1 The Basics.

In this section we shall try to lay something of a foundation for the rest of the course. In the process we shall begin a review of rings and modules that will continue though most of these notes.

We shall assume that the reader has a good grasp of the basic notions and results of ring theory. In particular, we expect the reader to be comfortable with the ideas like ring homomorphism, isomorphism, (left/right) ideal, factor ring, the isomorphism theorems, center, etc. Nevertheless, often when we first encounter a concept, even though it should be familiar, we will try to pause briefly to review it. Frequently, such review will be embedded in the exercises.

For us “ring” will mean “ring with identity”; that is, an identity is part of the defining structure of the ring. Thus, if \( R \) is a ring and \( S \) is a subring of \( R \), then not only must \( S \) have an identity, but it must be the same as the identity of \( R \). Similarly, if \( R \) and \( S \) are rings with identities \( 1_R \) and \( 1_S \), respectively, then for a map \( \varphi : R \rightarrow S \) to be a ring homomorphism, we must have \( \varphi(1_R) = 1_S \); that is, all ring homomorphisms are “unital”.

An “algebra” is a ring with some additional structure. Let \( K \) be a commutative ring, let \( R \) be a ring, and let \( \gamma : K \rightarrow \text{Cen} \, R \) be a ring homomorphism from \( K \) into the center of \( R \). Then the system \( (R, K, \gamma) \) is a \( K \)-algebra. Different homomorphisms \( \gamma \) produce different \( K \)-algebras. Still we often suppress explicit mention of \( \gamma \) and talk about the \( K \)-algebra \( R \). In this case we also usually write, for each \( a \in K \) and \( x \in R \),

\[
ax = \gamma(a)x.
\]

If \( (R, K, \gamma) \) is a \( K \)-algebra and if \( S \) is a subring of \( R \), then \( (S, K, \gamma) \) is a subalgebra of \( (R, K, \gamma) \) in case \( \gamma(K) \subseteq S \). When this is the case we tend to refer to \( S \) itself as a subalgebra of \( R \). If \( (R, K, \gamma) \) and \( (S, K, \gamma') \) are two \( K \)-algebras, then a ring homomorphism \( \varphi : R \rightarrow S \) is an algebra homomorphism in case \( \varphi \circ \gamma = \gamma' \), or suppressing \( \gamma \) and \( \gamma' \), for all \( a \in K \) and \( x \in R \),

\[
\varphi(ax) = a\varphi(x).
\]

The kernel of such a homomorphism is an ideal of the underlying ring \( R \). On the other hand, given an ideal \( I \) of \( R \), there is a natural \( K \)-algebra structure on the factor ring \( R/I \) given by

\[
a(x + I) = (ax) + I.
\]

Thus, the kernels of \( K \)-algebra homomorphisms are precisely the kernels of ring homomorphisms of the underlying rings. So the ideals of the algebra \( (R, K, \gamma) \) are the ideals of \( R \) and the factor algebras are the factor rings of \( R \) with the above \( K \)-structure.

Strictly speaking, we could deal exclusively with algebras. Indeed, you will note that each ring \( R \) is uniquely an algebra over the ring \( \mathbb{Z} \) of integers. However, if we drop our insistence on an identity, then
we may be able to view a ring $R$ in more than one way as an algebra over $\mathbb{Z}$. Thus, in the interest of caution we shall continue to maintain the distinction between rings and algebras.

As we have seen repeatedly in the past, it is natural to represent groups as permutations on sets. There is an analogous representation theory for rings. Thus, let $M$ be an abelian group. Then the set $\text{End}(M)$

of all endomorphisms of $M$ is a ring under the usual operations. These endomorphism rings provide a rich source of rings. Indeed, as we shall see shortly, we can realize every ring as a subring of such an endomorphism ring. But we must be alert for parity; $M$ actually has two endomorphism rings

$$\text{End}^\ell(M) \quad \text{and} \quad \text{End}^r(M)$$

where in $\text{End}^\ell(M)$ we view each endomorphism as a left operator and in $\text{End}^r(M)$ we treat each endomorphism as a right operator. So in these two rings the operations are given by

