Q \longrightarrow Q'$ is a homomorphism with $q' = hq$, then h is an isomorphism.* ■

When projective covers exist, dual properties hold for them. We will leave the details of the following result to the reader.

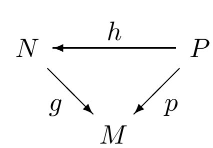
9.5. Theorem. *Assume that a module ${}_RM$ has a projective cover. Then for every epimorphism $p : P \longrightarrow M$, the following are equivalent:*

(a) *$p : P \longrightarrow M$ is a projective cover;*

(b) *P is projective and for every epimorphism $p' : P' \longrightarrow M$ with ${}_RP'$ projective, there exists a, necessarily split, epimorphism $f : P' \longrightarrow P$ with $pf = p'$:*



(c) *$p : P \longrightarrow M$ is superfluous and for every superfluous epimorphism $g : N \longrightarrow M$ there exists an epimorphism $h : P \longrightarrow N$ with $gh = p$*



9.6. Corollary. *Let $p : P \longrightarrow M$ and $p' : P' \longrightarrow M$ be projective covers of M. Then there is an isomorphism $f : P \longrightarrow P'$ such that $p = p'f$. In fact, if $h : P \longrightarrow P'$ is a homomorphism with $p = p'h$, then h is an isomorphism.*

Before we continue, maybe it would be in order to make a few general comments about Theorems 9.3 and 9.5. The first says that $q : M \longrightarrow Q$ is an injective envelope iff it is a minimal injective extension of M in the sense that Q is a direct summand of every injective extension of M iff it is a maximal essential extension in that it factors through any essential monomorphism from M. Dually, if M has a projective cover, then an epimorphism $p : P \longrightarrow M$ is a projective cover iff P is a minimal projective in the sense that it is a direct summand of any projective with M as a factor iff it is a maximal superfluous epimorphism in the sense that p factors through any superfluous epimorphism onto M.

Thanks to the essential uniqueness of injective envelopes, we sometimes think of "the" injective envelope of ${}_RM$ as some given overmodule $E(M)$ of M for which the inclusion map $M \hookrightarrow E(M)$ is an injective envelope. As we shall see in the Exercises this is fraught with danger; so we must use some caution when we speak of "the" injective envelope of a module. We make some more remarks on this after the next couple of lemmas on properties of injective envelopes.

9.7. Lemma. *If $N \leq M$, then $N \trianglelefteq M$ iff $E(N) = E(M)$.*

Proof. ($\Longrightarrow$) $N \trianglelefteq M \trianglelefteq E(M) \Longrightarrow N \trianglelefteq E(M)$ and $E(M)$ injective, so $E(N) = E(M)$.

($\Longleftarrow$) Since $N \leq M \leq E(N)$ and $N \trianglelefteq E(N)$, we have $N \trianglelefteq M$. ∎

The next result follows immediately from (b) of Theorem 9.3.

9.8. Lemma. *If $M \leq Q$ with Q injective, then $Q = E(M) \oplus Q'$ for some $Q' \leq Q$.* ∎

The statements of these last two lemmas are both rather informal, but they illustrate well our sometimes casual use of "the" injective envelope. Lemma 9.7 stated properly should say that $N \trianglelefteq M$ iff for every injective envelope $q : M \longrightarrow Q$ of M, the restriction $q|_N : N \longrightarrow Q$ is an injective envelope of N. Similarly, the second, Lemma 9.8, should say that if $M \leq Q$ with Q injective, then there is an injective direct summand $E \leq Q$ with $M \hookrightarrow E$ an injective envelope. We'll leave to you the task of rewording the next lemma to avoid any ambiguity.

9.9. Lemma. *If $M_1, \ldots, M_n \leq M$ with $M = M_1 \oplus \cdots \oplus M_n$, then $E(M) = E(M_1) \oplus \cdots \oplus E(M_n)$.*

Proof. It will suffice to prove this for $n = 2$, for then an obvious induction takes care of the rest. So suppose that $M = M_1 \oplus M_2$ and that $E = E(M)$. Then by Lemma 9.8 for $i = 1, 2$ we have $E = E(M_i) \oplus E_i$ for some $E_i \leq E$ and some essential injective extension $E(M_i)$ of M_i. Now $E(M_1) \cap E(M_2) \neq 0$ and $M_i \trianglelefteq E(M_i)$ would force $M_1 \cap M_2 \neq 0$. So $E(M_1)$ and $E(M_2)$ are independent. Thus $E(M_1) \oplus E(M_2)$ is an injective submodule of E , so $E = E(M_1) \oplus E(M_2) \oplus E'$ for some $E' \leq E$. But $M_1 \oplus M_2 \leq E(M_1) \oplus E(M_2)$ and $M_1 \oplus M_2 \trianglelefteq E$, so $E' = 0$. ∎

Exercises 9.

9.1. Prove Theorem 9.5.

9.2. Let R be a ring with $\operatorname{Rad} R = 0$. Prove that a finitely generated module ${}_RM$ has a projective cover iff M is already projective. [Hint: Proposition 8.8.]

9.3. Let $R = \mathbb{Z}_4$, so that ${}_RR$ is injective. (See Exercise 3.6.) Consider the injective module $E = R \times R$, and the submodule $M = \{(0,0),(2,2)\}$ of E. Prove that

(a) ${}_RM$ is not injective.

(b) ${}_RM$ is an intersection of injectives.

(c) There are many injective submodules of E that are essential extensions of M.

9.4. As we saw in the last exercise, a submodule M of an injective module E can have many injective envelopes in E. But

(a) Let E be injective. Prove that every submodule M of E has a unique injective envelope in E iff every intersection of two injective submodules of E is injective.

(b) If K and N are injective submodules of a module M such that $K \cap N$ is also injective, then prove that $K + N$ is injective.

(c) On the other hand consider the injective $\mathbb{Z}$ module $E = \mathbb{Q} \times \mathbb{Q}$. Show that E contains a couple of injective submodules K and N with $K + N$ injective, but $K \cap N$ not injective.

9.5. We have seen that projective generators have a prominent role to play. What about injective cogenerators? Well, it turns out that there is a minimal such module. Let $\mathcal{S}$ be a complete set of representatives of the simple left R-modules. (That is, each simple module is isomorphic to exactly one of the simples in the set $\mathcal{S}$.) Let

$$C = E(\coprod_{S \in \mathcal{S}} S).$$

(a) Prove that C is an injective cogenerator; that is, C is injective and cogenerates every left R module.

(b) Prove that if Q is another injective cogenerator, then there is a split monomorphism $C \longrightarrow Q$.

10 The Goldie Rank.

In this Section we consider the "Goldie rank" of a module. This rank measures in some sense the width of the module. Its definition hinges on the fact that if an injective module ${}_RE$ has a direct decomposition into a finite number, say n, of indecomposable submodules, then no direct decomposition of E has more than n non-zero terms. We call that number n, if it exists, the "Goldie rank" of E and of any module M for which E is an injective envelope. So we begin by characterizing those modules whose injective envelopes are indecomposable.

A module ${}_RM$ is **uniform** if every non-zero submodule of M is essential in M. For example, if the lattice of submodules of M is a chain, then M is uniform. So, in particular, the abelian groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ are uniform when p is a prime. More generally, if ${}_RM$ is Artinian, then it is uniform iff it has a simple socle. (See Proposition 8.9.)

10.1. Lemma. *For a module ${}_RM$ with $E = E(M)$ the following are equivalent:*

(a) *M is uniform;*

(b) *For every pair N_1, N_2 of non-zero submodules of M, $N_1 \cap N_2 \neq 0$;*

(c) *Every non-zero submodule of M is uniform;*

(d) *E is indecomposable;*

(e) *E is uniform.*

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) are trivial.

(b) $\Longrightarrow$ (d) If $E = E_1 \oplus E_2$, then $(M \cap E_1) \cap (M \cap E_2) = 0$ so by (b), $M \cap E_1 = 0$ or $M \cap E_2 = 0$. But $M \trianglelefteq E$, so either $E_1 = 0$ or $E_2 = 0$, and E is indecomposable.

(d) $\Longrightarrow$ (e) Let $0 \neq N \leq E$. Then by Lemma 9.8 there is some $E' \leq E$ with $E = E(N) \oplus E'$. Then by (d), we have $E' = 0$, so $E = E(N)$ and $N \trianglelefteq E$.

(e) $\Longrightarrow$ (a) By (b) every non-zero submodule of E is uniform. ■

Let ${}_RM$ be a left R-module and let $E = E(M)$ be an injective envelope of M. If E has a direct decomposition

$$E = E_1 \oplus \cdots \oplus E_n$$

with each $E_1, \ldots, E_n$ indecomposable and with n minimal among all such finite indecomposable decompositions, then n is the **Goldie rank** of M and we write

$$\operatorname{Grank}(M) = n.$$

In the literature you will also find the names **uniform** **rank** and just plain **rank**. If the injective envelope $E(M)$ has no finite Indecomposable decomposition, then M has infinite Goldie rank. Note, of course, that

$$\mathrm{Grank}(M) = \mathrm{Grank}(E(M)).$$

The (**left**) **Goldie** **rank** of the ring R is the Goldie rank of the regular module ${}_RR$. Similarly, the (**right**) **Goldie** **rank** of the ring R is the Goldie rank of the regular module R_R. Now the following result shows that this notion of rank of a module M does give us a legitimate measure of the width of the module M.

10.2. Theorem. *For an injective module ${}_RE$, if* $\mathrm{Grank}(E) = n$*, then for every direct decomposition*

$$E = Q_1 \oplus \cdots \oplus Q_m$$

with each Q_j non-zero, we have $m \leq n$.

Proof. Let $E = E_1 \oplus \cdots \oplus E_n$ with each E_i indecomposable. We induct on n to show that $m \leq n$. If $n = 1$, E is indecomposable, and trivially, $m \leq 1$. So assume the Theorem is false and that $n > 1$ is the least n for which the Theorem fails. So we may choose the above decomposition so that $m > n$. Set

$$Q = Q_1 \oplus \cdots \oplus Q_{m-1}.$$

Suppose that $Q \cap E_i = 0$ for some i. Then there is a split monomorphism $Q \longrightarrow E/E_i$. But clearly $\mathrm{Grank}(E/E_i) \leq n - 1$ contradicting $m - 1 > n - 1$. So each $Q \cap E_i$ is non-zero and since each E_i is indecomposable injective and thus uniform, $Q \cap E_i \trianglelefteq E_i$ for all i. Therefore, (see Exercise 8.3),

$$(Q \cap E_1) \oplus \cdots \oplus (Q \cap E_n) \trianglelefteq E_1 \oplus \cdots \oplus E_n = E.$$

But this contradicts that $Q \cap Q_m = 0$. ■

10.3. Corollary. *A module M has Goldie rank n iff every indecomposable direct decomposition of $E(M)$ has exactly n terms.* ■

Thanks to the characterization of indecomposable injectives in terms of uniform modules (Lemma 10.1) we can characterize the Goldie rank of a module internally without reference to its injective envelope.

10.4. Corollary. *A module M has Goldie rank* $\mathrm{Grank}(M) = n$ *iff there exists an independent set $M_1, \ldots, M_n$ of uniform submodules of M with $M_1 \oplus \cdots \oplus M_n \trianglelefteq M$.*

Proof. ($\Longrightarrow$) Since $\mathrm{Grank}(M) = n$, $E = E(M)$ has an indecomposable direct decomposition $E = E_1 \oplus \cdots \oplus E_n$. Since $M \trianglelefteq E$ and since each E_i is uniform, each $M \cap E_i$ is uniform and each $M \cap E_i \trianglelefteq E_i$. So (see Exercise 8.3) $(M \cap E_1) \oplus \cdots \oplus (M \cap E_n) \trianglelefteq E$, and threfore, $(M \cap E_1) \oplus \cdots \oplus (M \cap E_n) \trianglelefteq M$.

($\Longrightarrow$) Since each M_i is uniform, each $E(M_i)$ is indecomposable. But then (see Lemma 9.9)

$$E(M) = E(M_1 \oplus \cdots \oplus M_n) = E(M_1) \oplus \cdots \oplus E(M_n)$$

is an indecomposable direct decomposition of $E(M)$. ■

10.5. Corollary. *If $K, N \leq M$ with $K \oplus N \trianglelefteq M$, and if* Grank$(M)$ *is finite, then*

$$\text{Grank}(M) = \text{Grank}(K) + \text{Grank}(N).$$ ■

10.6. Lemma. *Let $N \leq M$. Then there exists a submodule $K \leq M$ with $N \cap K = 0$ and $N \oplus K \trianglelefteq M$.*

Proof. Let $K \leq M$ be maximal w.r.t. $K \cap N = 0$. Then $N \oplus K \trianglelefteq M$. ■

10.7. Corollary. *If $N \leq M$ and* Grank$(M) = n$, *then* Grank$(N) \leq n$.

Proof. By Lemma 10.6 there exists some submodule $K \leq M$ with $N + K = N \oplus K \trianglelefteq M$, so apply Corollary 10.5. ■

Often we will be interested in just whether a module has finite Goldie rank rather than in knowing the exact rank. The following provides a particularly useful test.

10.8. Corollary. *For a module M, if $n \in \mathbb{N}$, then* Grank$(M) \leq n$ *iff for every independent set $M_1, \ldots, M_m$ of non-zero submodules of M, $m \leq n$. In particular, M has finite Goldie rank iff M has no infinite independent set of non-zero submodules.*

Proof. A set $M_1, \ldots, M_m \leq M$ is independent iff $M_1 + \cdots + M_m = M_1 \oplus \cdots \oplus M_m$ iff (See Lemma 9.9) $E(M) = E(M_1) \oplus \cdots \oplus E(M_m)$ iff $m \leq n$. ■

Exercises 10.

10.1. Prove that ${}_RM$ has finite Goldie rank if

(a) ${}_RM$ is Artinian (b) ${}_RM$ is Noetherian.

10.2. Find an example of a module

(a) That is indecomposable with infinite Goldie rank.

(b) That has no uniform submodules

10.3. Prove that if ${}_RM$ has finite Goldie rank, then a monomorphism $f : M \longrightarrow N$ is essential iff $\mathrm{Grank}(N) = \mathrm{Grank}(M)$.

10.4. Let k be a field and V_k a right vector space over k. Let $S = \mathrm{End}(V_k)$ be its endomorphism ring. Finally, let R be the matrix ring

$$R = \begin{bmatrix} S & V \\ 0 & k \end{bmatrix}.$$

Prove that the left Goldie rank of R is 1 but that the right Goldie rank of R is $\dim(V_k)$.

10.5. A ring R is **left perfect** in case every left R-module ${}_RM$ has a projective cover. (See [1] Section 28 for characterizations of these rings.) A module ${}_RM$ is **co-uniform** if every $N < M$ is superfluous in M. Let R be left perfect. Prove that for a module ${}_RM$ the following are equivalent:

(a) ${}_RM$ is co-uniform;

(b) For every pair $N_1, N_2 < M$, $N_1 + N_2 < M$.

(c) Every factor module of M is co-uniform.

(d) If $p : P \longrightarrow M$ is a projective cover of M, then P is indecomposable.

(e) If $p : P \longrightarrow M$ is a projective cover of M, then P is co-uniform.

10.6. Prove that if ${}_RP$ is a progenerator for R, then R has finite left Goldie rank iff ${}_RP$ has finite Goldie rank.

10.7. Let $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ be an additive category equivalence.

(a) Prove that $\mathrm{Grank}({}_RM) = \mathrm{Grank}({}_SF(M))$ for every left R-module M.

(b) Prove that R has finite left Goldie rank iff S has finite left Goldie rank.

(c) Show that the left Goldie ranks of R and S need not be equal.

11 Annihilators.

In this Section we take a brief look at the important notion of annihilators. Although we shall use these in only very limited contexts, we will give a fairly general initial treatment, since the latter is actually easier than the special cases and is readily applied in the special cases.

Suppose that R, S, and T are rings, that ${}_RP_S$, ${}_SQ_T$, and ${}_RU_T$ are bimodules, and finally, that

$$\mu : P \otimes_S Q \longrightarrow U$$

is an (R, T) bimodule homomorphism. Then for each set $X \subseteq Q$ its **left annihilator** (in P) is

$$\mathbf{l}_P(X) = \{p \in P \ : \ \mu(p \otimes x) = 0 \ \forall x \in X\}.$$

For each $A \subseteq P$, its **right annihilator** (in Q) is

$$\mathbf{r}_Q(A) = \{q \in Q \ : \ \mu(a \otimes q) = 0 \ \forall a \in A\}.$$

It is clear that $\mathbf{l}_P(X)$ is a submodule of ${}_RP$ and that $\mathbf{r}_Q(A)$ is a submodule of Q_T. Thus, $\mathbf{l}_P$ defines a function from the power set of Q to the set of submodules of ${}_RP$, and $\mathbf{r}_Q$ defines a function from the power set of P to the set of submodules of Q_T.

Example. Our two principal examples of this will occur when R and S are rings and ${}_RM_S$ is a bimodule. The module multiplication $\mu : R \otimes_R M \longrightarrow M$ by $\mu : (r \otimes x) \longmapsto rx$ is an (R, S) bimodule homomorphism. Then the annihilators $\mathbf{l}_R$ and $\mathbf{r}_M$ are maps from the subsets of M to the left ideals of R and from the subsets of R to the submodules of M_S, respectively.

