11 Annihilators.

In this Section we take a brief look at the important notion of annihilators. Although we shall use these in only very limited contexts, we will give a fairly general initial treatment, since the latter is actually easier than the special cases and is readily applied in the special cases.

Suppose that R, S, and T are rings, that ${}_RP_S$, ${}_SQ_T$, and ${}_RU_T$ are bimodules, and finally, that

$$\mu : P \otimes_S Q \longrightarrow U$$

is an (R, T) bimodule homomorphism. Then for each set $X \subseteq Q$ its **left annihilator** (in P) is

$$\mathbf{l}_P(X) = \{p \in P \ : \ \mu(p \otimes x) = 0 \ \forall x \in X\}.$$

For each $A \subseteq P$, its **right annihilator** (in Q) is

$$\mathbf{r}_Q(A) = \{q \in Q \ : \ \mu(a \otimes q) = 0 \ \forall a \in A\}.$$

It is clear that $\mathbf{l}_P(X)$ is a submodule of ${}_RP$ and that $\mathbf{r}_Q(A)$ is a submodule of Q_T. Thus, $\mathbf{l}_P$ defines a function from the power set of Q to the set of submodules of ${}_RP$, and $\mathbf{r}_Q$ defines a function from the power set of P to the set of submodules of Q_T.

Example. Our two principal examples of this will occur when R and S are rings and ${}_RM_S$ is a bimodule. The module multiplication $\mu : R \otimes_R M \longrightarrow M$ by $\mu : (r \otimes x) \longmapsto rx$ is an (R, S) bimodule homomorphism. Then the annihilators $\mathbf{l}_R$ and $\mathbf{r}_M$ are maps from the subsets of M to the left ideals of R and from the subsets of R to the submodules of M_S, respectively.

On the other side, module multiplication $\mu : M \otimes_S S \longrightarrow M$ via $\mu : x \otimes s \longmapsto xs$ is an (R, S) homomorphism. So here the annihilators $\mathbf{l}_M$ and $\mathbf{r}_S$ are maps from the subsets of S to the submodules of ${}_RM$ and from the subsets of M to the right ideals of S, respectively. ■

Returning to the general case, our immediate interest is in the behavior of the annihilator functions $\mathbf{l}_P$ and $\mathbf{r}_Q$. We begin with the easy and very basic

11.1. Lemma. *For an (R, T) bimodule homomorphism $\mu : {}_R P \otimes_S Q_T \longrightarrow U$ the annihilators $\mathbf{l}_P$ and $\mathbf{r}_Q$ satisfy for all $X, Y \subseteq Q$ and $A, B \subseteq P$,*

(1) $X \subseteq Y \Longrightarrow \mathbf{l}_P(X) \supseteq \mathbf{l}_P(Y)$, *and* $A \subseteq B \Longrightarrow \mathbf{r}_Q(A) \supseteq \mathbf{r}_Q(B)$;

(2) $X \subseteq SXT \subseteq \mathbf{r}_Q\mathbf{l}_P(X)$, *and* $A \subseteq RAS \subseteq \mathbf{l}_P\mathbf{r}_Q(A)$;

(3) $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) = \mathbf{l}_P(X)$, *and* $\mathbf{r}_Q\mathbf{l}_P\mathbf{r}_Q(A) = \mathbf{r}_Q(A)$.

Proof. The first two are trivial. For (3) from (1) applied to (2) we have that $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) \subseteq \mathbf{l}_P(X)$. But the second part of (2) applied to $\mathbf{l}_P(X)$ gives $\mathbf{l}_P\mathbf{r}_Q\mathbf{l}_P(X) \supseteq \mathbf{l}_P(X)$. ■

11.2. Lemma. *For an (R,T) bimodule homomorphism $\mu :_R P \otimes_S Q_T \longrightarrow U$ the annihilators $\mathbf{l}_P$ and $\mathbf{r}_Q$ satisfy for all indexed sets of additive subgroups $(P_\alpha)_{\alpha\in\Omega}$ in P and $(Q_\alpha)_{\alpha\in\Omega}$ in Q,*

(1) $\mathbf{l}_P(\sum_\Omega Q_\alpha) = \bigcap_\Omega \mathbf{l}_P(Q_\alpha)$ *and* $\mathbf{r}_Q(\sum_\Omega P_\alpha) = \bigcap_\Omega \mathbf{r}_Q(P_\alpha)$;

(2) $\sum_\Omega \mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$, *and* $\sum_\Omega \mathbf{r}_Q(P_\alpha) \subseteq \mathbf{r}_Q(\bigcap_\Omega P_\alpha)$.

Proof. (1) Since $Q_\beta \subseteq \sum_\Omega Q_\alpha$ for all β, we have from Lemma 11.1 that $\mathbf{l}_P(\sum_\Omega Q_\alpha) \subseteq \bigcap_\Omega \mathbf{l}_P(Q_\alpha)$. But trivially $\bigcap_\Omega \mathbf{l}_P(Q_\alpha)$ annihilates every Q_α and hence the sum $\sum_\Omega(Q_\alpha)$.