In $\text{End}^\ell(M)$:

$$\begin{align*}
(f + g)(x) &= f(x) + g(x) \\
(fg)(x) &= f(g(x))
\end{align*}$$

and in $\text{End}^r(M)$:

$$\begin{align*}
(x)(f + g) &= (x)f + (x)g \\
(x)(fg) &= ((x)f)g
\end{align*}$$

Thus, it is clear that the map $f \mapsto f$ is an anti-isomorphism of $\text{End}^\ell(M)$ onto $\text{End}^r(M)$, so that

$$\text{End}^r(M) \cong (\text{End}^\ell(M))^{\text{op}}.$$  

Since the natural source for rings are the endomorphisms of abelian groups, the natural representation theory for a ring would be determined by its action as endomorphisms on some abelian group. Indeed, that is the approach that we take. Thus, let $R$ be a ring, let $M$ be an abelian group, and let $\lambda : R \to \text{End}^\ell(M)$ be a ring homomorphism. Then the pair $(M, \lambda)$ is a left $R$-representation of $R$. Different choices of $\lambda$ determine different representations, but we tend to suppress the mention of $\lambda$ and simply write $ax$ for $\lambda(a)(x)$. Thus, alternatively, we can characterize a left representation of $R$ as an abelian group $M$ together with a map $\mu : R \times M \to M$ abbreviated

$$\mu(a, x) = ax,$$

satisfying for all $a, b \in R$ and all $x, y \in M$,

$$\begin{align*}
a(x + y) &= ax + ay \\
(ab)x &= a(bx) \\
(a + b)x &= ax + bx \quad \text{and} \quad 1x = x.
\end{align*}$$

These, of course, are just the axioms for a left $R$-module. So the left representations of a ring $R$ are simply the left $R$-modules. For such a module we often refer to the map $\mu$ as scalar multiplication.
and to the ring \( R \) as the **scalar** ring of the module. As another bit of convenient shorthand we will often write just

\[ R^M \]

to indicate that \( M \) is a left \( R \)-module. This poses some small danger since a given abelian group \( M \) may admit many different left \( R \)-module structures, so we should not invoke this shorthand if there is any possibility of serious ambiguity.

A left representation \( \lambda : R \to \text{End}^l(M) \), or the corresponding module \( R^M \), is **faithful** if \( \lambda \) is injective. Equivalently \( R^M \) is faithful iff for each \( a \in R \)

\[ ax = 0 \text{ for all } x \in M \implies a = 0. \]

The left representations of a ring \( R \) are the left \( R \)-modules. There is a dual right-hand version. A pair \((M, \rho)\) consisting of an abelian group \( M \) and a ring homomorphism \( \rho : R \to \text{End}^r(M) \) is a **right representation** of \( R \) Again we often downplay the homomorphism \( \rho \) and write

\[ xa = (x)((a)\rho). \]

Such a right representation can be characterized as an abelian group \( M \) together with a map \( \nu : M \times R \to M \) abbreviated

\[ \nu(x, a) = xa, \]

satisfying for all \( a, b \in R \) and all \( x, y \in M \),

\[ (x + y)a = xa + ya \quad x(ab) = (xa)b \]
\[ x(a + b) = xa + xb \quad x1 = x; \]

that is, \( M \) is a **right \( R \)-module**. So the right representations of \( R \) are just the right \( R \)-modules. Our shorthand for a right \( R \)-module \( M \) is

\[ M_R. \]

For each ring \( R \) there are two special modules. Indeed, the maps \( \mu : R \times R \to R \) and \( \nu : R \times R \to R \) defined by

\[ \mu(a, x) = ax \quad \text{and} \quad \nu(x, a) = xa \]

define a left \( R \)-module and a right \( R \)-module on the abelian group \( R \). These are the **regular** \( R \)-modules. Here we allow no ambiguity. That is, the symbols

\[ R^R \quad \text{and} \quad R_R \]

are reserved to represent these regular \( R \)-modules.
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Note that the regular modules $R_R$ and $R_R$ are both faithful. Thus, we have a “Cayley Theorem” for rings: Every ring is isomorphic to a subring of the endomorphism ring of an abelian group.