On the other side, module multiplication $\mu : M \otimes_S S \longrightarrow M$ via $\mu : x \otimes s \longmapsto xs$ is an (R, S) homomorphism. So here the annihilators $\mathbf{l}_M$ and $\mathbf{r}_S$ are maps from the subsets of S to the submodules of ${}_RM$ and from the subsets of M to the right ideals of S, respectively. ■

Returning to the general case, our immediate interest is in the behavior of the annihilator functions $\mathbf{l}_P$ and $\mathbf{r}_Q$. We begin with the easy and very basic

11.1. Lemma. *For an (R, T) bimodule homomorphism $\mu :_R P \otimes_S Q_T \longrightarrow U$ the annihilators $\mathbf{l}_P$ and $\mathbf{r}_Q$ satisfy for all $X, Y \subseteq Q$ and $A, B \subseteq P$,*

(1) $X \subseteq Y \Longrightarrow \mathbf{l}_P(X) \supseteq \mathbf{l}_P(Y)$, *and* $A \subseteq B \Longrightarrow \mathbf{r}_Q(A) \supseteq \mathbf{r}_Q(B)$;

(2) $X \subseteq SXT \subseteq \mathbf{r}_Q\mathbf{l}_P(X)$, *and* $A \subseteq RAS \subseteq \mathbf{l}_P\mathbf{r}_Q(A)$;

(3) $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) = \mathbf{l}_P(X)$, *and* $\mathbf{r}_Q\mathbf{l}_P\mathbf{r}_Q(A) = \mathbf{r}_Q(A)$.

Proof. The first two are trivial. For (3) from (1) applied to (2) we have that $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) \subseteq \mathbf{l}_P(X)$. But the second part of (2) applied to $\mathbf{l}_P(X)$ gives $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) \supseteq \mathbf{l}_P(X)$. ■

11.2. Lemma. *For an (R,T) bimodule homomorphism $\mu :_R P \otimes_S Q_T \longrightarrow U$ the annihilators $\mathbf{l}_P$ and $\mathbf{r}_Q$ satisfy for all indexed sets of additive subgroups $(P_\alpha)_{\alpha\in\Omega}$ in P and $(Q_\alpha)_{\alpha\in\Omega}$ in Q,*

(1) $\mathbf{l}_P(\sum_\Omega Q_\alpha) = \bigcap_\Omega \mathbf{l}_P(Q_\alpha)$ *and* $\mathbf{r}_Q(\sum_\Omega P_\alpha) = \bigcap_\Omega \mathbf{r}_Q(P_\alpha)$*;*

(2) $\sum_\Omega \mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$*, and* $\sum_\Omega \mathbf{r}_Q(P_\alpha) \subseteq \mathbf{r}_Q(\bigcap_\Omega P_\alpha)$*.*

Proof. (1) Since $Q_\beta \subseteq \sum_\Omega Q_\alpha$ for all β, we have from Lemma 11.1 that $\mathbf{l}_P(\sum_\Omega Q_\alpha) \subseteq \bigcap_\Omega \mathbf{l}_P(Q_\alpha)$. But trivially $\bigcap_\Omega \mathbf{l}_P(Q_\alpha)$ annihilates every Q_α and hence the sum $\sum_\Omega(Q_\alpha)$.

(2) By Lemma 11.1 we certainly have $\mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$ for every α. But $\mathbf{l}_P(\bigcap_\Omega Q_\alpha)$ and $\mathbf{l}_P(Q_\alpha)$ are submodules of ${}_RP$, so $\sum_\Omega \mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$. ■

Thus, from these Lemmas we see that the functions $\mathbf{l}_P\mathbf{r}_Q$ and $\mathbf{r}_Q\mathbf{l}_R$ define closure operators on the sets P and Q, and in each case the closure of any set is a submodule. In general, not all submodules of ${}_RP$ or Q_T are closed under these operators. Still given the set up where $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ is a bimodule homomorphism, we set

$$\mathcal{L}_P(Q) = \{\mathbf{l}_P(X) \ : \ X \subseteq Q\},$$

and

$$\mathcal{R}_Q(P) = \{\mathbf{r}_Q(A) \ : \ A \subseteq P\}.$$

Thus, $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are the "closed" submodules of ${}_RP$ and Q_T, respectively. Of course, these sets are subposets of the sets of all submodules of ${}_RP$ and Q_T. Clearly, $P \in \mathcal{L}_P(Q)$ and $Q \in \mathcal{R}_Q(P)$, and by Lemma 11.2, $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are both closed under arbitrary intersections. Thus, we infer that

11.3. Proposition. *The posets $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are complete lattices and the pair of maps $\mathbf{r}_Q : \mathbf{l}_P(X) \longmapsto \mathbf{r}_Q(\mathbf{l}_P(X))$ and $\mathbf{l}_P : \mathbf{r}_Q(A) \longmapsto \mathbf{l}_P(\mathbf{r}_Q(A))$ are lattice anti-isomorphisms between $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$.* ■

11.4. Corollary. *The lattice $\mathcal{L}_P(Q)$ satisfies the Maximum (Minimum) Condition iff the lattice $\mathcal{R}_Q(P)$ satisfies the Minimum (Maximum) Condition.* ■

Given a bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ we call the modules in $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ the **annihilator modules** for μ. (Frequently, when the map μ is understood, we suppress mention of it.) Of particular interest are the cases where all submodules of ${}_RP$ and of Q_T are annihilator modules. That would mean that every submodule $M \leq_R P$ and $N \leq Q_T$ would satisfy $M = \mathbf{l}_R(X)$ and $N = \mathbf{r}_Q(A)$ for some $X \subseteq Q$ and $A \subseteq Q$. But then by Lemma 11.1 part (3),

11.5. Corollary. *Every submodule of* $_RP$ *and of* Q_T *is an annihilator module iff for each* $M \leq_R P$ *and every* $N \leq Q_T$

$$\mathbf{l}_P\mathbf{r}_Q(M) = M \qquad \textit{and} \qquad \mathbf{r}_Q\mathbf{l}_P(N) = N. \qquad \blacksquare$$

We say that the bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ satisfies the **double annihilator property** if the conditions of Corollary 11.5 are satisfied. Some very good things happen when we have the double annihilator property. We won't go into most of them here, but next we will show how that property tightens up the second part of Lemma 11.2

11.6. Corollary. *If the bimodule homomorphism* $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ *satisfies the double annihilator property, then for all indexed sets of additive subgroups* $(P_\alpha)_{\alpha\in\Omega}$ *in* P *and* $(Q_\alpha)_{\alpha\in\Omega}$ *in* Q,

$$\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \mathbf{l}_P(\bigcap\nolimits_\Omega Q_\alpha), \quad \textit{and} \quad \sum\nolimits_\Omega \mathbf{r}_Q(P_\alpha) = \mathbf{r}_Q(\bigcap\nolimits_\Omega P_\alpha).$$

Proof. By (1) of Lemma 11.2 and the fact that $\mathbf{r}_Q\mathbf{l}_P(Q_\alpha) = Q_\alpha$, we have

$$\mathbf{r}_Q(\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \bigcap\nolimits_\Omega \mathbf{r}_Q\mathbf{l}_P(Q_\alpha) = \bigcap\nolimits_\Omega Q_\alpha.$$

So from the double annihilator property

$$\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \mathbf{l}_P\mathbf{r}_Q\left(\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha)\right) = \mathbf{l}_P\left(\bigcap\nolimits_\Omega Q_\alpha\right). \qquad \blacksquare$$

Here is an important, and very possibly familiar, example of annihilators at play. Let R and S be rings and let $_RU_S$ be a bimodule. For each left R-module $_RM$ and each right S-module N_S their U-**duals** are the right S-module M^*_S and left R-module $_RN^*$ defined by

$$M^* = \operatorname{Hom}_R(M, U) \qquad \text{and} \qquad N^* = \operatorname{Hom}_S(N, U).$$

Here we want to be careful about how these homomorphisms operate. Thus, we want the elements of M^* to **act on the right** and those of N^* to **act on the left**. So we have bimodule homomorphisms

$$\mu : M \otimes_{\mathbb{Z}} M^* \longrightarrow {}_RU_S \qquad \text{and} \qquad \nu : N^* \otimes_{\mathbb{Z}} N \longrightarrow {}_RU_S$$

defined by

$$\mu : x \otimes f \longmapsto (x)f \qquad \text{and} \qquad \nu : g \otimes y \longmapsto g(y).$$

So each of these bimodule homomorphisms generates a pair of annihilator maps. For example, for the first, if $X \subseteq M$ and $A \subseteq M^*$, then

$$\mathbf{r}_{M^*}(X) = \{f \in M^* \ : \ X \subseteq \operatorname{Ker} f\} \qquad \text{and} \qquad \mathbf{l}_M(A) = \cap\{\operatorname{Ker} f \ : f \in A\}.$$

A problem of some significance is to determine those submodules of $_RM$ (M^*_S) that are annihilators. We'll look at this some more in the Exercises. On the other hand the special case of this set up that we're probably most familiar with is when $R = S$ is a field and $U = R$ is the regular bimodule. Then for each ${}_RM$, an R-vector space, M^* is just its usual dual of all linear functionals $f : M \longrightarrow R$ on M. In that special case every submodule of ${}_RM$ is a left annihilator.

As we mentioned in another earlier Example a particularly common example of all of this arises with a bimodule ${}_RM_S$ and the scalar multiplication map

$$\mu : R \otimes_R M \longrightarrow M.$$

In this case for each $X \subseteq M$ and $A \subseteq R$,

$$\mathbf{l}_R(X) = \{r \in R \ : \ rx = 0 \ \forall x \in X\} \qquad \text{and} \qquad \mathbf{r}_M(A) = \{x \in M \ : \ ax = 0 \ \forall a \in A\}.$$

So $\mathbf{l}_R(X)$ is a left ideal of R and $\mathbf{r}_M(A)$ is a right S submodule of M. Of course, if X is actually a submodule of M, then $\mathbf{l}_R(X)$ is an ideal of R, and if A is a right ideal of R, then $\mathbf{r}_M(A)$ is an (R, S) sub-bimodule of M. The left ideals in $\mathcal{L}_R(M)$ are often called M-**annihilator left ideals** and the S-submodules in $\mathcal{R}_M(R)$ are **annihilator** S-**submodules** of M.

For this last example of a bimodule ${}_RM_S$, there is a related notion and notation that is extremely useful. Thus, let X and Y be subsets of M. We then set

$$(X : Y) = \{r \in R \ : \ rY \subseteq X\}.$$

So one has that

$$(X : Y) \subseteq R \qquad \text{and} \qquad (X : Y)Y \subseteq X.$$

The case of most interest is when $N \leq M$ is an R-submodule of M, so that $(N : X)$ is a left ideal of R. In fact $(N : X)$ is the annihilator

$$(N : X) = \mathbf{l}_R(X + N/N)$$

in R of the subset $(N + X)/N$ of the R-module M/N. Thus, one often refers to $(N : X)$ as a **relative annihilator**. Finally, we note that for each $X \subseteq M$, the left ideal $(0 : X)$ is just the left annihilator of X and is also equal to the intersection of the annihilators of the $x \in X$. So for each $x \in M$ and $X \subseteq M$,

$$(0 : x) = \mathbf{l}_R(x) \qquad \text{and} \qquad (0 : X) = \mathbf{l}_R(X) = \bigcap_{x \in X} \mathbf{l}_R(x).$$

There is one particularly important special case of this last special case, the one where $M = R$. Here $\mu : R \otimes R \longrightarrow R$ is just multiplication in R. And in this case, for each $X \subseteq R$

$$\mathbf{l}_R(X) = \{r \in R \ : \ rX = 0\} \qquad \text{and} \qquad \mathbf{r}_R(X) = \{r \in R \ : \ Xr = 0\}.$$

The left ideals in $\mathcal{L}_R(R)$ are called the **annihilator left ideals** of R and the right ideals in $\mathcal{R}_R(R)$ are the **annihilator right ideals** of R.

We conclude with one simple instance in which we refer to annihilator ideals. Recall that a one-sided ideal I of R is **nilpotent** in case there is some $n \in \mathbb{N}$ with $I^n = 0$. The one sided ideal I is **nil** if each element of I is nilpotent. Of course, every nilpotent one sided ideal is nil, but the converse fails. But here is one case in which they are the same.

11.7. Theorem. *Let R be a ring with no non-zero nilpotent ideals. If $\mathcal{L}_R(R)$, the lattice of annihilator left ideals, satisfies the Maximum Condition, then R has no non-zero nil one-sided ideals.*

Proof. If I is a non-zero nil left ideal, then for each $x \in I$ the right ideal xR is nil. So it will suffice to show that there is no non-zero nil principal right ideal xR. Suppose, on the contrary, a non-zero right ideal xR is nil. By hypothesis the set

$$\{\mathbf{l}_R(xa) \ : \ 0 \neq xa \in xR\}$$

has a maximal element, say $\mathbf{l}_R(xa)$. Let $y = xa$. We claim that RyR is nilpotent, so that $y = 0$ contrary to $y \neq 0$. Suppose then that $b \notin \mathbf{r}_R(y)$, so that $yb \neq 0$. Now $y \in xR$, so yb is nilpotent, so there is a least $n \in \mathbb{N}$ with $(yb)^n = 0$. Of course, $n > 1$, so $(yb)^{n-1} \neq 0$. But clearly, $\mathbf{l}_R(y) \leq \mathbf{l}_R((yb)^{n-1})$, so by the maximality of $\mathbf{l}_R(y)$, we have $\mathbf{l}_R(y) = \mathbf{l}_R((yb)^{n-1})$. So $yb \in \mathbf{l}_R((yb)^{n-1}) = \mathbf{l}_R(y)$, and $yby = 0$. But that means that $yRy = 0$, so that RyR is nilpotent, and $y = 0$. ■

Exercises 11.

11.1. Prove that the inclusion in Lemma 11.2 part (2) can be strict. [Hint: Consider the ring $R = \mathbb{Z}[x, y]/(xy)$. Let ${}_RM = {}_RR$ be the regular module, and consider the submodules Rx and Ry of M. Finally compute annihilators w.r.t. the multiplication map $\mu : R \otimes_R M \longrightarrow M$.]

11.2. Given a bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ we know that $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are complete lattices. (See Proposition 11.3.) Moreover, we know, for example, that in each case the greatest lower bound of any set of submodules is the intersection of the set.

(a) Let $\mathcal{A} \subseteq \mathcal{L}_P(Q)$. Find a formula for the least upper bound of $\mathcal{A}$.

(b) Show that $\mathcal{L}_P(Q)$ need not be a sublattice of the lattice of submodules of P.

11.3. Consider a bimodule ${}_RU_S$ and a module ${}_RM$. Compute annihilators w.r.t. the bimodule homomorphism $\mu : M \otimes_{\mathbb{Z}} M^* \longrightarrow U$ where $M^* = \mathrm{Hom}_R(M, U)$ is the U-dual of M. Prove that a submodule N of ${}_RM$ is in the lattice $\mathcal{L}_M(M^*)$ iff U cogenerates M/N.

11.4. Prove that if R is a ring for which the regular module ${}_RR$ cogenerates all left R modules, then ${}_RR$ is injective. [Hint: Let $E = E({}_RR)$ be the injective envelope of ${}_RR$. Since ${}_RR$ cogenerates E, there is a monomorphism $\varphi : E \longrightarrow R^\Omega$ for some set Ω. For each $\alpha \in \Omega$ let $e_\alpha = \pi_\alpha(1)$ where the $\pi_\alpha : R^\Omega \longrightarrow R$ are the coordinate projections. Let I be the right ideal $I = \sum e_\alpha R$. Then $\mathbf{l}_R(I) = 0$ so $I = R$, and $1 = \sum e_\alpha a_\alpha$. Then the map $x \longmapsto \sum \pi_\alpha(x)e_\alpha$ is a split epi, and ${}_RR$ is injective.]

11.5. A ring R is a **cogenerator ring** in case ${}_RR$ cogenerates all left R-modules and R_R cogenerates all right R-modules. By Exercise 11.4 if R is a cogenerator ring, then ${}_RR$ and R_R are both injective. Prove that R is a cogenerator ring iff ${}_RR$ and R_R are both injective and R satisfies the double annihilator property in the sense that for every left ideal I and every right ideal J,

$$\mathbf{l}_R\mathbf{r}_R(I) = I \qquad \text{and} \qquad \mathbf{r}_R\mathbf{l}_R(J) = J.$$

11.6. The type of duality that is given by pairs of annihilator maps $\mathbf{l}_P$ and $\mathbf{r}_Q$ are almost densely distributed in mathematics albeit in many different guises. Such pairings are at the heart of Galois theory and algebraic geometry. Here is another one that occurs in elementary analysis. Let X be a compact Hausdorff space and let $C = C(X, \mathbb{R})$ be the ring of all continuous functions $f : X \longrightarrow \mathbb{R}$. For each set $A \subseteq C$ and each set $Y \subseteq X$ let

$$\mathbf{r}_X(A) = \{x \in X \ : \ f(x) = 0 \ \forall f \in A\} \qquad \text{and} \qquad \mathbf{l}_C(Y) = \{f \in C \ : \ f(x) = 0 \ \forall x \in Y\}.$$

You might check that these "annihilators" satisfy the properties of Lemma 11.1. Then prove that

(a) A subset $Y \subseteq X$ is closed iff it is an annihilator; that is, iff $Y = \mathbf{r}_X(A)$ for some $A \subseteq C$;

(b) A subset $I \subseteq C$ is an ideal iff it is an annihilator; that is iff $I = \mathbf{l}_C(Y)$ for some $Y \subseteq X$.

(c) For every $Y \subseteq X$, its closure is $\mathbf{r}_X\mathbf{l}_C(Y)$ and for each $A \subseteq C$ the ideal generated by A is $\mathbf{l}_C\mathbf{r}_X(A)$.

(d) Let $\mathcal{M}$ be the set of all maximal ideals of C. For each subset $\mathcal{I} \subseteq \mathcal{M}$ let

$$\overline{\mathcal{I}} = \{M \in \mathcal{M} \ : \ \cap\mathcal{I} \subseteq M\}.$$

Prove that $\mathcal{I} \longmapsto \overline{\mathcal{I}}$ defines a closure operator on $\mathcal{M}$ and that in the resulting topology $\mathcal{M}$ is homeomorphic to X.

12 The Maximal Ring of Quotients.

Let R be a commutative ring with set Δ of non-zero divisiors. In 600 Algebra we learn that there is a commutative overring Q of R in which each $d \in \Delta$ is invertible and for which each $q \in Q$ can be written in the form $q = d^{-1}a$ for some $a \in R$ and $d \in \Delta$; moreover, this ring Q is unique to within isomorphism over R. For non-commutative rings, however, such rings of fractions need not exist. Sometimes they do, but in any case there are various alternatives. In the next few Sections we shall look at some of these. In this one we shall look at what is the largest of the viable alternatives, one that exists for every ring.

We begin with a not necessarily commutative ring R. Let $E = E({}_RR)$ be the injective envelope of the regular left module; we will adopt the point of view that ${}_RR$ is a submodule of ${}_RE$. Next, let

$$S = \mathrm{End}({}_RE)$$

be the endomorphism ring of ${}_RE$. So ${}_RE_S$ is a bimodule. Now for each $x \in E$ right multiplication $\rho_x : a \longmapsto ax$ is an R-homomorphism from R to E. Since E is an injective extension of R, there is an extension $\overline{\rho}_x : E \longrightarrow E$ of ρ_x over R. Then since $(1)\rho_x = x$, this shows that each element $x \in E$ has the form $(1)s$ for at least one $s \in S$. Thus,

12.1. Lemma. *The map $s \longmapsto (1)s$ is an S-epimorphism $S_S \longrightarrow E_S$.* ∎

Now the ring that we are really looking for is

$$Q_{max} = \mathrm{End}(E_S) = \mathrm{BiEnd}({}_RE),$$

the biendomorphism ring of ${}_RE$. We call the ring Q_{max} the **maximal ring of left quotients of** R. Later we shall have to contend with several other "rings of left quotients", but here the maximal ring Q_{max} will be the only one we consider. So, at least in the remainder of this Section, we shall defer to notational efficiency and abbreviate

$$Q = Q_{max}\,.$$

Since ${}_RE$ is faithful, the map $\lambda : R \longrightarrow Q$ defined by $\lambda_a : x \longmapsto ax$ for all $x \in E$ and $a \in R$ is an injective ring homomorphism. We usually identify R with its image in Q, and so treat R as a subring of Q. All of this identification could be dangerous except fortunately, we have

12.2. Lemma. *The map $\eta : {}_RQ \longrightarrow {}_RE$ defined by $\eta : q \longmapsto q(1) \in E$ is an R-monomorphism.*

Proof. The identification of R with a subring of Q is via the map λ, so for each $r \in R$ and $q \in Q$, as endomorphisms of E_S, $rq = \lambda_r \circ q$. Thus,

$$\eta(rq) = (\lambda_r \circ q)(1) = \lambda_r(q(1)) = r(q(1)) = r\eta(q),$$

and η is an R-homomorphism. So suppose $\eta(q) = q(1) = 0$. Then for each $s \in S$

$$(q(1))s = q((1)s) = 0$$

Thus, by Lemma 12.1, $q(E) = 0$ and $q = 0$. ■

Thus, we shall often think of Q as being a Q-submodule of E. Of course, if at any time we want to be careful to distinguish between $q \in Q$ as an element of E and as a bi-endomorphism of E, we shall treat the former as $\eta(q) = q(1)$. With that in mind we want to characterize Q as a submodule of E.

12.3. Proposition. *For an element $x \in E$, we have $x \in Q$ iff $\mathbf{r}_S(R) \subseteq \mathbf{r}_S(x)$. In particular,*

$$Q = \mathbf{l}_E\mathbf{r}_S(R).$$

Proof. ($\Longrightarrow$) Consider $\eta(q) = q(1)$. Then $Rs = 0$ gives that

$$qs = \eta(q)s = (q(1))s = q((1)s) = q(0) = 0.$$

($\Longleftarrow$) Let $x \in E$ and suppose $\mathbf{r}_S(R) \subseteq \mathbf{r}_S(x)$. Since $E = (1)S$ (see Lemma 12.1), this means there is $q \in Q$ with $q : (1)s \longmapsto xs$. Thus $xs = q(1s) = q(1)s$, so if $s = 1_E$, then $x = q(1)$.

For the final statement observe that we have proved that $\mathbf{r}_S(R) \subseteq \mathbf{r}_S(Q)$. On the other hand since $R \subseteq Q$, we have $\mathbf{r}_S(Q) \subseteq \mathbf{r}_S(R)$. ■

Now we come to one of the key notions in the study of Q. A left ideal $D \leq R$ is said to be **dense** in case

$$\mathbf{r}_S(D) = \mathbf{r}_S(R).$$

Note that if D is dense and $D \leq I \leq R$ are left ideals, then (see Lemma 11.1)

$$\mathbf{r}_S(R) \subseteq \mathbf{r}_S(I) \subseteq \mathbf{r}_S(D)$$

and since D is dense, $\mathbf{r}_S(D) = \mathbf{r}_S(R)$, so $\mathbf{r}_S(I) = \mathbf{r}_S(R)$ and I is dense.

12.4. Lemma. *Let D be a dense left ideal of R. Then*

(1) *$(D : q) = \{r \in R \ : \ rq \in D\}$ is dense for every $q \in Q$;*

(2) *If $x \in E$, then $Dx = 0 \Longrightarrow x = 0$.*

Proof. For (1) suppose that $(D : q)s = 0$ for some $s \in S$. Then for each $d \in D$ and $r \in R$ if $d + rq = 0$, then $rq \in D$, so that $rs = 0$. So there is an R-homomorphism from the submodule $D + Rq$ of E to E given by $d + rq \longmapsto rs$. Then since ${}_RE$ is injective, there is some $s' \in S$ such that $(d + rq)s' = rs$ for all $d + rq \in D + Rq$. But D is dense and $Ds' = 0$, so $Rs' = 0$. Therefore,

$$1s = qs' = (q(1))s' = q(1s') = q(0) = 0,$$

and so $Rs = (R1)s = R((1)s) = 0$. Now apply Proposition 12.3.

For (2), suppose that $Dx = 0$. Now by Lemma 12.1, there is an $s \in S$ with $x = (1)s$. So $Ds = D((1)s)) = Dx = 0$, and thus, $0 = Rs = (R1)s = R((1)s)) = Rx$, so that $x = 0$. ■

It follows at once from part (1) of Lemma 12.4 that if $q \in Q$, then $(R : q)$ is dense. Thus, right multiplication by the element q in Q is an R-homomorphism from the dense left ideal $(R : q)$ into R. Moreover, by part (2) of that Lemma, q is uniquely determined by its behavior on any dense left ideal. In fact, we have

12.5. Corollary. *Let D be a dense left ideal of R and let $f \in \mathrm{Hom}_R(D, R)$. Then there is a unique $q \in Q$ with $q(d) = f(d)$ for all $d \in D$.*

Proof. Since E is an injective extension of both D and R, the homomorphism extends to an endomorphism of E. That is, there is some $s \in S$ with $f(d) = (d)s$ for all $d \in D$. We claim that $(1)s \in Q$, and to show that we use Lemma 12.3 and show that $\mathbf{r}_S(R) \subseteq \mathbf{r}_S((1)s)$. So suppose that $s' \in S$ with $Rs' = 0$; but then $D(1s)s' = ((D)s)s' = (f(D))s' \subseteq Rs' = 0$. Thus, $Dss' = 0$; so since D is dense, $Rss' = 0$. But then by Lemma 12.3, we have $Q((1)s)s' = Qss' = 0$. Thus, $\mathbf{r}_S(R) \subseteq \mathbf{r}_S(Qs)$, so by Lemma 12.3, again, $Q((1))s \subseteq Q$. That means $(1)s = q \in Q$. Then $q(d) = d(1)s = (d)s = f(d)$ for all $d \in D$. Uniqueness of q follows immediately from Lemma 12.4. ■

Thus, thanks to the last two results alone, the dense left ideals generate some real interest. As we shall see in the next Section they form one of a class of very special left ideals. So next, we seek an "internal" characterization of the dense ideals. First, though, let's give them a label. So we let

$$\mathcal{D} = \{D \leq_R R \ : \ D \text{ is dense in } R\}$$

be the set of all dense left ideals of R.

12.6. Lemma. *For a left ideal I of R, the following statements are equivalent:*

(a) *$I \in \mathcal{D}$;*

(b) *There is a $D \in \mathcal{D}$ such that $\mathbf{r}_R(I : d) = 0$ for all $d \in D$;*

(c) *For every $d \in R$, $\mathbf{r}_R(I : d) = 0$.*

Proof. (a) $\Longrightarrow$ (c) Let $d \in R$ and $a \in \mathbf{r}_R(I : d)$. By Lemma 12.1 there is an $s \in S$ with $(1)s = a$, so $s \in \mathbf{r}_S(I : d)$. Since $I \in \mathcal{D}$, $(I : d) \in \mathcal{D}$ for all $d \in R$ by Lemma 12.4, and we have $Rs = 0$. But then $Ra = R(1)s = Rs = 0$, so $a = 0$.

(c) $\Longrightarrow$ (b) Let $D = R$.

(b) $\Longrightarrow$ (a) Suppose that there is some $s \in S$ with $Is = 0$ and $Rs \neq 0$. Since D is dense, $Ds \neq 0$.

But $R \trianglelefteq E$, so there is some $0 \neq r \in Ds \cap R$. Say $r = ds$ with $d \in D$. Then

$$(I : d)r = (I : d)ds \subseteq Is = 0,$$

contradicting (b). So $Rs = 0$ and $I \in \mathcal{D}$. ∎

Here is a useful little Corollary of this characterization.

12.7. Corollary. *Every left ideal $D \leq R$ that is dense in R is essential in R.*

Proof. Suppose that D is a left ideal and $x \neq 0$ in R with $D \cap Rx = 0$. Then $(D : x) = \mathbf{l}_R(x)$, so that $\mathbf{r}_R(D : x) = \mathbf{r}_R\mathbf{l}_R(x)$. But $x \in \mathbf{r}_R\mathbf{l}_R(x)$, so by Lemma 12.6, D is not dense. ∎

We conclude with a lemma that will be of significance in the next Section.

12.8. Lemma. *Let $I \leq R$ be a left ideal and let $D \in \mathcal{D}$ be dense. If $(I : d) \in \mathcal{D}$ for each $d \in D$, then $I \in \mathcal{D}$.*

Proof. Let $a \in R$; we claim that $\mathbf{r}_R(I : a) = 0$, so that by Lemma 12.6 we would have that I is dense. Suppose that $x \in R$ is not zero. Since D is dense, we have by Lemma 12.6, $(D : a)x \neq 0$, say that $ra \in D$ and $rx \neq 0$. But $ra \in D$, so $(I : ra) \in \mathcal{D}$, and $(I : ra)rx \neq 0$ by Lemma 12.4. However, $(I : ra)r \subseteq (I : a)$, so $(I : a)x \neq 0$. Since this holds for all $a \in R$, we have $I \in \mathcal{D}$. ∎

Exercises 12.

12.1. The goal of this exercise is to show that if Q is the maximal ring of left quotients of R, then Q is also the maximal ring of left quotients of Q. So let $E = E({}_RR)$ and $S = \mathrm{End}({}_RE)$.

(a) Show that ${}_QE$ is an injective envelope of its submodule ${}_QQ$. [Hint: Let $I \leq {}_QQ$ be a left ideal and let $\varphi : I \longrightarrow E$ be a Q-homomorphism. So there is an R-endomorphism $\overline{\varphi} : E \longrightarrow E$ extending φ. For each $q \in Q$ define the R-endomorphism $s_q : E \longrightarrow E$ by $(x)s_q = (xq)\overline{\varphi} - x(q)\overline{\varphi}$. Note that $(R)s_q = 0$, so by 12.3 $(Q)s_q = 0$ and $\overline{\varphi}$ is a Q-endomorphism, and ${}_QE$ is injective. Then easily $Q \trianglelefteq {}_QE$.]

(b) Conclude that Q is it own maximal ring of left quotients. [Hint: The main thing is to show that $S = \mathrm{End}({}_QE)$.]

12.2. Prove that the poset $\mathcal{D}$ of dense left ideals of R is closed under intersection. Thus, deduce that $\mathcal{D}$ is a lattice. [Hint: Let $D_1, D_2 \in \mathcal{D}$. Show that $(D_1 \cap D_2 : d) \in \mathcal{D}$ for all $d \in D_2$. Use Lemma 12.8.]

12.3. Suppose that the lattice $\mathcal{D}$ of dense left ideals of R has a least element, D_0. Prove that the maximal ring Q of left quotients of R is isomorphic to $\mathrm{End}({}_RD_0)$. [Hint: For each $q \in Q$, the dense left ideal $(D_0 : q)$ contains D_0.]

12.4. Prove that if R is commutative, then its maximal ring Q of leftt quotients is isomorphic to the center of $S = \mathrm{End}({}_R E)$. In particular, Q is commutative.

12.5. Let K be a field and let R be the 3-dimenional commutative K algebra $R = K1 + Kx + Ky$ with $x^2 = xy = y^2 = 0$.

(a) Show that if $S = \mathrm{End}({}_R E)$, then S is not commutative. Thus, the maximal ring Q of left quotients is not self-injective.

(b) Since R is Artinian it has a least dense left ideal D_0. Determine D_0 and find the maximal ring Q of left quotients of R. (See Exercises 12.2 and 12.3.)

13 Topologies, and Torsion Theory.

As we mentioned in Section 12, unlike the case for commutative rings, a non-commutative ring R need not have a ring of fractions in which all non-zero divisors are invertible. But we claimed that there are alternatives, and we saw that one of these is the maximal ring of left quotients. Here we shall see that there are other viable choices, all subrings of the maximal ring. It also turns out that these various rings of quotients are characterized by topologies on R and by "torsion" functors on R-modules. Here we will give a brief introduction to topologies and torsion theories for R and give some indication of how they are connected. (See Stenström [11] for a thorough treatment of these topics.) Then in the next Section we shall show how they characterize rings of quotients.

Consider a set $\mathcal{T}$ of left ideals of R, and consider the two properties

(T.1) For all $D \in \mathcal{T}$ and all $a \in R$, $(D : a) \in \mathcal{T}$;

(T.2) Let I be a left ideal of R and $D \in \mathcal{T}$. If $(I : d) \in \mathcal{T}$ for all $d \in D$, then $I \in \mathcal{T}$.

A non-empty set $\mathcal{T}$ of left ideals of R is said to be a a **topology** (or a **Gabriel topology** or an **additive topology**) for R in case it satisfies both conditions (T.1) and (T.2).

As we shall see in the Exercises, such an additive topology $\mathcal{T}$ does indeed produce a topology on R relative to which it is a topological ring with $\mathcal{T}$ a system of neighborhoods of zero. Before we continue with our general discussion of topologies, we observe that the set $\mathcal{D}$ of dense left ideals of R forms such a topology.

13.1. Theorem. *The set $\mathcal{D}$ of all dense left ideals of a ring R is a topology for R.*

Proof. Lemmas 12.4 and 12.8. ■

The topology $\mathcal{D}$ is sometimes called the **Lambek topology**. Each topology for R is what is sometimes called a **filter** of left ideals for R. That is,

13.2. Proposition. *If $\mathcal{T}$ is a topology for R, then $R \in \mathcal{T}$, and*

(T.3) *If $D \leq I \leq R$ and $D \in \mathcal{T}$, then $I \in \mathcal{T}$;*

(T.4) *If $D_1, D_2 \in \mathcal{T}$, then $D_1 \cap D_2 \in \mathcal{T}$.*

Proof. Since $\mathcal{T}$ is not empty, there is some $D \in \mathcal{T}$ and $0 \in D$. So by (T.1), $R = (D : 0) \in \mathcal{T}$. For (T.3) suppose that $I \leq D \leq R$ and $D \in \mathcal{T}$. Then $(I : d) = R \in \mathcal{T}$ for all $d \in D$, so by (T.2), $I \in \mathcal{T}$. Finally, for (T.4), we have $(D_1 \cap D_2 : d) = (D_1 : d) \in \mathcal{T}$ for all $d \in D_2$, so by (T.2) $D_1 \cap D_2 \in \mathcal{T}$. ■

A non-empty collection $\mathcal{T}$ of left ideals of R is a **pre-topology** for R in case it satisfies (T.1), (T.3), and (T.4).