The collection of all left representations of a ring $R$, that is, the collection of all left $R$-modules, forms a very rich and interesting category. Let $(M, \lambda)$ and $(M', \lambda')$ be two left $R$-modules. A group homomorphism $f : M \rightarrow M'$ is an $R$-homomorphism in case for each $a \in R$ the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\lambda(a)} & M \\
\downarrow{f} & & \downarrow{f} \\
M' & \xrightarrow{\lambda'(a)} & M'
\end{array}
$$

commutes. Thus, a map $f : M \rightarrow M'$ is an $R$-homomorphism, also called an $R$-linear map, iff for all $a, b \in R$ and all $x, y \in M$,

$$f(ax + by) = af(x) + bf(y).$$

The identity map on each $R_M$ is an $R$-homomorphism and compositions of $R$-homomorphisms are $R$-homomorphisms, so that one readily checks that the collection of all left $R$-modules together with all $R$-homomorphisms forms a category. We denote this category of left $R$-modules by $\text{RMod}$. In an entirely analogous fashion the collection of all right $R$-modules forms a category. Of course, if $M$ and $M'$ are right $R$-modules, then an $R$-homomorphism $f : M \rightarrow M'$ is characterized as a map $f : M \rightarrow M'$ such that for all $a, b \in R$ and all $x, y \in M$,

$$f(xa + yb) = f(x)a + f(y)b.$$ 

We denote this category of right $R$-modules by $\text{Mod}R$.

In general, the categories $\text{RMod}$ and $\text{Mod}R$ are quite different. Rings for which they are essentially the same (definitely a non-technical statement) often have some very strong properties. Certainly as an important special case we have that if $R$ is commutative (a very strong property), then $\text{RMod} = \text{Mod}R$.

The two categories $\text{RMod}$ and $\text{Mod}R$ have considerable structure and it is that, the representation theory of $R$, that will be the focus of much of this course. We will tend to concentrate on the category $\text{RMod}$ and will let you hold that up to the mirror to learn about the category $\text{Mod}R$. In this Section, we will look at just one of the unusual structural features of these module categories.
Let $M$ and $N$ be two left $R$-modules. Then the collection of all $R$-homomorphisms from $M$ to $N$ is a collection of functions from $M$ to $N$ and is, thus, a set. We denote this set by

$$\text{Hom}_R(M, N).$$

If there is no question about the ring $R$ of scalars, we may even omit one piece of decoration and denote this set by $\text{Hom}(M, N)$. Each of these hom sets comes with the structure of an abelian group. That is, if $f, g \in \text{Hom}_R(M, N)$, then we define $f + g : M \to N$ by

$$(f + g)(x) = f(x) + g(x) \quad (x \in M).$$

Then it is easily checked that $f + g$ is $R$-linear and that under the operation $(f, g) \mapsto f + g$ the set $\text{Hom}_R(M, N)$ is an abelian group. Moreover, if $M, M', M''$ are left $R$-modules, then composition

$$\circ : (g, f) \mapsto g \circ f$$

is a bilinear map $\text{Hom}(M', M'') \times \text{Hom}(M, M') \to \text{Hom}(M, M'')$. That is, for all $g, \overline{g} \in \text{Hom}(M', M'')$ and all $f, \overline{f} \in \text{Hom}(M, M')$,

$$(g + \overline{g})f = gf + \overline{g}f \quad \text{and} \quad g(f + \overline{f}) = gf + g\overline{f}.$$ 

For the record, we will state this formally in

**1.1. Theorem.** If $R$ is a ring, then in the category $R\text{Mod}$ (and in $\text{Mod}R$) each hom set $\text{Hom}_R(M, N)$ is an abelian group and composition is bilinear. In other words the module categories $R\text{Mod}$ and $\text{Mod}R$ are additive.

This has one particularly important consequence.

**1.2. Corollary.** If $R$ is a ring, then for each left (right) $R$-module $M$, the set $\text{Hom}_R(M, M)$ is a ring w.r.t. the operations of pointwise addition and composition.

If $M$ is an $R$-module, then an $R$-homomorphism $f : M \to M$ is an $R$-endomorphism. (A bijective endomorphism is an automorphism.) Thanks to Corollary 1.2 the set of all endomorphisms of $M$ is actually a ring. Now each $R$-endomorphism of $M$ is a group endomorphism, so can be viewed as residing in either $\text{End}^R(M)$ or $\text{End}^\ell(M)$. Here we adopt a very useful convention. If $R \cdot M$ is a left $R$-module, then its $R$-endomorphism ring is the subring

$$\text{End}(R \cdot M)$$

of $\text{End}^\ell(M)$ consisting of all $R$-endomorphisms of $R \cdot M$. Dually, if $M \cdot R$ is a right $R$-module, then its $R$-endomorphism ring is the subring

$$\text{End}(M \cdot R)$$
of \( \text{End}^R(M) \) consisting of all \( R \)-endomorphisms of \( M_R \). The point is that the \( R \)-endomorphism ring of a module \textit{operates on the side opposite the scalars}. So for a module \( R \cdot M \), the ring \( R \) acts on the left and the endomorphism ring \( \text{End}(R \cdot M) \) act on the right, and they act jointly as follows: for each \( r \in R \), \( x \in M \), and \( s \in \text{End}(R \cdot M) \)

\[(rx)s = r(xs).\]

This leads us to yet another important concept.