13.3. Theorem. *The collection $\mathcal{E}$ of all essential left ideals of R is a pre-topology for R.*

Proof. Since $\mathcal{E}$ clearly satisfies (T.3) and (T.4), it will suffice to prove that it satisfies (T.1). So suppose that $D \trianglelefteq R$, that $a \in R$, and that $Rx \cap (D : a) = 0$. If $x \neq 0$, then clearly $Rxa \neq 0$, so $Rxa \cap D \neq 0$ and there is a non-zero $rxa \in D$. But then $0 \neq rx \in (D : a)$, a contradiction. ■

Let $\mathcal{T}$ be a pre-topology for R. Then for each left ${}_RM$ module, define

$$t_{\mathcal{T}}(M) = \{x \in M \ : \ (0 : x) \in \mathcal{T}\}.$$

Observe that if ${}_RM_S$ is a bimodule, and if $x, y \in t_{\mathcal{T}}(M)$, then $(0 : x), (0 : y) \in \mathcal{T}$, and by Lemma 11.2, $(0 : x + y) \supseteq (0 : x) \cap (0 : y)$, so by (T.3) and (T.4), $(0 : x + y) \in \mathcal{T}$, so $x + y \in t_{\mathcal{T}}(M)$. Also, if $x \in M$, $a \in R$, and $s \in S$, then $(0 : axs) = ((0 : xs) : a) \in \mathcal{T}$ by (T.1), so $axs \in t_{\mathcal{T}}(M)$. That is,

13.4. Lemma. *If $\mathcal{T}$ is a pre-topology for R, then for each bimodule ${}_RM_S$, $t_{\mathcal{T}}(M)$ is an (R, S)-submodule.* ■

The submodule $t_{\mathcal{T}}(M)$ is called the $\mathcal{T}$-**pre-torsion submodule** of M. If $\mathcal{T}$ is a topology, then $t_{\mathcal{T}}(M)$ is the $\mathcal{T}$-**torsion submodule** of M.

13.5. Corollary. *If $\mathcal{T}$ is a pre-topology for R, then the $\mathcal{T}$-torsion submodule $t_{\mathcal{T}}(R)$ of ${}_RR$ is an ideal.* ■

Not surprisingly, the ideal $t_{\mathcal{T}}(R)$ of R is called the $\mathcal{T}$-**torsion ideal** of R.

13.6. Lemma. *Let $\mathcal{T}$ be a pre-topology of left ideals for R. Then the class*

$$\mathbf{T}_{\mathcal{T}} = \{M \in R\mathbf{Mod} \ : \ t_{\mathcal{T}}(M) = M\}$$

is closed under submodules, factor modules, and coproducts.

Proof. That the class $\mathbf{T}_{\mathcal{T}}$ is closed under submodules and factor modules is almost trivial. So suppose that $M_1, \ldots, M_n$ are in $\mathbf{T}_{\mathcal{T}}$, and let $x = (x_1, \ldots, x_n)$ be in their coproduct. Then by (T.4), the intersection $I = (0 : x_1) \cap \cdots \cap (0 : x_n)$ is in $\mathcal{T}$, and $I \leq (0 : x)$, so by (T.3), $(0 : x) \in \mathcal{T}$. ■

If $\mathcal{T}$ is a topology, then the modules in the class $\mathbf{T}_{\mathcal{T}}$ are known as the $\mathcal{T}$-**torsion modules**, and $\mathbf{T}_{\mathcal{T}}$ as the $\mathcal{T}$-**torsion class**.

13.7. Proposition. *Let $\mathcal{T}$ be a topology for R. Then the $\mathcal{T}$-torsion class $\mathbf{T}_{\mathcal{T}}$ is closed under coproducts, and satisfies for every short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$, in $R\mathbf{Mod}$, $M \in \mathbf{T}_{\mathcal{T}}$ iff $K, N \in \mathbf{T}_{\mathcal{T}}$.*

Proof. By Lemma 13.6 $\mathbf{T}_{\mathcal{T}}$ is closed under coproducts, and if $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$, with

M in $\mathbf{T}_{\mathcal{T}}$, then $K, N \in \mathbf{T}_{\mathcal{T}}$. Conversely, suppose that $K, N \in \mathbf{T}_{\mathcal{T}}$. Let $x \in M$. Then since $N \in \mathbf{T}_{\mathcal{T}}$, $(K : x) \in \mathcal{T}$. But for each $a \in (K : x)$, since $K \in \mathbf{T}_{\mathcal{T}}$, we have $((0 : x) : a) = (0 : ax) \in \mathcal{T}$. Thus, by (T.2), $(0 : x) \in \mathcal{T}$, and it follows that $M \in \mathbf{T}_{\mathcal{T}}$. ■

13.8. Corollary. *If $\mathcal{T}$ is a topology for R, then for every submodule $N \leq M \in R\mathbf{Mod}$*

$$t_{\mathcal{T}}(N) = N \cap t_{\mathcal{T}}(M).$$

Proof. Clearly, $t_{\mathcal{T}}(N) \subseteq N \cap t_{\mathcal{T}}(M)$. But, also, $N \cap t_{\mathcal{T}}(M)$ as a submodule of $t_{\mathcal{T}}(M)$ is torsion and a submodule of N, so $t_{\mathcal{T}}(N) \supseteq N \cap t_{\mathcal{T}}(M)$. ■

In fact, the converse of Proposition 13.7 is true, and so the conditions (1) and (2) of that Proposition characterize those classes that are torsion classes of some topology. (See Stenström [11], Theorem 5.1, p. 146.)

Dually, given a topology $\mathcal{T}$ for R, a module ${}_RM$ is $\mathcal{T}$-**torsion free** in case $t_{\mathcal{T}}(M) = 0$. The class

$$\mathbf{F}_{\mathcal{T}} = \{M \in R\mathbf{Mod} \ : \ t_{\mathcal{T}}(M) = 0\}$$

of all $\mathcal{T}$-torsion free modules is a **torsion free class**. It is an easy little exercise to see that the classes of torsion and torsion free modules for a topology $\mathcal{T}$ characterize each other. That is,

13.9. Proposition. *Let $\mathcal{T}$ be a topology for R. Then*

(1) $\mathbf{F}_{\mathcal{T}} = \{M \in R\mathbf{Mod} \ : \ \mathrm{Hom}_R(N, M) = 0 \ \forall \ N \in \mathbf{T}_{\mathcal{T}}\}$;

(2) $\mathbf{T}_{\mathcal{T}} = \{M \in R\mathbf{Mod} \ : \ \mathrm{Hom}_R(M, N) = 0 \ \forall \ N \in \mathbf{F}_{\mathcal{T}}\}$.

Proof. . We'll leave this as an exercise. ■

Thus, dual to Proposition 13.7 we have

13.10. Proposition. *Let $\mathcal{T}$ be a topology for R. Then the $\mathcal{T}$-torsion free class $\mathbf{F}_{\mathcal{T}}$ is closed under submodules and products and satisfies for every short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$, in $R\mathbf{Mod}$, if $K, N \in \mathbf{F}_{\mathcal{T}}$, then $M \in \mathbf{F}_{\mathcal{T}}$.* ■

13.11. Corollary. *If $\mathcal{T}$ is a topology for R, then for every $\mathcal{T}$-torsion free module M, its injective envelope $E(M)$ is also $\mathcal{T}$-torsion free.*

Proof. Since $M \trianglelefteq E(M)$ and $0 = t_{\mathcal{T}}(M) = M \cap t_{\mathcal{T}}(E(M))$ by Corollary 13.8, we must have $t_{\mathcal{T}}(E(M)) = 0$. ■

As one would hope, a module modulo its torsion submodule is torsion free. That is,

13.12. Lemma. *If $\mathcal{T}$ is a topology for R, then for every module ${}_RM$, we have $t_{\mathcal{T}}(M/t_{\mathcal{T}}(M)) = 0$.*

Proof. Let $x \in M$ with $(t_{\mathcal{T}}(M) : x) \in \mathcal{T}$. Then for each $r \in (t_{\mathcal{T}}(M) : x)$, we have $rx \in t_{\mathcal{T}}(M)$, so $((0 : x) : r) = (0 : rx) \in \mathcal{T}$, and thus, by (T.4), $x \in t_{\mathcal{T}}(M)$. ■

Before we leave our discussion of torsion, we want to record a couple of useful characterizations of $\mathcal{T}$-torsion and $\mathcal{T}$-torsion free modules when $\mathcal{T}$ is a topology for R. So let $\mathcal{T}$ be a topology for R. Consider the set

$$\mathbf{C} = \{I \leq {}_RR \ : \ R/I \text{ is } \mathcal{T}\text{-torsion free}\}.$$

So the set $\{R/I \ : \ I \in \mathbf{C}\}$ contains a copy of each cyclic $\mathcal{T}$-torsion free module. Now let

$$E_{\mathcal{T}} = \prod_{I \in \mathbf{C}} E(R/I).$$

Then $E_{\mathcal{T}}$ is injective.

13.13. Lemma. *Let $\mathcal{T}$ be a topology for R. For an R-module ${}_RM$ the following are equivalent:*

(a) *M is $\mathcal{T}$-torsion;*

(b) *M is the sum of its cyclic $\mathcal{T}$-torsion submodules;*

(c) $\mathrm{Hom}_R(M, E_{\mathcal{T}}) = 0$.

Proof. The equivalence of (a) and (b) is simply from the definition. For the equivalence of (a) and (c) observe that $E_{\mathcal{T}}$ is injective, so $\mathrm{Hom}(M, E_{\mathcal{T}}) \neq 0$ iff there is an $x \in M$ and a non-zero R-homomorphism $\varphi : Rx \longrightarrow R/I$ for some $I \in \mathbf{C}$ iff the cyclic module $R\varphi(x)$ is $\mathcal{T}$-torsion free iff the cyclic submodule $Rx \leq M$ is not $\mathcal{T}$-torsion. ■

13.14. Corollary. *Let $\mathcal{T}$ be a topology for R. A finitely generated module M is $\mathcal{T}$-torsion iff $DM = 0$ for some $D \in \mathcal{T}$. In particular, a cyclic module R/I is $\mathcal{T}$-torsion iff $I \in \mathcal{T}$.*

Proof. Clearly, if $DM = 0$ for some $D \in \mathcal{T}$, then trivially M is torsion. On the other hand, if M is finitely generated and $\mathcal{T}$-torsion, then there is some finite set $D_1, \ldots, D_n \in \mathcal{T}$ and homomorphisms $\varphi_i : R/D_i \longrightarrow M$ with $\sum_{i=1}^{n} \mathrm{Im}\, \varphi_i = M$. But then with $D = D_1 \cap \cdots \cap D_n \in \mathcal{T}$, we have $DM = 0$. ■

We now have a dual set of characerizations of $\mathcal{T}$-torsion free modules.

13.15. Lemma. *Let $\mathcal{T}$ be a topology for R. For a module ${}_RM$ the following are equivalent*

(a) *M is $\mathcal{T}$-torsion free;*

(b) $\mathrm{Hom}_R(R/D, M) = 0$ *for all $D \in \mathcal{T}$;*

(c) *$E_{\mathcal{T}}$ cogenerates M.*

Proof. The module M is $\mathcal{T}$-torsion free iff it has no non-zero torsion elements, that is no non-zero elements x with $\mathbf{l}_R(x) \in \mathcal{T}$. This gives (a) $\iff$ (b). Since each R/I for $I \in \mathbf{C}$ is $\mathcal{T}$-torsion free, each $E(R/I)$ is $\mathcal{T}$-torsion free, so the product $E_{\mathcal{T}}$ and every product of copies of $E_{\mathcal{T}}$ is $\mathcal{T}$-torsion free. Since (c) implies that M is isomorphic to a submodule of a product of copies of $E_{\mathcal{T}}$, M is also $\mathcal{T}$-torsion free, so (c) $\Longrightarrow$ (a). Finally, for (a) $\Longrightarrow$ (c), suppose that M is $\mathcal{T}$-torsion free. Then for each non-zero $x \in M$, the cyclic module Rx is $\mathcal{T}$-torsion free and so there is an R-monomorphism $\varphi_x : Rx \longrightarrow E_{\mathcal{T}}$. Since $E_{\mathcal{T}}$ injective, w.m.a. that $\varphi_x : R \longrightarrow E_{\mathcal{T}}$ with $Rx \cap \mathrm{Ker}\,\varphi_x = 0$. Thus, the product map

$$\prod_{x \in M} \varphi_x : M \longrightarrow E_{\mathcal{T}}^M$$

is a monomorphism, and M is $\mathcal{T}$-torsion free. ■

In the literature the torsion theory induced by a topology on R is usually referred to as an **<u>hereditary</u>** torsion theory. Such an hereditary torsion theory is one that satisfies the dual conditions of Corollaries 13.8 and 13.11. There are torsion theories that do not come from topologies and are not hereditary. Our primary interest is in torsion theories that arise from toplogies on R and that have the additional property that ${}_RR$ is torsion free. So let's return to topologies.

It is easy to see that the set of all topologies on R forms a poset that is closed under arbitrary intersections. In particular, this poset of topologies is a complete lattice with the largest topology the topology of all left ideals of R. But this also means that if $\mathcal{S}$ is any collection of left ideals of R, then the intersection of all topologies containing $\mathcal{S}$ is a topology. Now it might be fair to wonder where all of these topologies come from. It turns out that every injective module determines such a topology. We next see how that works.

Let ${}_RE$ be an injective module. Then we set

$$\mathcal{T}_E = \{D \leq {}_RR \ : \mathrm{Hom}_R(R/D, E) = 0\}.$$

So since E is injective, $\mathcal{T}_E$ consists of all cyclic modules R/D for which E cogenerates no non-zero submodule. We now see that every topology for R arises as a $\mathcal{T}_E$ for some injective module E.

13.16. Theorem. *If E is an injective module, then $\mathcal{T}_E$ is a topology for R. Moreover, for each topology $\mathcal{T}$ for R we have $\mathcal{T} = \mathcal{T}_{(E_{\mathcal{T}})}$.*

Proof. Let ${}_RE$ be injective. We show that $\mathcal{T}_E$ satisfies (T.1) and (T.2). Let $D \in \mathcal{T}_E$ and $a \in R$. Then there is an R-monomorphism $\rho : R/(D : a) \longrightarrow R/D$ with $\rho : x + (D : a) \longmapsto xa + D$. So there is an $\mathbb{Z}$-epimorphism $\mathrm{Hom}_R(R/D, E) \longrightarrow \mathrm{Hom}_R(R/(D : a), E)$; but $D \in \mathcal{T}_E$, so $\mathrm{Hom}_R(R/D, E) = 0$. Thus, $\mathrm{Hom}_R(R/(D : a), E) = 0$ and $(D : a) \in \mathcal{T}_E$. That is, $\mathcal{T}_E$ satisfies (T.1).

Suppose next that $I \leq {}_RR$, $D \in \mathcal{T}_E$, and $(I : d) \in \mathcal{T}_E$ for all $d \in D$. Now for each $d \in D$ there is an R-monomorphism $\rho_d : R/(I : d) \longrightarrow R/I$ with $\rho_d : x + (I : d) \longmapsto xd + I$. Since $\mathrm{Hom}(R/(I : d), E) = 0$ for every $d \in D$, for every $f : R/I \longrightarrow E$, we have $f(I + D) = 0$. Thus, every such f factors through $R/(I + D)$, and since $I + D \in \mathcal{T}$, we conclude that $\mathrm{Hom}(R/I, E) = 0$ and $I \in \mathcal{T}_E$. Therefore, $\mathcal{T}_E$ satisfies (T.2) and is a topology for R.

Finally, let $\mathcal{T}$ be a topology, and let $D \leq {}_RR$ be a left ideal. Then $D \in \mathcal{T}$ iff R/D is $\mathcal{T}$-torsion iff (by Lemma 13.13) $\mathrm{Hom}(R/D, E_\mathcal{T}) = 0$ iff $D \in \mathcal{T}_{E_\mathcal{T}}$. ■

If E is an injective module, then the topology $\mathcal{T}_E$ for R is the topology **cogenerated** by E. So an interesting question is to learn what we can about the topology cogenerated by the injective envelope $E(M)$ of some module M. We begin with the following simple little fact.