Suppose that \( R \) and \( S \) are two rings and that \( M \) is an abelian group that has structures \( R \cdot M \) and \( M \cdot S \) as a left \( R \) and a right \( S \) module. This then gives \( M \) the structure of an \((R,S)\)-\textbf{bimodule} in case for all \( r \in R \), \( x \in M \), and \( s \in S \)

\[(rx)s = r(xs).\]

If \( M \) is such an \((R,S)\)-bimodule, then we abbreviate that fact by \( R \cdot M \cdot S \).

The rub here is that \( M \) is an \((R,S)\) bimodule iff each \( s \in S \) acts as an \( R \)-endomorphism of \( R \cdot M \) and each \( r \in R \) acts as an \( S \)-endomorphism of \( M \cdot S \). Maybe we should state that formally:

\[1.3. \text{ Proposition.} \] Suppose that \( R \) and \( S \) are rings, and that an abelian group \( M \) is a left \( R \)-module and a right \( S \)-module under the actions \( \lambda : R \rightarrow \text{End}^R(M) \) and \( \rho : R \rightarrow \text{End}^S(M) \). Then \( R \cdot M \cdot S \) iff \( \lambda(r) \in \text{End}(M \cdot S) \) for all \( r \in R \) iff \( \rho(s) \in \text{End}(R \cdot M) \) for all \( s \in S \). \( \blacksquare \)

Of course, it follows immediately that if \( R \cdot M \) is a left \( R \)-module and \( N \cdot R \) is a right \( R \)-module, then there are natural bimodule structures

\[ R \cdot M \cdot \text{End}(R \cdot M) \quad \text{and} \quad \text{End}(M \cdot S) \cdot N \cdot R. \]

Let \( M \) and \( N \) be left \( R \)-modules. Then it is tempting to suspect that the abelian group \( \text{Hom}_R(M, N) \) is actually an \( R \)-module via a scalar multiplication \( (a, f) \mapsto af \) where

\[(af)(x) = a(f(x)).\]

But in general this does not work. (See Exercise 1.1.) However, as we shall now see, \( \text{Hom}_R(M, N) \) has a very rich module structure.
1.4. Proposition. Suppose that $R$, $S$ and $T$ are rings, and that $R_M$ and $R_N$ are bimodules. Then

$$\text{Hom}_R(R_M, R_N)$$

is an $(S, T)$ bimodule where for each $\varphi \in \text{Hom}_R(M, N)$ and each $m \in M$

$$(s\varphi t)(m) = (\varphi(ms))t$$

for all $s \in S$ and $t \in T$.

The proof of this is just a matter of checking that the action as stated works. So we shall omit the details. We might mention that there are other versions, and they are all simply consequences of the behavior of the Hom functors that we shall discuss in a later lecture. For now, though, we shall use this to prove a simple little fact that will be useful in the next couple of sections.

An element $e$ of a ring $R$ is **idempotent** in case $e^2 = e$. If $e$ is a non-zero idempotent of $R$, then the set

$$eRe = \{exe \mid x \in R\}$$

is, under the addition and multiplication of $R$, a ring; note, though, that unless $e = 1$, the ring $eRe$ is not a subring of $R$. Also, if $RM$ is an $R$-module, then

$$eM = \{ex \mid x \in M\}$$

is an $eRe$ module in the obvious way. Similarly, $Re$ is a right $eRe$ module. So thanks to Proposition 1.4 $\text{Hom}_R(Re, M)$ is a left $eRe$-module. Now we have the following important fact, a very special version of the Yoneda Lemma.

1.5. Proposition. Suppose that $e$ is a non-zero idempotent in the ring $R$. Then for each left $R$-module $RM$ right multiplication

$$\rho : eM \to \text{Hom}_R(Re, M)$$

is an $eRe$ module isomorphism.