13.17. Lemma. *Let $E(M)$ be the injective envelope of a module ${}_RM$. If I and K are left ideals of R with $I \leq K$, then $\mathrm{Hom}_R(K/I, E(M)) = 0$ iff $\mathbf{r}_M(I : k) = 0$ for all $k \in K$.*

Proof. Since $E(M)$ is injective, $\mathrm{Hom}(K/I, E(M)) = 0$ iff $\mathrm{Hom}(Rk + I/I, M) = 0$ for all $k \in K$ iff $\mathrm{Hom}(R/(I : k), M) = 0$ for all $k \in K$ iff $\mathbf{r}_M(I : k) = 0$ for all $k \in K$. ■

Now the key fact about $\mathcal{T}_{E(M)}$ is

13.18. Lemma. *Let $E(M)$ be the injective envelope of a module ${}_RM$. For a left ideal $I \leq R$ the following are equivalent:*

- **(a)** *$I \in \mathcal{T}_{E(M)}$;*
- **(b)** *$\mathbf{r}_M(I : a) = 0$ for all $a \in R$;*
- **(c)** *There is a $D \in \mathcal{T}_{E(M)}$ with $\mathbf{r}_M(I : d) = 0$ for all $d \in D$.*

Proof. (a) $\iff$ (b) and (a) $\iff$ (c) follow immediately from Lemma 13.17. ■

13.19. Theorem. *Let $E(M)$ be the injective envelope of a module ${}_RM$. Then the topology $\mathcal{T}_{E(M)}$ is the (unique) largest topology relative to which M is torsion free.*

Proof. Trivially $E(M)$ cogenerates M, and so by Lemma 13.15, M is $\mathcal{T}_{E(M)}$ torsion free. Suppose that $\mathcal{T}$ is a topology relative to which M is torsion free. Let $I \in \mathcal{T}$. Then by (T.1), $(I : a) \in \mathcal{T}$ for all $a \in R$. But $t_\mathcal{T}(M) = 0$. So $\mathbf{r}_M(I : a) = 0$ for all $a \in R$, and $I \in \mathcal{T}_{E(M)}$ by Lemma 13.18. ■

13.20. Corollary. *For a ring R the Lambek topology $\mathcal{D}$ of all dense left ideals of R is cogenerated by the injective envelope $E(R)$ of R, and so is the (unique) largest topology relative to which ${}_RR$ is torsion free.*

Proof. It follows immediately from Lemma 13.18 and Lemma 12.6 that $\mathcal{D}$ is the topology $\mathcal{T}_{E(R)}$. So the final assertion follows from Theorem 13.19. ■

Exercises 13.

13.1. Let $\mathcal{C}$ a non-empty class of left R-modules. Set

$$\begin{aligned}\mathbf{F} &= \{{}_RN \ : \ \mathrm{Hom}_R(C,N) = 0 \text{ for all } C \in \mathcal{C}\},\\ \mathbf{T} &= \{{}_RM \ : \ \mathrm{Hom}_R(M,N) = 0 \text{ for all } N \in \mathbf{F}\}.\end{aligned}$$

The pair $(\mathbf{T},\mathbf{F})$ is the **torsion theory generated** by $\mathcal{C}$.

(a) Prove that $\mathbf{T}$ is closed under factor modules, coproducts, and extensions (*i.e.,* for every short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0,$$

if $K, N \in \mathbf{T}$ then $M \in \mathbf{T}$).

(b) Prove that $\mathbf{F}$ is closed under submodules, products, and extensions.

13.2. With the notation of Exercise 13.1, suppose that $\mathcal{C}$ is closed under submodules and factor modules. Prove that there is a topology $\mathcal{T}$ for R such that $\mathbf{T} = \mathbf{T}_{\mathcal{T}}$ and $\mathbf{F} = \mathbf{F}_{\mathcal{T}}$. [Hint: Show that $\mathbf{F}$ is closed under injective envelopes, and then consider the injective envelopes of cylcic modules in $\mathbf{F}$.]

13.3. Let M be a non-zero cyclic $\mathbb{Z}$-module, let $E = E(M)$ be its injective envelope, and let $\mathcal{T}$ be the topology on $\mathbb{Z}$ cogenerated by E. Prove that the regular module $\mathbb{Z}$ is $\mathcal{T}$-torsion free.

13.4. Let R be a P.I.D., and let $\mathbf{P}$ be the set of all primes of R. For each of the following injective R-modules E, describe the ideals in the topology $\mathcal{T} = \mathcal{T}_E$, and the class $\mathbf{T}_{\mathcal{T}}$ of $\mathcal{T}$-torsion modules.

(a) For some $p \in \mathbf{P}$ let $E = E(R/(p))$;

(b) For some $p \in \mathbf{P}$, let $E = \prod_{q\in\mathbf{P}}\{E(R/q) \ : q \neq p\}$.

13.5. Let $\mathcal{T}$ be a topology for R. Prove that the set $\mathcal{U} = \{x + D \ : \ x \in R \text{ and } D \in \mathcal{T}\}$ is a system of neighborhoods for a topology on R relative to which the operations of addition and multiplication are continuous maps $R \times R \longrightarrow R$ and the operation of negation is a continuous map $R \longrightarrow R$.

13.6. Let $\mathcal{T}$ be a topology for R. A left R-module V is $\mathcal{T}$-**injective** if $\mathrm{Hom}_R(\_, V)$ is exact on all short exact sequences $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ with N $\mathcal{T}$-torsion.

(a) Prove the $\mathcal{T}$-Injective Test Lemma: An R-module V is $\mathcal{T}$-injective iff for every $D \in \mathcal{T}$ and every R-homomorphism $f : D \longrightarrow V$ there is an extension $\overline{f} : R \longrightarrow V$.

(b) Let V be a left R-module and let $E(V)$ be its injective envelope. Prove that V is $\mathcal{T}$-injective iff $V = \{x \in E(V) \ : \ (V : x) \in \mathcal{T}\}$.

13.7. Let $\mathcal{T}$ be a topology for R and let M be a left R-module. An essential monomorphism $\varphi : M \longrightarrow V$ is a $\mathcal{T}$-**injective** **envelope** for M in case V is $\mathcal{T}$-injective.

(a) Prove that if $E(M)$ is the injective envelope of M, then there is a $\mathcal{T}$-injective envelope $M \longrightarrow V$ where $V = \{x \in E(M) \ : \ (M : x) \in \mathcal{T}\}$.

(b) Prove that if $\varphi : M \longrightarrow V$ and $\varphi' : M \longrightarrow V$; are $\mathcal{T}$-injective envelopes of M, then there is an isomorphism $\psi : V \longrightarrow V'$ with $\psi \circ \varphi = \varphi'$.

13.8. Let $\mathcal{T}$ be a topology for R. A left R-module V is $\mathcal{T}$-**closed** if for every short exact sequence $0 \longrightarrow K \stackrel{f}{\longrightarrow} M \longrightarrow N \longrightarrow 0$ with N $\mathcal{T}$-torsion, $\mathrm{Hom}_R(f, V)$ is an isomorphism.

(a) Prove that V is $\mathcal{T}$-closed iff V is $\mathcal{T}$-torsion free and $\mathcal{T}$-injective.

(b) Let M be a $\mathcal{T}$-torsion free left R-module, and let $M \longrightarrow V$ be a $\mathcal{T}$-injective envelope. Prove that V is a $\mathcal{T}$-closed R-module.

14 Rings of Left Quotients.

Although we did not provide all of the proofs, in the last Section we sketched the fact that every topology for R determines an hereditary torsion for R**Mod** and conversely. Moreover, we saw that every topology, or equivalently every hereditary torsion, determines an injective module which, in turn, characterizes the topology. It also turns out that every topology for R can be characterized by a ring of left quotients for R. We shall restrict our attention to those topologies $\mathcal{T}$ for which R is $\mathcal{T}$-torsion free; for the more general cases we refer the reader to Stenström's book [11].

Let $\mathcal{T}$ be a topology for R. We say that $\mathcal{T}$ is **faithful** in case the regular module ${}_RR$ is $\mathcal{T}$-torsion free. We remind the reader that the correspondence between topologies and injective modules is not bijective. But if E cogenerates $\mathcal{T}$ and if $\mathcal{T}$ is faithful, then E cogenerates R (see Lemma 13.15), and therefore E cogenerates $E(R)$. Appealing then to Corollary 13.20, we have

14.1. Proposition. *For a topology $\mathcal{T}$ on R the following are equivalent:*

(a) *$\mathcal{T}$ is faithful;*

(b) *$\mathcal{T} = \mathcal{T}_E$ for some injective module E with $E(R) \leq E$;*

(c) *$\mathcal{T} \subseteq \mathcal{D}$, the topology of dense left ideals of R.* ■

Let ${}_RE$ be an injective module with $E(R) \leq E$ and let $\mathcal{T} = \mathcal{T}_E$ be the faithful topology it cogenerates. Then the biendomorphism ring

$$Q_{\mathcal{T}} = \mathrm{BiEnd}({}_RE)$$

is a **ring of left quotients of** R. So the maximal ring of left quotients is just the biendomorphism ring

$$Q_{max} = Q_{\mathcal{T}_{E(R)}} = \mathrm{BiEnd}(E(R)).$$

Our immediate goal is to show that each ring of left quotients $Q_{\mathcal{T}}$ is isomorphic to a subring of the maximal ring Q_{max}. But first, an important lemma about general biendomorphism rings.

14.2. Lemma. *Let $M = N \oplus K$. Then the restriction* $\mathrm{Res} : f \longmapsto f\big|_N$ *is a ring homomorphism*

$$\mathrm{Res} : \mathrm{BiEnd}({}_RM) \longrightarrow \mathrm{BiEnd}({}_RN).$$

Moreover, if N generates or cogenerates K, then this homomorphism is injective. If N both generates and cogenerates K, then the homomorphism Res *is an isomorphism.*

Proof. Let $S = \mathrm{End}({}_RM)$ so there is an idempotent $e \in S$ with $N = Me$ and $\rho : eSe \longrightarrow \mathrm{End}(N)$

a ring isomorphism. So for each $f \in \text{BiEnd}(M)$ the restriction $\text{Res}(f)$ is given by

$$\text{Res}(f) : x \longmapsto f(xe) = f(x)e,$$

so clearly $\text{Res}(f) : N \longrightarrow N$. It is then easy to see that each $\text{Res}(f)$ is an eSe homomorphism, so that $\text{Res}(f) \in \text{BiEnd}(N)$. But for $f, g \in \text{BiEnd}(M)$ and $x \in N$, we have

$$\text{Res}(f+g)(x) = (f+g)(x)e = f(x)e + g(x)e = \text{Res}(f)(x) + \text{Res}(g)(x) = (\text{Res}(f) + \text{Res}(g))(x),$$

and

$$\text{Res}(fg)(x) = (fg)(x)e = f(g(x)e)e = \text{Res}(f)\text{Res}(g)(x),$$

so that $\text{Res} : \text{BiEnd}(M) \longrightarrow \text{BiEnd}(N)$ is a ring homomorphism. If N generates K, then it generates M, so that $MeS = NS = M$. So if $f \in \text{Ker}(\text{Res})$, then $f(M) = f(MeS) = f(M)eS = 0$, and we have $f = 0$. Next, if N cogenerates K, it cogenerates M so that $\mathbf{l}_M(Se) = 0$, so if $\text{Res}(f) = 0$, then $f(M)Se = f(MSe) = 0$, and so $f(M) = 0$.

Finally, suppose that N generates and cogenerates K; then it generates and cogenerates M. Thus, $NS = M$ and $\mathbf{l}_M(Se) = 0$. It will suffice, then, to prove that Res is surjective. So let $f \in \text{BiEnd}(N)$. Suppose that some $x_1, \ldots, x_n \in N$ and $s_i, \ldots, s_n \in S$ satisfy $\sum x_i s_i = 0$. Then

$$(\sum f(x_i)s_i)Se = \sum f(x_ie)(s_iSe) = \sum f(x_i)es_iSe = \sum f(x_ies_iSe) = f(\sum(x_is_iSe) = 0,$$

so there is an S-homomorphism $\overline{f} : NS = M \longrightarrow M$ via

$$\overline{f}(\sum x_i s_i) = \sum f(x_i)s_i.$$

But then $\text{Res}(\overline{f}) = f$, and Res is surjective. ■

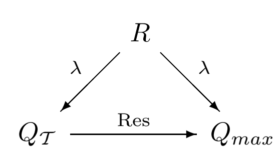
In what follows we shall study a topology $\mathcal{T}$ for R that is faithful. So by Proposition 14.1, $\mathcal{T}$ is cogenerated by some injective E with $E(R) \leq E$. So w.m.a. $E = E(R) \oplus E'$. Let $H = \text{End}(_RE)$, and let $e \in H$ be an idempotent with $E(R) = Ee$. Then

$$Q_{\mathcal{T}} = \text{BiEnd}(_RE) \qquad \text{and} \qquad Q_{max} = \text{BiEnd}(_RE(R)).$$

So we let Res be the restriction map

$$\text{Res} : Q_{\mathcal{T}} \longrightarrow Q_{max}\,.$$

14.3. Proposition. *Let $\mathcal{T}$ be a faithful topology for R cogenerated by an injective module E with $E(R) \leq E$. Then $\text{Res} : Q_{\mathcal{T}} \longrightarrow Q_{max}$ is an injective ring homomorphism for which*



commutes where each λ is left multiplication by R.

Proof. Since $\mathcal{T}$ is cogenerated by an injective of the form $E = E(R) \oplus E'$, and $\mathcal{D}$ is cogenerated by $E(R)$, Lemma 14.2 will give us the desired result if we prove that $E(R)$ generates every injective. However, since R generates every left R-module, its injective envelope $E(R)$ generates every injective left R-module. ∎

Next, we want to characterize the ring $Q_{\mathcal{T}}$ as a subring of Q_{max}. That is, we want to describe $\text{Res}(Q_{\mathcal{T}})$ in Q_{max}. Since ${}_RE$ is injective, clearly, the map $h \longmapsto (1)h$ is an H-epimorphism $H \longrightarrow E$. Also, recall from Lemma 12.2 that there is an R-monomorphism $\eta : Q_{max} \longrightarrow E(R)$ given by $\eta : q \longmapsto q(1)$. With that we have the desired characterization.

14.4. Theorem. *Let $\mathcal{T}$ be a faithful topology for R cogenerated by an injective module E with $E(R) \leq E$. Let $H = \text{End}({}_RE)$. Then*

$$\eta(\text{Res}(Q_{\mathcal{T}})) = \mathbf{l}_{E(R)}\mathbf{r}_H(R).$$

Proof. We claim first that for $q \in Q_{max}$, $q \in \text{Res}(Q_{\mathcal{T}})$ iff $\mathbf{r}_H(R) \subseteq \mathbf{r}_H(q(1))$. So suppose that $q \in \text{Res}(Q_{\mathcal{T}})$ and $Rh = 0$. Then

$$q(1)h = q((1)h) = q(0) = 0.$$

Conversely, suppose that $\mathbf{r}_H(R) \subseteq \mathbf{r}_H(q(1))$. Then there exists a $\overline{q} \in Q_{\mathcal{T}}$ with $\overline{q} : (1)h \longmapsto (q(1))h$ for all $h \in H$. So

$$(\text{Res})(\overline{q})(e) = \overline{q}(1)e = q(1)e = q((1)e) = q(1).$$

From this claim we now have that

$$\eta(\text{Res}(Q_{\mathcal{T}})) \subseteq \mathbf{l}_{E(R)}\mathbf{r}_H(Q_{\mathcal{T}}) = \mathbf{l}_{E(R)}\mathbf{r}_H(R).$$

But for each $x \in \mathbf{l}_{E(R)}\mathbf{r}_H(Q_{\mathcal{T}}) = \mathbf{l}_{E(R)}\mathbf{r}_H(R)$ we have $\mathbf{r}_H(R) \subseteq \mathbf{r}_H(x)$, so that $\mathbf{r}_S(R) \subseteq \mathbf{r}_S(x)$. From the latter and Proposition 12.3 we have that $x = q(1)$ for some $q \in Q_{max}$ and from the former and our above claim we have that $q \in \text{Res}(Q_{\mathcal{T}})$. ∎

14.5. Corollary. *Let $\mathcal{T}$ be a faithful topology for R. Then*

$$\text{Res}(Q_{\mathcal{T}}) = \{q \in Q_{max} \ : (R : q) \in \mathcal{T}\}.$$

Proof. Let $x \in E(R)$. Then by Theorem 14.4, we have $x \in \eta(\text{Res}(Q_{\mathcal{T}}))$ iff $\mathbf{r}_H(R) \subseteq \mathbf{r}_H(x)$ iff $\text{Hom}_R(Rx/R, E(R)) = 0$ iff $\text{Hom}_R(R/(E : x), E(R)) = 0$ iff $(R : x) \in \mathcal{T}$. ∎

We conclude this section by looking at how rings of left quotients behave under Morita equivalence. So suppose that R and S are Morita equivalent rings under the inverse equivalences

$$F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod} \qquad \text{and} \qquad G : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}\,.$$

Let $\mathcal{T}$ be a topology for R cogenerated by some injective module ${}_RE$. Then $F(E)$ must be an injective S-module and so cogenerates a topology $\mathcal{T}_{F(E)}$ for S. But note that for every S-module N,

$$\mathrm{Hom}_S(N, F(E)) = 0 \iff \mathrm{Hom}_R(G(N), E) = 0,$$

so that E and $GF(E)$ cogenerate the same topology $\mathcal{T}$ for R. Thus, the equivalences F and G define a bijection between the topologies for R and those for S. Moreover, the associated rings of quotients are equivalent. That is,

14.6. Theorem. *Let $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ be a Morita equivalence. If E is an injective R-module, then the biendomorphism rings $\mathrm{BiEnd}({}_RE)$ and $\mathrm{BiEnd}({}_SF(E))$ are Morita equivalent.*

Proof. By Theorem 6.8 w.m.a. that there is a progenerator P_R for R with $S = \mathrm{End}(P_R)$ and $F(\_) = {}_SP \otimes_R (\_)$. Thus, there are natural numbers m and n with

$$P^n \cong R \oplus R_0 \qquad \text{And} \qquad R^m \cong P \oplus P_0$$

for some right R-modules R_0 and P_0. Set $H = \mathrm{End}({}_RE)$, so that

$$H \cong \mathrm{End}({}_SF(E)) = \mathrm{End}({}_SP \otimes E).$$

Thus, as right H-modules we have

$$(P \otimes_R E)^n \cong (E \oplus (R_0 \otimes_R E))^n \cong E \oplus N$$

for some N_H, and

$$E^m \cong (R \otimes_R E)^m \cong ((P \otimes E) \oplus (P_0 \otimes E))^m \cong (P \otimes_R E) \oplus M$$

for some M_H. Thus, (see Exercise 6.5)

$$\mathrm{BiEnd}({}_RE) = \mathrm{End}(E_H) \approx \mathrm{End}((P \otimes E)_H) = \mathrm{End}(F(E)_H) = \mathrm{BiEnd}({}_SF(E)). \qquad \blacksquare$$

14.7. Corollary. *Let $\mathcal{T}$ be a faithful topology for R cogenerated by an injective E module with $E(R) \leq E$. Let $F : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ be a Morita equivalence. Then $\mathcal{T}_{F(E)}$ is a faithful topology for R and*

$$Q_{\mathcal{T}_E} \approx Q_{\mathcal{T}_{F(E)}}.$$

Proof. Thanks to Theorem 14.6 it will suffice to prove that $\mathcal{T}_{F(E)}$ is faithful. But R is $\mathcal{T}_E$-torsion free iff every progenerator is $\mathcal{T}_E$ torsion free iff every S-progenerator is $\mathcal{T}_{F(E)}$ torsion free. $\blacksquare$

Exercises 14.

14.1. Let R be a P.I.D., and let P be some non-empty set of primes of R. Let

$$E = \prod_{p \in P} \{E(R/(p)) \ : \ p \in P\}.$$

Prove that if $\mathcal{T}$ is the topology cogenerated by E, then $\mathcal{T}$ is faithful.

14.2. Let R be a P.I.D., and let F be the field of fractions of R. Let $\mathbf{P}$ be the set of all primes of R. For each of the following injective R-modules E, characterize the subring of F that is isomorphic to the ring of quotients $Q_{\mathcal{T}_E}$

(a) For some $p \in \mathbf{P}$ let $E = E(R/(p))$;

(b) For some $p \in \mathbf{P}$, let $E = \prod_{q \in \mathbf{P}} \{E(R/q) \ : q \neq p\}$.

14.3. Let $\mathcal{T}$ be a faithful topology for R, let $Q_{\mathcal{T}}$ be the corresponding ring of left quotients for R, and finally let V be a $\mathcal{T}$-closed module. That is, V is $\mathcal{T}$-torsion free and $\mathcal{T}$-injective; see Exercise 13.8. We claim that $Q_{\mathcal{T}}$ acts on V so that V is a left $Q_{\mathcal{T}}$ module. So let $q \in Q_{\mathcal{T}}$ and let $x \in V$. Set $D = (R : q)$, so that $D \in \mathcal{T}$. If $\rho_x : R \longrightarrow V$ is right multiplication by x, then $\rho_x \circ : D \longrightarrow V$ is an R-homomorphism. But then since V is $\mathcal{T}$ closed, there is a unique $y \in V$ with $\rho_x \circ q = \rho_y$ on D; that is, $q(d)x = sy$ for all $d \in D$ We set $qx = y$. Prove that under this action V is a left $Q_{\mathcal{T}}$ module.

14.4. Let $\mathcal{T}$ be a faithful topology for R, and let $Q_{\mathcal{T}}$ be the corresponding ring of left quotients for R. For each module ${}_RM$ let $E_{\mathcal{T}}(M)$ be a $\mathcal{T}$-injective envelope of M, and let $M_{\mathcal{T}} = E_{\mathcal{T}}(M/t_{\mathcal{T}}(M))$. Since $M/t_{\mathcal{T}}(M)$ is $\mathcal{T}$-torsion free, $M_{\mathcal{T}}$ is $\mathcal{T}$-closed, (see Exercise 13.8), so that by the previous exercise, $M_{\mathcal{T}}$ is a left $Q_{\mathcal{T}}$-module. This is the **module of left quotients** for M over $Q_{\mathcal{T}}$.

(a) Let M and N be $\mathcal{T}$-torsion free and let $f : M \longrightarrow N$ be an R-homomorphism. Prove that there is a unique $Q_{\mathcal{T}}$-homomorphism $\hat{f} : M_{\mathcal{T}} \longrightarrow N_{\mathcal{T}}$ extending f.

(b) Deduce that there is an additive covariant functor $\mathbf{Q}_{\mathcal{T}} : \mathbf{RMod} \longrightarrow \mathbf{Q}_{\mathcal{T}}\mathbf{Mod}$ such that for each ${}_RM$, $\mathbf{Q}_{\mathcal{T}} : \mathbf{M} \longmapsto \mathbf{M}_{\mathcal{T}}$.

15 The Classical Ring of Left Quotients.

In Section 12 we introduced the maximal ring Q_{max} of left quotients of a ring R, and in the last section we considered the collection of all rings of left quotients of R and how they arise from topologies on R. Fortunately, Q_{max} is the largest of those rings of left quotients in which we are mostly interested. Here we begin the investigation of several important special types of rings of quotients that will finish up the material in these lectures. Specifically, in this section we look at the so-called classical ring of left quotients that generalizes the familiar ring of quotients for commutative rings that we studied in graduate algebra. The one downside of our study is that these classical rings of quotients need not exist for many rings, and so part of our assignment is to learn when they do exist.

Let R be a ring, and let Δ be the set of non-zero divisors of R. A ring Q and an injective ring homomorphism $\varphi : R \longrightarrow Q$ is a **classical ring of left quotients of** R if

(Cl.1) $\varphi(d)$ is invertible in Q for every $d \in \Delta$;

(Cl.2) Each $q \in Q$ can be factored $q = \varphi(d)^{-1}\varphi(a)$ for some $d \in \Delta$ and $a \in R$.

It is easy, but slightly tedious, to see that if R has a classical ring of left quotients, then it is unique in a fairly stong sense. In fact it satisfies she following universal property. We will leave the details as an exercise.

15.1. Proposition. *Suppose that R has a classical ring of left quotients, $\varphi : R \longrightarrow Q$. If S is a ring and $\psi : R \longrightarrow S$ is a ring homomorphism such that $\psi(d)$ is invertible in S for each $d \in \Delta$, then there is a unique ring homomophism $\sigma : R \longrightarrow S$ with $\psi = \sigma \circ \varphi$.* ■

As we shall see in the Exercises not every ring has a classical ring of left quotients. So let's check for some necessary conditions for R to have a classical ring of left quotients, $\varphi : R \longrightarrow Q$. Let $a \in R$ and $d \in \Delta$, so that both $\varphi(a)$ and $\varphi(d)^{-1}$ are in Q. Thus, by (Cl.2) there must exist $b \in R$ and $d' \in \Delta$ with

$$\varphi(a)\varphi(d)^{-1} = \varphi(d')^{-1}\varphi(b),$$

or $\varphi(d'a) = \varphi(d')\varphi(a) = \varphi(b)\varphi(d) = \varphi(bd)$, so that

$$d'a = bd.$$

We say that R satisfies the **left Ore condition** or simply is **left Ore** in case for each $a \in R$ and $d \in \Delta$,

$$Rd \cap \Delta a \neq \emptyset.$$

Thus, we have that the left Ore condition is necessary for R to have a ring of left quotients. Shortly, we shall be able to prove that it is also sufficient, and our first step will be to find a topology.

15.2. Lemma. *If R left Ore, then for every $d \in \Delta$, the left ideal Rd is dense.*

Proof. Recall (Lemma 12.6) that Rd is dense iff for every $a \in R$, $\mathbf{r}_R(Rd : a) = 0$. But since R is left Ore, there exists some $d' \in \Delta$ with $d'a \in Rd \cap \Delta a$, so $d' \in (Rd : a)$ and $\mathbf{r}_R(Rd : a) \subseteq \mathbf{r}_R(d') = 0$. ■

This suggests that we check whether the set of left ideals of the form Rd with $d \in \Delta$ can produce a topology. So we consider the set

$$\mathcal{O} = \{I \leq {}_R R \ : \ I \cap \Delta \neq \emptyset\}.$$

15.3. Proposition. *For a ring R the following are equivalent:*

(a) *$\mathcal{O}$ is a topology;*

(b) *$\mathcal{O}$ satisfies (T.1);*

(c) *R is left Ore.*

Proof. (a) $\Longrightarrow$ (b) is clear. For (b) $\Longrightarrow$ (a) we need only show that $\mathcal{O}$ satisfies (T.2). So suppose that $D \in \mathcal{O}$ and that I is a left ideal such that $(I : a) \in \mathcal{O}$ for all $a \in D$. But there is some $d \in D \cap \Delta$, so $(I : d) \cap \Delta \neq \emptyset$, say $d' \in (I : d) \cap \Delta$. Then $d'd \in I \cap \Delta$ and $I \in \mathcal{O}$.

(b) $\iff$ (c) Let $a \in R$ and $d \in \Delta$. Then $(Rd : a) \in \mathcal{O} \iff (Rd : a) \cap \Delta \neq \emptyset \iff \exists\ d'a \in Rd \iff Rd \cap \Delta a \neq \emptyset$. ■

15.4. Theorem. *If R is left Ore, then $\mathcal{O}$ is a topology and $Q_{\mathcal{O}}$ is a classical ring of left quotients for R.*

Proof. By Proposition 15.3, $\mathcal{O}$ is a topology, and by Lemma 15.2, $\mathcal{O} \subseteq \mathcal{D}$. So $Q_{\mathcal{O}}$ is a subring of Q_{max}, and we can view R as a subring of $Q_{\mathcal{O}}$. Let $d \in \Delta$, so there exists an R-homomorphism $Rd \longrightarrow R$ such that $ad \longmapsto a$ for all $a \in R$. But $Rd \in \mathcal{O}$, so that there is a unique $q \in Q_{max}$ with $q(d) = 1$. (See Corollary 12.5.) But then $Rd \subseteq (R : q)$, so by Corollary 14.5, $q \in Q_{\mathcal{O}}$. But then $dq = dq(1) = q(d) = 1$. On the other hand

$$d(q(1)d - 1) = (dq)d - d = d - d = 0,$$

so that $Rd(q(1)d - 1) = Rd(qd - 1) = 0$. However, $Rd \in \mathcal{D}$, so $qd = 1$, and d is invertible in $Q_{\mathcal{O}}$. Finally, let $q \in Q_{\mathcal{O}}$. Then by Corollary 14.5 again $(R : q) \in \mathcal{O}$, so that $d \in (R : q)$ for some $d \in \Delta$. Thus, $dq = a \in R$ and $q = d^{-1}a$. ■

15.5. Corollary. *A ring R has a classical ring of left quotients iff R is left Ore.* ■

Let R be a left Ore ring, so by Corollary 15.5, it has a classical ring of left quotients. By Proposition 15.1 this is unique to within isomorphism through R. Thus, we will take $Q_{\mathcal{O}}$ to be the classical ring of left quotients of R and we will denote it simply by Q_{cl}.

A (not necessarily commutative) integral domain is a **left Ore domain** in case it satisfies the left Ore condition. This bring us to where much of this all started with the following result due originally to Ore.

15.6. Corollary. *A ring R has a classical division ring of left quotients if and only if R is a left Ore domain.* ■

You should be able to note that the left Ore condition, also sometimes referred to as the **common left multiple property**, is really a substitute for commutativity. We shall see how this works in the Exercises. But for one particularly valuable illustration of this we prove that in Q_{cl} there exist "common denominators".

15.7. Lemma. *Let E be a left Ore ring. If*

$$d_1^{-1}a_1, \ldots, d_n^{-1}a_n \in Q_{cl},$$

then there exist $d \in \Delta$ and $b_1, \ldots, b_n \in R$ with

$$d_i^{-1}a_i = d^{-1}b_i$$

for all $i = 1, \ldots, n$.

Proof. Since $\mathcal{O}$ is a topology $Rd_1 \cap \cdots \cap Rd_n \in \mathcal{O}$, so there is some $d \in \Delta \cap Rd_1 \cap \cdots \cap Rd_n$. So there are $c_1, \ldots, c_n \in \Delta$ with $d = c_i d_i$ for all $i = 1, \ldots, n$. But then $d_i^{-1}a_i = d^{-1}c_i a_i$ for $i = 1, \ldots, n$. ■

Exercises 15.

15.1. Let $\psi : R \longrightarrow S$ be an injective ring homomorphism that satisfies (Cl.1).

(a) Prove that for each pair $d_1, d_2 \in \Delta$, $\psi(d_1 d_2)^{-1} = \psi(d_2)^{-1}\psi(d_1)^{-1}$.

(b) Let $\varphi : R \longrightarrow Q$ be a classical ring of left quotients for R. Prove that there exists a unique injective ring homomorphism $\sigma : Q \longrightarrow S$ with $\sigma \circ \varphi = \psi$.

15.2. Let A be a commutative integral domain and let R be the ring of 2×2 matrices over A.

(a) Prove that R is left Ore.

(b) Find the classical ring Q_{cl} of left quotients of R.

15.3. Let $\mathbb{M}_2(\mathbb{Z})$ be the ring of all 2×2 matrices over $\mathbb{Z}$. Let

$$R = \{A \in \mathbb{M}_2(\mathbb{Z}) \ : \ A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } r \in \mathbb{Z}\}.$$

(a) Prove that $A \in R$ iff $[\,1 \quad -1\,]\,A = s\,[\,1 \quad -1\,]$ for some $s \in \mathbb{Z}$.

(b) Prove that R is a subring of $\mathbb{M}_2(\mathbb{Z})$.

(c) Prove that R is left Ore.

(d) Find Q_{cl} for R.

15.4. Let R be the ring of all 2×2 upper triangular matrices over $\mathbb{Z}$.

(a) Show that R is left Ore.

(b) Find the classical ring Q_{cl} of left quotients of R.

15.5. Let R be an integral domain, not necesarily commutative.

(a) Prove that R is left Ore iff every non-zero principal left ideal Rd is essential in R.

(b) Using the above fact show that the ring $k\langle x, y\rangle$ of polynomials over a field k in the non-commuting indeterimants x and y is not left Ore.

15.6. Let R be a left noetherian integral domain. Prove that R is left Ore. [Hint: Let $x, y \in R$ be non-zero. For each $n \in \mathbb{N}$ let

$$I_n = Rx + Rxy + Rxy^2 + \cdots + Rxy^n.$$

Then for some $n \in \mathbb{N}$ we have $I_n = I_{n+1}$. Now use the previous exercise.]

15.7. Let R be an integral domain. Prove that if every finitely generated left ideal is principal (such a ring is **left Bezout**), then R is left Ore. [Hint: Let $x, y \in R$ be non-zero and suppose that $Rx \cap Ry = 0$. Then $Rx + Ry = Rz$ for some $z \in R$. In particular, $y = rz$ for some $r \in R$. Now show that $R \cong Rx \cong R/Rr$, a contradiction.]

15.8. Let R satisfy the DCC for principal left ideals. Prove that $R = Q_{cl}$. [Hint: Suppose $d \in \Delta$. Then there is some $n \in \mathbb{N}$ with $Rd^n = Rd^{n+1}$, so $d^n = ad^{n+1}$ for some $a \in R$. Show that $a = d^{-1}$.]

16 Non-singular Rings.

In this Section we consider the ring of left quotients when the set $\mathcal{E}$ of all essential left ideals of R is a topology. As we shall see, the set $\mathcal{E}$ is a faithful topology for R iff Q_{max} is a left self-injective von Neumann regular ring, an amazingly strong property. (See Exercise 6.3.)

In Theorem 13.3 we saw that for any ring R the set $\mathcal{E}$ of essential left ideals is a pre-topology, satisfying (T.1). Thus, every module M has a pre-torsion submodule for $\mathcal{E}$. This, submodule is usually called the **singular** **submodule** of M and is denoted by $Z(M)$. That is,

$$Z(M) = \{x \in M \ : \ \mathbf{l}_R(x) \trianglelefteq {}_R R\}.$$

The module M is said to be **non-singular** if $Z(M) = 0$. As we saw in Lemma 13.4, if ${}_R M_S$ is a bimodule, then $Z(M)$ is also an (R, S) bimodule. In particular, the singular submodule of the regular module ${}_R R$ is an ideal. We call this ideal,

$$Z({}_R R) = \{a \in R \ : \ \mathbf{l}_R(a) \trianglelefteq R\},$$

the **left** **singular** **ideal** of R. We say that R is **left** **non-singular** if $Z({}_R R) = 0$. If $Z({}_R R) \neq 0$, then R is **left** **singular**. Of course, there are also right hand versions of this.