**Proof.** Here the map $\rho$ satisfies, for all $x \in M$ and all $a \in R$

$$\rho(ex) : ae \mapsto aex.$$ 

So clearly, $\rho(ex) \in \text{Hom}_R(Re, M)$ and the map $\rho$ is additive. Then for each $ere \in eRe$

$$\rho(ere)(ae) = aerex = (aere)(ex) = \rho(ex)(aere) = ((ere)\rho(ex))(ae),$$
where the last equality is an application of Proposition 1.4. So \( \rho \) is an \( eRe \) homomorphism. Finally, one checks that \( \rho \) is surjective since for each \( \varphi \in \text{Hom}_R(Re, M) \)

\[
\varphi = \rho(e\varphi(e)),
\]

and \( \rho \) is injective since if \( ex \in \text{Ker} \rho \), then \( \rho(ex) : e \mapsto ex = 0. \)

As we claimed this is actually just a piece of a much more general and powerful result. The main ingredient missing here is that the isomorphism \( \rho \) is \textbf{natural}. In this case this means that if \( R \) and \( N \) are left \( R \)-modules and if \( f : M \rightarrow N \) is an \( R \)-homomorphism, then

\[
\xymatrix{ eM \ar[r]^{f|_{eM}} & N \ar[d]^{ho_N} \\
\text{Hom}(Re, M) \ar[r]^{f_*} & \text{Hom}(Re, N) \ar[u]_{\rho_M}
}
\]

commutes where \( f_*(\varphi) = f \circ \varphi \) for each \( \varphi \in \text{Hom}_R(Re, M) \). But this follows since for each \( x \in M \) and \( a \in R \)

\[
f_*(\rho_M(ex))(ae) = f \circ \rho_M(ex)(ae) = f(aex) = aef(x) = \rho_N(ef(x))(ae).
\]

---

**Exercises 1.**

1.1. Here is a simple example of a ring \( R \) and left \( R \)-modules \( M \) and \( N \) for which \( \text{Hom}_R(M, N) \) is not a left \( R \)-module. Let \( R \) be the ring of all polynomials in \( x \) and \( y \), non-commuting, over \( \mathbb{Z} \) of the form \( a + bx + cy + dyx \) subject to the relations \( x^2 = y^2 = xy = 0 \). [Alternatively, let \( \mathbb{Z}\langle x, y \rangle \) be the ring of all polynomials in \( x \) and \( y \), non-commuting indeterminants, let \( I = \langle x^2, y^2, xy \rangle \) be the ideal generated by \( x^2, y^2, \) and \( xy \), and let \( R = \mathbb{Z}\langle x, y \rangle/I \).] Let \( M = N = rR \) both be the regular left \( R \)-module. Then the identity function \( f : rR \rightarrow rR \) is a left \( R \)-homomorphism. But show that the map \( (yf) : a + bx + cy + dyx \mapsto ay + byx \) is not \( R \)-linear. [Hint: Show that \( (yf)(xa) \neq x(yf)(a) \).]

1.2. Let \( R \) be a ring and \( M \) a left \( R \)-module. A subgroup \( N \) of \( M \) is a \textbf{submodule} in case it is stable under the action of \( R \). We abbreviate the fact that \( N \) is a submodule of \( M \) by

\[
N \leq M \quad \text{or} \quad N \leq_R M
\]

to emphasize that \( N \) is an \( R \)-submodule. Then it is easy to see that the collection \( S(M) \) of all submodules of \( M \) is a poset (= partially ordered set) w.r.t. the relation \( \leq \) with least element the
zero submodule and with greatest element \( M \). Let \( \mathcal{A} = \{ M_\alpha \mid \alpha \in A \} \) be some set of submodules of \( M \). Then we let

\[
\sum \mathcal{A} = \sum \mathcal{A} M_\alpha = \bigcup \{ M_{\alpha_1} + \cdots + M_{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in A, n \in \mathbb{N} \}
\]

be the set of all finite sums from the union of the set \( \mathcal{A} \).

(a) Show that \( \mathcal{S}(M) \) is a complete lattice where for each set \( \mathcal{A} \) in \( \mathcal{S}(M) \)

\[
\text{glb } \mathcal{A} = \bigcap \mathcal{A} \quad \text{and} \quad \text{lub } \mathcal{A} = \sum \mathcal{A}.
\]

(b) Show that the lattice \( \mathcal{S}(M) \) is modular; that is, for \( H, K, L \leq M \)

\[
H \geq K \implies H \cap (K + L) = K + (H \cap L).
\]

(c) Show that \( \mathcal{S}(M) \) need not be distributive; that is, it need not satisfy

\[
H \cap (K + L) = H \cap K + H \cap L
\]

for all \( H, K, L \in \mathcal{S}(M) \).