In an integral domain R we have $\mathbf{l}_R(x) = 0$ for every non-zero $x \in R$, so R is non-singular. On the the hand, the ring $\mathbb{Z}_4$ has singular submodule $2\mathbb{Z}_4$ and so is singular.

16.1. Lemma. *If M is a left R-module, then $Z(M) = 0$ iff $Z(E(M)) = 0$.*

Proof. Simply observe that $M \trianglelefteq E(M)$ and $Z(M) = Z(E(M)) \cap M$. ■

Now let R be a ring for which the set $\mathcal{E}$ of essential left ideals forms a faithful topology. Then (see Proposition 14.1) we must have $\mathcal{E} \subseteq \mathcal{D}$. On the other hand (see Corollary 12.7), for any ring R we have $\mathcal{D} \subseteq \mathcal{E}$. This gives us part of the following characterization of rings for which $\mathcal{E}$ is a faithful topology.

16.2. Theorem. *For a ring R the following are equivalent:*

(a) *R is left non-singular;*

(b) *$E(R)$ is left non-singular;*

(c) *$\mathcal{E}$ is a faithful topology for R;*

(d) *$\mathcal{E} = \mathcal{D}$.*

Proof. (a) $\iff$ (b) is Lemma 16.1.

(c) $\iff$ (d) follows from our above remarks.

(d) $\Longrightarrow$ (a) follows from Lemma 12.4.

(a) $\Longrightarrow$ (d) Thanks to Corollary 12.7 it will suffice to show that $\mathcal{E} \subseteq \mathcal{D}$. So let $I \in \mathcal{E}$. Since $\mathcal{E}$ satisfies (T.1), $(I : a) \in \mathcal{E}$ for all $a \in R$. But then since R is left non-singular, this means that $\mathbf{r}_R(I : a) = 0$ for all $a \in R$. Thus, by Lemma 12.6, $I \in \mathcal{D}$. ■

It follows that a left non-singular ring R is characterized by the property that the largest imaginable faithful topology, namely $\mathcal{E}$, actually is a topology. So not surprisingly, the corresponding quotient ring is the whole injective envelope $E({}_RR)$. That is,*

16.3. Theorem. *Let R be a left non-singular ring. Then as an R-module*

$$Q_{max} = E({}_RR).$$

Moreover, Q_{max} is left self-injective.

Proof. To see that $Q_{max} = E(R)$, it will suffice by Theorem 14.4 to show that if $H = \text{End}({}_RE(R))$, then $\mathbf{r}_H(R) \subseteq \mathbf{r}_H(E(R))$. Suppose that $Rh = 0$ for some $h \in H$. Since $R \trianglelefteq E(R)$, for each $x \in E(R)$, $(R : x) \trianglelefteq R$. Now $Z(E(R)) = 0$ and $(R : x)((x)h) = 0$, so $(x)h = 0$. Thus, $h \in \mathbf{r}_H(E(R))$.

To show that Q_{max} is left self-injective we first abbreviate $Q = Q_{max}$. Let I be a left ideal of Q and suppose that $f : I \longrightarrow Q$ is a left Q-homomorphism. Since $Q = E(R)$ is R-injective, there is an R-homomorphism $\overline{f} : Q \longrightarrow Q$ extending f. We claim that $\overline{f}$ is actually a Q-homomorphism. For each $x \in Q$ let $\overline{f}_x : Q \longrightarrow Q$ be the R-homomorphism defined by

$$\overline{f}_x(q) = \overline{f}(qx) - q\overline{f}(x)$$

for all $q \in Q$. Now $D = (R : q) \trianglelefteq R$ and $D\overline{f}_x(q) = 0$ for all $x \in Q$. Thus, since ${}_RQ$ is non-singular, $\overline{f}_x(q) = 0$ for all $x, q \in Q$, so $\overline{f}$ is a Q-homomorphism. and ${}_QQ$ is injective. ■

A ring R is **<u>von</u> <u>Neumann</u> <u>regular</u>** in case for each $a \in R$ we have $a \in aRa$. These rings have a very strong decomposition theory. This is clear from the following multiple characterization of von Neumann regular rings. (See Exercise 6.3.)

16.4. Proposition. *For a ring R the following are equivalent:*

(a) *R is von Neumann regular;*

(b) *For every $a \in R$ the inclusion homomorphism $Ra \hookrightarrow {}_RR$ splits;*

(c) *For every $a \in R$ the inclusion homomorphism $aR \hookrightarrow R_R$ splits;*

(d) *Every finitely generated left (right) ideal is a direct summand.* ■

*Recall that a ring R is left self-injective if the regular module ${}_RR$ is injective.

16.5. Theorem. *Let R be left non-singular. Then Q_{max} is a left self-injective von Neumann regular ring.*

Proof. Thanks to Theorem 16.3 it will suffice to prove that Q_{max} is von Neumann regular. Since R is left non-singular, Q_{max} is left non-singular (see Lemma 16.1), so it will suffice to prove that if R is left non-singular and left self-injective, then R is von Neumann regular. So let $a \in R$. Then there is some left ideal $K \leq R$ with $D = \mathbf{l}_R(a) \oplus K \trianglelefteq {}_RR$. Then there exists an R-homomorphism $f : Ka \longrightarrow R$ with $f : ka \longmapsto k$ for all $k \in K$. Since R is left self-injective, w.m.a. that $f : R \longrightarrow R$. Let $x = f(1)$. Then $f(ka) = kaf(1) = kax$. Thus, $kax = k$ for all $k \in K$. Therefore, $D(axa - a) = 0$ and since R is left non-singular, $a = axa$. ∎

Finally, for a left non-singular ring R, we ask when is Q_{max} a semisimple ring. Thanks to Theorem 16.5 this amounts to determining which left self-injective von Neumann regular rings are semisimple. Not surprisingly this is easily characterized in terms of the Goldie rank.

16.6. Proposition. *A left self-injective von Neumann rigular ring R is semisimple iff it has finite Goldie rank.*

Proof. Trivially every semisimple ring has finite Goldie rank. Conversely, if $\text{Grank}({}_RR) = n < \infty$, then since R is left self-injective, there exist indecomposable left ideals $I_1, \ldots, I_n$ with ${}_RR = I_1 \oplus \cdots \oplus I_n$. We claim that each I_i is simple. Indeed, let $x \in I_i$ be non-zero. Then by Proposition 16.4, $Rx \leq I_i$ is a direct summand of R, and so since I_i is indecomposable $Rx = I_i$. That is, each I_i is simple and R is semisimple. ∎

To apply this Proposition to Q_{max} for a left non-singular ring, we'll need

16.7. Lemma. *If R is left non-singular and if $Q = Q_{max}$, then* $\text{Grank}({}_RR) = \text{Grank}({}_QQ)$.

Proof. Since $\text{Grank}({}_RR) = \text{Grank}(E(R)) = \text{Grank}({}_RQ)$, it will suffice to prove that $\text{Grank}({}_RQ) \leq \text{Grank}({}_QQ)$. So suppose that $M_1, \ldots, M_n$ are independent in ${}_RQ$, but that there is some element $x \in QM_1 \cap (QM_2 + \cdots + QM_n)$, say $x = q_1m_1 = q_2m_2 + \cdots + q_nm_n$. For each $i = 1, \ldots, n$, $D_i = (R : q_i) \trianglelefteq R$, so $D = D_1 \cap D_2 \cap \cdots \cap D_n \trianglelefteq R$. But $Dx \in M_1 \cap (M_2 + \cdots + M_n) = 0$, so since R is left non-singular, $x = 0$, and $QM_1, \ldots, QM_n$ are independent in Q. ∎

16.8. Theorem. *For a left non-singular ring R, its maximal ring Q_{max} of left quotients is semisimple iff R has finite left Goldie rank.*

Proof. By Theorem 16.5, Q_{max} is left self-injective and von Neumann regular. So by Proposition 16.6, Q_{max} is semisimple iff it has finite Goldie rank. Finally, by Lemma 16.7, Q_{max} has finite Goldie rank iff ${}_RR$ has finite Goldie rank. ∎

This has an interesting corollary that will be of considerable importance a bit later.

16.9. Corollary. *If R is a left non-singular ring of finite Goldie rank, then R satisfies the A.C.C. and D.C.C. on left annihilators.*

Proof. By Theorem 16.8, $Q = Q_{max}$ is semisimple and hence, of finite length. Suppose that $\mathbf{l}_R(X_1) \leq \mathbf{l}_R(X_2) \leq \cdots$ is an increasing sequence of left annihilators in R. Then in Q we have $\mathbf{r}_Q\mathbf{l}_R(X_1) \geq \mathbf{r}_Q\mathbf{l}_R(X_2) \geq \cdots$, so for some $n \in \mathbb{Z}$, $\mathbf{r}_Q\mathbf{l}_R(X_n) = \mathbf{r}_Q\mathbf{l}_R(X_{n+1})$. Then intersecting with R gives that $\mathbf{r}_R\mathbf{l}_R(X_n) = \mathbf{r}_R\mathbf{l}_R(X_{n+1})$, so that $\mathbf{l}_R(X_n) = \mathbf{l}_R\mathbf{r}_R\mathbf{l}_R(X_n) = \mathbf{l}_R\mathbf{r}_R\mathbf{l}_R(X_{n+1}) = \mathbf{l}_R(X_{n+1})$. That R also satisfies the D.C.C. on left annihilators is dual. ■

Exercises 16.

16.1. (a) Prove that the ring $R = \begin{bmatrix} \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & \mathbb{Z} \end{bmatrix}$ is left non-singular but not right non-singular.

(b) Prove that if R is a PID, then the factor ring R/I is non-singular (both left and right) iff $I = Rd$ where $d \in R$ has no prime square as a divisor.

16.2. Let $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$.

(a) Prove that R is left non-singular.

(b) Find the maximal ring Q_{max} of left quotients of R.

16.3. Consider a short exact sequence $0 \longrightarrow K \xrightarrow{f} M \longrightarrow N \longrightarrow 0$ of left R-modules with M non-singular. Prove that N is non-singular iff f is not an essential monomorphism. [Hint: ($\Longrightarrow$) Use the fact that if $K \trianglelefteq M$ and $x \in M$, then $(K : x) \trianglelefteq R$.]

16.4. Prove that if I and J are essential left ideals in a left non-singular ring R, then IJ is an essential left ideal.

16.5. Let R be von Neumann regular. Prove that R is semisimple iff R is left noetherian.

16.6. Let V_D be a vector space over a division ring D, and let $R = \mathrm{End}(V_D)$.

(a) Prove that R is von Nemann regular.

(b) Prove that R is right self-injective.

(c) Show that if V is not finite dimensional, then R is not left self-injective.

17 Semiprime Rings.

Our next goal is to characterize those rings whose *classical* ring of left quotients is semisimple. This requires that we degress briefly to discuss the notion of a prime ideal in a non-commutative setting. This, in turn, leads to a generalization of the prime radical from commutative algebra.

Let R be an arbitrary ring. An ideal P of R is **prime** in case for every pair $a, b \in R$, if $aRb \subseteq P$, then $a \in P$ or $b \in P$. Of course, if R is commutative, this reduces to the usual definition of primeness.

17.1. Proposition. *For an ideal in the ring R the following are equivalent:*

(a) *P is prime;*

(b) *For every pair I, J of left ideals of R, $IJ \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$;*

(c) *For every pair I, J of right ideals of R, $IJ \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$;*

(d) *For every pair I, J of ideals of R, $IJ \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$.*

Proof. (a) $\Longrightarrow$ (d) Let $IJ \subseteq P$ and suppose that $J \nsubseteq P$. Then there is some $b \in J \setminus P$. But then $aRb \subseteq P$ for all $a \in I$, so by (a), $a \in P$ for all $a \in I$.

(d) $\Longrightarrow$ (c) If I, J are right ideals with $IJ \subseteq P$, then $RIRJ \subseteq P$, so by (d) $I \subseteq RI \subseteq P$ or $J \subseteq RJ \subseteq P$. (c) $\Longrightarrow$ (b) is similar.

(b) $\Longrightarrow$ (a) If $aRb \subseteq P$, then $RaRb \subseteq P$, so by (b) $a \in R$ or $b \in R$. ■

A ring R is **prime** if 0 is a prime ideal. Thus, an ideal P is prime iff the ring R/P is a prime ring. More generally, an ideal I of R is **semiprime** if it is an intersection of prime ideals. A ring R is **semiprime** in case 0 is a semiprime ideal of R. So an ideal I is semiprime iff R/I is a semiprime ring. The smallest of all the semiprime ideals,

$$N(R) = \cap\{P \ : \ P \text{ is a prime ideal of } R\}$$

is the **prime radical** of R. So R is semiprime iff $N(R) = 0$. Our immediate goal is to find an internal characterization of $N(R)$.

An element a of a ring R is **strongly nilpotent** in case for every sequence $a_0, a_1, \ldots$ in R if

$$a_0 = a \quad \text{and} \quad a_{n+1} \in a_n R a_n \text{ for all } n \geq 0,$$

there must exist some $n \in \mathbb{N}$ with $a_n = 0$. That this really is a statement about the nilpotence of a may be made a bit more clear with the following characterization whose easy proof we leave to the reader.

17.2. Lemma. *An element $a \in R$ is strongly nilpotent iff for every sequence $x_1, x_2, \ldots$ in R, there exists some $n \in \mathbb{N}$ with*

$$a(x_1 a(x_2 a(\cdots a(x_{n-1} a(x_n) a x_{n-1}) a \cdots a) x_2) a x_1) a = 0. \qquad \blacksquare$$

17.3. Theorem. *For a ring R, its prime radical is*

$$N(R) = \{a \in R \ : \ a \text{ is strongly nilpotent } \}.$$

Proof. Suppose that $a \notin N(R)$, so there is some prime ideal P with $a \notin P$. Since P is prime, there is a function $\sigma : R \setminus P \longrightarrow R \setminus P$ such that $\sigma(x) \in xRx \setminus P$ for each $x \in R \setminus P$. Now define a sequence $a_0, a_1, \ldots$ in $R \setminus P$ recursively by

$$a_0 = a \quad \text{and} \quad a_{n+1} = \sigma(a_n), \text{ for all } n \geq 0.$$

Since $a_n \neq 0$ for all $n \in \mathbb{N}$, a is not strongly nilpotent.

On the other hand suppose that $a \in R$ is not strongly nilpotent. Then there is a sequence

$$a = a_0, a_1, a_2, \ldots$$

in R with $0 \neq a_{n+1} \in a_n R a_n$ for all $n \geq 0$. Then there is an ideal P maximal w.r.t. $a_n \notin P$ for all $n \in \mathbb{N}$. We claim that P is prime. Indeed, suppose that I, J are two ideals neither contained in P. Then by maximality, there must be some $n \in \mathbb{N}$ with $a_n \in I + P$ and $a_n \in J + P$. But then $a_{n+1} \in a_n R a_n \subseteq IJ + P$. So $IJ \not\subseteq P$, and P is prime. Of course, this means that $a \notin N(R)$. ■

17.4. Corollary. *For every ring R, its prime radical $N(R)$ is a nil ideal.* ■

We can now use Theorem 17.3 to characterize semiprime rings.

17.5. Corollary. *For a ring R the following are equivalent:*

(a) *R is semiprime;*

(b) *$N(R) = 0$;*

(c) *$I^2 = 0$ implies $I = 0$ for every left (right/two-sided) ideal I of R;*

(d) *$aRa = 0$ implies $a = 0$;*

(e) *R has no non-zero nilpotent left (right/two-sided) ideals;*

(f) *$IJ = 0$ implies $I \cap J = 0$ for every pair I, J of left (right/two-sided) ideals.*

Proof. (a) $\iff$ (b) This is simply by definition.

(f) $\Longrightarrow$ (c) $\Longrightarrow$ (d) and (c) $\iff$ (e) are all trivial.

(b) $\Longrightarrow$ (f) If $IJ = 0$, then $IJ \subseteq P$ for every prime ideal P, so $I \cap J \subseteq P$ for every prime ideal P. Thus, by (b), $I \cap J = 0$.

(d) $\Longrightarrow$ (b) By (d) there is a map $\sigma : R \setminus \{0\} \longrightarrow R \setminus \{0\}$ such that $\sigma(a) \in aRa$ for each $a \neq 0$ in R. Then for each $a \neq 0$ in R the sequence $a, \sigma(a), \sigma^2(a), \ldots$ is never 0, so a is not strongly nilpotent. Thus, by Theorem 17.3, $N(R) = 0$. ∎

In turn this gives us various characterizations of semiprime ideals.

17.6. Corollary. *For an ideal H of R the following are equivalent:*

(a) *H is semiprime;*

(b) *$I^2 \subseteq H$ implies $I \subseteq H$ for all left (right/two-sided) ideals I;*

(c) *$aRa \subseteq H$ implies that $a \in H$.* ∎

If R is left noetherian, then the prime radical $N(R)$ has a particularly nice form. Indeed, since R is left noetherian, it has a maximal nilpotent left ideal, say I. But then clearly the ideal IR is also nilpotent, so by maximality, I is a maximal two-sided nilpotent ideal. We now see that this maximal nilpotent ideal is actually $N(R)$.

17.7. Theorem. *If R is a left noetherian ring, then $N(R)$ is the unique largest nilpotent ideal of R.*

Proof. As we saw above, R has a maximal nilpotent ideal I. If J is another nilpotent ideal, then there is some n with $I^n = J^n = 0$. So $(I + J)^{2n} = 0$ and then by maximality $I + J = I$, so $J \subseteq I$. That is, I is the unique largest nilpotent ideal. Of course, then, $I \subseteq N(R)$. But if J is an ideal nilpotent modulo I, then it must be nilpotent and hence in I. So by Corollary 17.6, the ideal I is semiprime, so $N(R) \subseteq I$. ∎

Exercises 17.

17.1. For a ring R and an ideal P,

(a) Prove that every maximal ideal P is prime.

(b) Prove that every primitive ideal P is prime.

(c) Deduce that $N(R)$ is contained in the Jacobson radical $J(R)$.

(d) Show that $N(R)$ need not equal $J(R)$.

(e) Prove that if R is artinian, then every prime ideal is maximal, so that $N(R) = J(R)$.

17.2. For a ring R prove that it contains a unique largest nil ideal. This ideal $U(R)$ is the **upper nil radical** of R. The prime radical $N(R)$ is sometimes called the **lower nil radical** of R. So $N(R) \subseteq U(R)$.

(a) Show that $N(R)$ need not be $U(R)$.

(b) Prove that if R is left noetherian, then $N(R) = U(R)$.

17.3. Let R be a ring.

(a) Let $(P_\alpha)_{\alpha\in\Omega}$ be a chain of prime ideals in R. Prove that both $\cup_{\alpha\in\Omega}P_\alpha$ and $\cap_{\alpha\in\Omega}P_\alpha$ are prime ideals of R.

(b) Prove that any non-empty set $\mathcal{S}$ of prime ideals of R contains a maximal element and a minimal element.

(c) Prove that if P is a prime ideal in R, then P is contained in a maximal prime ideal and contains a minimal prime ideal.

17.4. (a) Prove that if R is left noetherian, then there exists a finite sequence $P_1, P_2, \ldots, P_n$ of prime ideals with $P_1P_2\cdots P_n = 0$. [Hint: Suppose false. So R contains an ideal maximal w.r.t. I contains no finite product of primes. Then w.m.a. $I = 0$. So every non-zero ideal of R contains a finite product of primes. But R is prime.]

(b) Find such a prime factorization of 0 in $\mathbb{Z}_{72}$.

17.5. Let $R = C([0,1],\mathbb{R})$ be the ring of all continuous real-valued functions on $[0,1]$. For each $f \in R$, let $Z(f) = \{x \in [0,1] \ : \ f(x) = 0\}$. So each $Z(f)$ is a closed set.

(a) Prove that for each $x \in [0,1]$ the set $M_x = \{f \in R \ : \ x \in Z(f)\}$ is a maximal ideal of R.

(b) For each $x > 0$ and each $y < 1$ in $[0,1]$, set

$$P_x = \{f \in R \ : \ (a,x) \subseteq Z(f) \text{ for some } 0 < a < x\} \quad \text{and}$$

$$Q_y = \{f \in R \ : \ (y,b) \subseteq Z(f) \text{ for some } y < b < 1\}.$$

Prove that each P_x and Q_y is a prime ideal.

(c) Prove that for each $x > 0$ and $y < 1$, R/P_x and R/Q_y is a local ring.

(d) Prove that for each open set $U \subseteq [0,1]$, $\{f \in R \ : \ U \subseteq Z(f)\}$ is a semiprime ideal.

17.6. Let R and S be Morita equivalent rings. Prove that if R is semiprime, then so is S. [Hint: See Theorem 6.12.]

17.7. Let R be a semiprime ring. For each ideal I of R set $I' = \mathbf{l}_R(I)$. So I' is an ideal; such an ideal is called an **annihilator ideal** of R. Denote by $\mathcal{A} = \mathcal{A}(R)$ the set of all annihilator ideals of R.

(a) Prove that for each ideal I of R, $I' = \cap\{P \ : \ P$ is a prime ideal of R with $I \not\subseteq P\}$.

(b) Deduce that for each ideal I of R, $I' = \mathbf{r}_R(I)$.

(c) Deduce further that for each ideal I of R, $I''' = I'$.

(d) Show that the set $\mathcal{A}$ of all annihilator ideals of R is a complete complemented lattice. [Hint: The complement of any annihilator ideal I' is just I''. That $\mathcal{A}$ is a complete lattice is essentially the same as part of Proposition 11.3.]

(e) Finally, deduce that $\mathcal{A}$ is a complete Boolean algebra. [That is, show that $\mathcal{A}$ is a complete, complemented, distributive lattice.]

17.8. Let R be a left noetherian semiprime ring and let $\mathcal{A}$ be its Boolean algebra of annihilator ideals.

(a) Prove that $\mathcal{A}$ is finite. [Hint: A Boolean algebra with the Maximum Condition must be finite.]

(b) Prove that every maximal element of $\mathcal{A}$ is a prime ideal.

18 The Goldie Theorems.

Now we are in a position to characterize those rings with a semisimple classical ring of left quotients. As a bonus we also characterize rings with simple artinian classical ring of left quotients. These characterizations are due to A. W. Goldie who published them in a series of papers in the late 1950's.

We begin with some standard terminology. A subring R of a ring Q is said to be a **left order in** Q in case Q is a ring of left quotients for R. Thus, our goal for this sectiion is to characterize those rings R that are left orders in semisimple (or simple artinian) rings. Of course, R has a classical ring of left quotients iff R is left Ore. (See Corollary 15.5.) So for openers we want to find out what we can about left Ore rings that are left orders in semisimple rings.

18.1. Lemma. *Let R be a left Ore ring with Q_{cl} semisimple. Then $\mathcal{O} = \mathcal{E}$ and ${}_R Q_{cl} = {}_R E(R)$, so in particular,*

$$Q_{max} = Q_{cl}\,.$$

Proof. It will suffice to prove that $\mathcal{E} \subseteq \mathcal{O}$. So let $I \trianglelefteq R$. Then ${}_R I \trianglelefteq {}_R Q_{cl}$. We claim that $Q_{cl}\, I \trianglelefteq Q_{cl}$. For if not, then there is some $0 \neq q \in Q_{cl}$ with $Q_{cl}\, q \cap Q_{cl}\, I = 0$. But there is some $d \in \Delta$ with $dq \in R$, and so $Rdq \cap I \neq 0$, a contradiction. Thus, as claimed, $Q_{cl}\, I \trianglelefteq Q_{cl}$; but Q_{cl} is semisimple, so $Q_{cl}\, I = Q_{cl}$. Therefore, there are $q_1, \ldots, q_n \in Q_{cl}$ and $a_1, \ldots, a_n \in I$ with $1 = q_1 a_1 + \cdots + q_n a_n$. But then by Lemma 15.7, there is some $d \in \Delta$ and $a \in I$ with $1 = d^{-1}a$. So $d = d(d^{-1}a) = a \in I$. Thus, $\mathcal{E} \subseteq \mathcal{O}$. ■

Note that this Lemma means that if R is a left order in a semisimple ring, then its maximal ring of left quotients is semisimple, so that $E(R)$ and hence, R must be left non-singular (see Theorem 16.2), and so by Theorem 16.8, R must have finite Goldie rank. With that and Corollary 16.9 we have the desired necessary conditions for a ring R to be a left order in a semisimple ring. That is,

18.2. Proposition. *If R is a left order in a semisimple ring, then R is semiprime ring of finite left Goldie rank that satisfies the A.C.C. on left annihilators.*

Proof. By Lemma 18.1 and Theorem 16.2, R is non-singular, and by Corollary 16.9, R satisfies the A.C.C. on left annihilators. So it will suffice to show that R is semiprime. Suppose then that $aRa = 0$ for some $a \in R$. We claim that $\mathbf{r}_R(RaR) \trianglelefteq {}_R R$. Indeed, for every $0 \neq x \in R$, we have $RaRx \leq \mathbf{r}_R(RaR) \cap Rx$. So if $\mathbf{r}_R(RaR) \cap Rx = 0$, then $RaRx = 0$, and $x \in \mathbf{r}_R(RaR) \cap Rx$, a contradition. So $\mathbf{r}_R(RaR) \trianglelefteq {}_R R$. But by Lemma 18.1, $\mathcal{E} = \mathcal{O}$, so there is some $d \in \mathbf{r}_R(RaR) \cap \Delta$. So $ad = 0$, and hence, $a = 0$, so by Corollary 17.5 R is semiprime. ■

A ring R is **left Goldie** in case it has finite left Goldie rank and satisfies the A.C.C. on left annihilators. So Proposition 18.2 states that if R is a left order in a semisimple ring, then R is a semiprime left Goldie ring. The icing on the cake is that the converse is true. To prove that we begin with.

18.3. Lemma. *If R is a semiprime left Goldie ring, then R is left non-singular.*

Proof. Let $Z = Z({}_RR)$ be the left singular ideal of R. Since R is semiprime, it will suffice to prove that Z is nilpotent. (See Corollary 17.5.) But we do have $Z \geq Z^2 \geq Z^3 \geq \cdots$, so that (see Lemma 11.1), $\mathbf{l}_R(Z) \leq \mathbf{l}_R(Z^2) \leq \mathbf{l}_R(Z^3) \leq \cdots$. Thus, since R satisfies the A.C.C. on left annihilators, $\mathbf{l}_R(Z^{n+1}) = \mathbf{l}_R(Z^n)$ for some $n \in \mathbb{N}$. We claim that $\mathbf{l}_R(Z^n) = R$. If not, then the set

$$\{\mathbf{l}_R(x) \ : \ x \in R \setminus \mathbf{l}_R(Z^n)\}$$

of left annihilators has a maximal element, say $\mathbf{l}_R(x)$. Let $a \in Z$, so that $\mathbf{l}_R(a) \trianglelefteq R$. But then there is some $0 \neq rx \in \mathbf{l}_R(a)$. So then, $r \in \mathbf{l}_R(xa) \setminus \mathbf{l}_R(x)$; then by the maximality of $\mathbf{l}_R(x)$, this means that $xa \in \mathbf{l}_R(Z^n)$ or that $x \in \mathbf{l}_R(aZ^n)$. Since this is true for every $a \in Z$, we conclude that $x \in \mathbf{l}_R(Z \cdot Z^n) = \mathbf{l}_R(Z^{n+1}) = \mathbf{l}_R(Z^n)$, a contradiction. Thus, $\mathbf{l}_R(Z^n) = R$, and $Z^n = 0$. ∎

Now here is the key lemma for the whole business. The impact of it is that if R is semiprime left Goldie, then Q_{cl} equals Q_{max} reducing the whole issue to the case of a left non-singular ring.

18.4. Lemma. *If R is a semiprime left Goldie ring, then $\mathcal{O} = \mathcal{E}$; that is, if R is a semiprime left Goldie ring, then a left ideal I of R is essential iff I contains a non-zero divisor.*

Proof. ($\Longleftarrow$) Let $d \in \Delta$. Then right multiplication $\rho_d : R \longrightarrow R$ is a monomorphism, so since $\mathrm{Grank}({}_RR) < \infty$, $Rd \trianglelefteq R$. (See Exercise 10.3.) Therefore, $\mathcal{O} \subseteq \mathcal{E}$.

($\Longrightarrow$) By Lemma 18.3 R is left non-singular, so by Corollary 16.9 R has the minimum condition on left annihilators. Let $I \trianglelefteq R$ be an essential left ideal of R. The set $\{\mathbf{l}_R(x) \ : \ x \in I\}$ has a minimal element, say $\mathbf{l}_R(d)$. We claim that $Rd \trianglelefteq I$ whence $Rd \trianglelefteq R$. Indeed, let $a \in I$ and suppose that $Ra \cap Rd = 0$. Then for all $r \in R$,

$$\mathbf{l}_R(ra + d) = \mathbf{l}_R(ra) \cap \mathbf{l}_R(d) \leq \mathbf{l}_R(d),$$

so by the minimality of $\mathbf{l}_R(d)$, we have $\mathbf{l}_R(d) = \mathbf{l}_R(ra + d)$ and hence $\mathbf{l}_R(d)ra = 0$. Therefore, $(Ra\mathbf{l}_R(d))^2 = 0$, and since R is semiprime, $Ra\mathbf{l}_R(d) = 0$, and $RaR \cap \mathbf{l}_R(d) = 0$. (See Corollary 17.5.) Thus, right multiplication $\rho_d : RaR \longrightarrow RaR$ is a monomorphism. Therefore, since R has finite left Goldie rank, $RaRd \trianglelefteq RaR$. (See Exercise 10.3.) But $Ra \leq RaR$ and $Ra \cap RaRd = 0$, so $Ra = 0$. Thus, as claimed $Rd \trianglelefteq I$, and hence $Rd \trianglelefteq R$.

Now by Theorem 16.8, $Q = Q_{max}$ is semisimple. And since $Rd \trianglelefteq R$, we have $Qd \trianglelefteq Q$, so since ${}_QQ$ is semisimple, $Qd = Q$. Thus, d is invertible in Q and hence a non-zero divisor in R. ∎

18.5. Theorem. **[Goldie]** *A ring R is a left order in a semisimple ring iff R is a semiprime left Goldie ring.*

Proof. The necessity is Proposition 18.2. Conversely, if R is semiprime left Goldie, then R is left non-singular (see Lemma 18.3), so by Theorem 16.8, Q_{max} is semisimple. But, finally, by Lemma 18.4, $Q_{cl} = Q_{max}$, and R is a left order in the semisimple ring $Q_{cl} = Q_{max}$. ∎

Now it is easy to specialize this to characterize rings that are left orders in simple artinian rings. Specifically,

18.6. Theorem. **[Goldie]** *A ring R is a left order in a simple artinian ring iff R is a prime left Goldie ring.*

Proof. In either case by Theorem 18.5, R is a left order in a semisimple ring $Q = Q_{cl}$, so it will suffice to prove that Q is simple iff R is prime.

So suppose that $Q = Q_{cl}$ is simple and suppose that $a, b \in R$ with $aRb = 0$ and $b \neq 0$. Then by the simplicity of Q, we have $QbQ = Q$, so there exist $q_i, q'_i \in Q$ with $1 = \sum_{i=1}^{n} q_i b q'_i$. By Lemma 15.7, there exist $d \in \Delta$ and $a_i \in R$ with $q_i = d^{-1}a_i$ for $i = 1, \ldots, n$. Thus,

$$d = d \cdot 1 = \sum_{i=1}^{n} a_i b q'_i,$$

and so

$$aRd = \sum_{i=1}^{n} aRa_i b q'_i \subseteq aRbQ = 0.$$

But $d \in \Delta$, so $aR = 0$ and $a = 0$. Thus, R is prime.

Conversely, suppose that $Q = Q_{cl}$ is not simple. Then (see Theorem 7.4.e) there exist non-zero central idempotents $e, f \in Q$ with $ef = 0$. By Lemma 15.7 there exist some $d \in \Delta$ and $e_0, f_0 \in R$ with $e = d^{-1}e_0$ and $f = d^{-1}f_0$. Then $e_0 \neq 0$ and $f_0 \neq 0$, but

$$Re_0Rf_0 = RdeRfd = 0,$$

so that R is not prime. ∎

Exercises 18.

18.1. Let R be a left Goldie semiprime ring. Show that if R_R has finite Goldie rank, then R is also right Goldie.

18.2. Prove that if R and S are Morita equivalent rings, and if R satisfies the A.C.C. for left annihilators, then S satisfies the A.C.C. for left annihilators. [Hint: See Corollary 6.11.]

18.3. Let R and S be Morita equivalent rings.

(a) Prove that if R is a semiprime left Goldie ring, then so is S. [Hint: See Exercises 17.6, 18.2, and 10.6.]

(b) Prove that if R is a semiprime left Goldie ring, then $Q_{cl}(R)$ and $Q_{cl}(S)$ are Morita equivalent. [Hint: See Corollary 14.7.]

18.4. Let $K = \mathbb{Q}(t)$ be the $\mathbb{Q}$-algebra of rational functions in the indeterminant t over the field $\mathbb{Q}$. Let $\alpha : K \longrightarrow K$ be the injective algebra homomorphism with $\alpha : t \longmapsto t^2$. Let R be the algebra $K[X]$ of all polynomials

$$a_0 + a_1 X + \cdots + a_n X^n$$

over K in the indeterminant X with the "skew" multiplication given by $X^n a = \alpha^n(a) X^n$.

(a) Prove that for all $f, g \in R$, $\deg(fg) = \deg(f) + \deg(g)$, and infer that R is an integral domain.

(b) Prove that for all $f, g \in R$ with $g \neq 0$ there exist unique $q, r \in R$ with $f = qg + r$ and $0 \leq \deg r < \deg g$.

(c) Prove that every left ideal of R is principal, so that R is left noetherian.

(d) Deduce that R is left Goldie. [Hint: See Exercise 15.6.]

(e) Prove that R is not right Goldie.

(f) Finally, infer that R is a left order in a division ring, but is not a right order in any division ring.

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