1.3. Let \( M \) be a left \( R \)-module. If \( X \subseteq M \) is any subset of \( M \), then we set

\[
RX = \{ \sum_{i=1}^{n} a_i x_i \mid a_i \in R, x_i \in X \text{ and } n \in \mathbb{N} \},
\]

the set of all finite \( R \)-linear combinations of \( X \). (Note: The element 0 is the only linear combination of the empty set.) It is elementary (if tedious) to prove that \( RX \) is a submodule of \( M \). We say that \( RX \) is spanned or generated by \( X \) and that \( X \) is a set of generators or is a spanning set for \( RX \). We say that a module is finitely generated if it has a finite spanning set. It is cyclic if it is spanned by a single element. A module \( M \) is simple if its lattice of submodules consists of just 0 and \( M \).

(a) Prove that a module \( M \) is finitely generated iff for every set \( \mathcal{A} \) of submodules of \( M \) with

\[
\sum \mathcal{A} = M,
\]

there is a finite subset \( \mathcal{F} \subseteq \mathcal{A} \) with \( \sum \mathcal{F} = M \).

(b) Prove that a left \( R \)-module \( M \) is cyclic iff it is a factor of the regular module \( \mathbb{R}R \).

(c) Prove that every module \( M \) is the join (= glb) of its cyclic submodules.

(d) Prove that a module \( \mathbb{R}M \) is simple iff it is the factor of \( \mathbb{R}R \) module some maximal left ideal.

(e) Prove that if \( \mathbb{R}M \) is finitely generated, then \( M \) has a maximal submodule. [Yes, folks, Professor Zorn and the Axiom of Choice are welcome participants in this course.]
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1.4. A module $_RM$ is **Noetherian** in case the lattice $\mathcal{S}(M)$ satisfies the A.C.C. Prove that the following are equivalent for a module $M$:

(a) $M$ is Noetherian;
(b) Every submodule of $M$ is finitely generated;
(c) Every non-empty set in $\mathcal{S}(M)$ has a maximal element.

1.5. There is a perfect dual to what we found in the previous exercise. A module $_RM$ is **finitely cogenerated** in case for every set $\mathcal{A}$ of submodules of $M$ with $\bigcap \mathcal{A} = 0$, there is some finite set $\mathcal{F} \subseteq \mathcal{A}$ with $\bigcap \mathcal{F} = 0$. A module is **Artinian** in case its lattice $\mathcal{S}(M)$ of submodules satisfies the D.C.C. Prove that the following are equivalent for a module $M$:

(a) $M$ is Artinian;
(b) Every factor module of $M$ is finitely cogenerated;
(c) Every non-empty set in $\mathcal{S}(M)$ has a minimal element.

1.6. Recall that a category $\mathcal{C}$ is **additive** in case, first, that for each pair $(A, B)$ of objects in $\mathcal{A}$ the morphism set $\text{Mor}_A(A, B)$ comes equipped with the structure of an additive abelian group, and, second, that composition in the category is bilinear w.r.t. to these additions. Thus, for each object $A$ in such a category, the set $\text{Mor}_\mathcal{C}(A, A)$ is a ring. This gives us an entirely different characterization of rings. Indeed, we could define a ring simply to be an additive category with a single object. (Note that leads to a sweeping generalization of the notion of a ring, namely, a small additive category.) Let’s pursue this version of the notion of a ring a bit further; in particular, let’s see if we can figure out what modules and homomorphisms should be in this setting. For this we introduce the category $\mathbf{Ab}$ of all abelian groups. (Of course, this is just the category $\mathbb{Z}\text{Mod}$.) Let $R$ be a ring viewed as an additive category with a single object.

(a) Show that $M$ is a left $R$-module iff there is a covariant additive functor $F_M : R \rightarrow \mathbf{Ab}$ whose image lies in the full subcategory with object set $\{M\}$.

(b) Show that if $M$ and $N$ are left $R$-modules, then $f \in \text{Hom}_{\mathbf{Ab}}(M, N)$ is an $R$-homomorphism iff $f$ is a natural transformation of the functors $F_M$ to $F_N$.

(c) Show that $M$ is a right $R$-module iff there is a contravariant additive functor $F_M : R \rightarrow \mathbf{Ab}$ whose image lies in the full subcategory with object set $\{M\}$. 