(2) By Lemma 11.1 we certainly have $\mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$ for every α. But $\mathbf{l}_P(\bigcap_\Omega Q_\alpha)$ and $\mathbf{l}_P(Q_\alpha)$ are submodules of $_RP$, so $\sum_\Omega \mathbf{l}_P(Q_\alpha) \subseteq \mathbf{l}_P(\bigcap_\Omega Q_\alpha)$. ■

Thus, from these Lemmas we see that the functions $\mathbf{l}_P\mathbf{r}_Q$ and $\mathbf{r}_Q\mathbf{l}_R$ define closure operators on the sets P and Q, and in each case the closure of any set is a submodule. In general, not all submodules of $_RP$ or Q_T are closed under these operators. Still given the set up where $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ is a bimodule homomorphism, we set

$$\mathcal{L}_P(Q) = \{\mathbf{l}_P(X) \ : \ X \subseteq Q\},$$

and

$$\mathcal{R}_Q(P) = \{\mathbf{r}_Q(A) \ : \ A \subseteq P\}.$$

Thus, $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are the "closed" submodules of $_RP$ and Q_T, respectively. Of course, these sets are subposets of the sets of all submodules of $_RP$ and Q_T. Clearly, $P \in \mathcal{L}_P(Q)$ and $Q \in \mathcal{R}_Q(P)$, and by Lemma 11.2, $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are both closed under arbitrary intersections. Thus, we infer that

11.3. Proposition. *The posets $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are complete lattices and the pair of maps $\mathbf{r}_Q : \mathbf{l}_P(X) \longmapsto \mathbf{r}_Q(\mathbf{l}_P(X))$ and $\mathbf{l}_P : \mathbf{r}_Q(A) \longmapsto \mathbf{l}_P(\mathbf{r}_Q(A))$ are lattice anti-isomorphisms between $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$.* ■

11.4. Corollary. *The lattice $\mathcal{L}_P(Q)$ satisfies the Maximum (Minimum) Condition iff the lattice $\mathcal{R}_Q(P)$ satisfies the Minimum (Maximum) Condition.* ■

Given a bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ we call the modules in $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ the **annihilator modules** for μ. (Frequently, when the map μ is understood, we suppress mention of it.) Of particular interest are the cases where all submodules of $_RP$ and of Q_T are annihilator modules. That would mean that every submodule $M \leq_R P$ and $N \leq Q_T$ would satisfy $M = \mathbf{l}_R(X)$ and $N = \mathbf{r}_Q(A)$ for some $X \subseteq Q$ and $A \subseteq Q$. But then by Lemma 11.1 part (3),

11.5. Corollary. *Every submodule of* ${}_RP$ *and of* Q_T *is an annihilator module iff for each* $M \leq_R P$ *and every* $N \leq Q_T$

$$\mathbf{l}_P\mathbf{r}_Q(M) = M \qquad \textit{and} \qquad \mathbf{r}_Q\mathbf{l}_P(N) = N. \qquad \blacksquare$$

We say that the bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ satisfies the **double annihilator property** if the conditions of Corollary 11.5 are satisfied. Some very good things happen when we have the double annihilator property. We won't go into most of them here, but next we will show how that property tightens up the second part of Lemma 11.2

11.6. Corollary. *If the bimodule homomorphism* $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ *satisfies the double annihilator property, then for all indexed sets of additive subgroups* $(P_\alpha)_{\alpha\in\Omega}$ *in* P *and* $(Q_\alpha)_{\alpha\in\Omega}$ *in* Q*,*

$$\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \mathbf{l}_P(\bigcap\nolimits_\Omega Q_\alpha), \quad \textit{and} \quad \sum\nolimits_\Omega \mathbf{r}_Q(P_\alpha) = \mathbf{r}_Q(\bigcap\nolimits_\Omega P_\alpha).$$

Proof. By (1) of Lemma 11.2 and the fact that $\mathbf{r}_Q\mathbf{l}_P(Q_\alpha) = Q_\alpha$, we have

$$\mathbf{r}_Q(\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \bigcap\nolimits_\Omega \mathbf{r}_Q\mathbf{l}_P(Q_\alpha) = \bigcap\nolimits_\Omega Q_\alpha.$$

So from the double annihilator property

$$\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha) = \mathbf{l}_P\mathbf{r}_Q\left(\sum\nolimits_\Omega \mathbf{l}_P(Q_\alpha)\right) = \mathbf{l}_P\left(\bigcap\nolimits_\Omega Q_\alpha\right). \qquad \blacksquare$$

Here is an important, and very possibly familiar, example of annihilators at play. Let R and S be rings and let ${}_RU_S$ be a bimodule. For each left R-module ${}_RM$ and each right S-module N_S their U-**duals** are the right S-module M^*_S and left R-module ${}_RN^*$ defined by

$$M^* = \operatorname{Hom}_R(M, U) \qquad \text{and} \qquad N^* = \operatorname{Hom}_S(N, U).$$

Here we want to be careful about how these homomorphisms operate. Thus, we want the elements of M^* to **act on the right** and those of N^* to **act on the left**. So we have bimodule homomorphisms

$$\mu : M \otimes_{\mathbb{Z}} M^* \longrightarrow {}_RU_S \qquad \text{and} \qquad \nu : N^* \otimes_{\mathbb{Z}} N \longrightarrow {}_RU_S$$

defined by

$$\mu : x \otimes f \longmapsto (x)f \qquad \text{and} \qquad \nu : g \otimes y \longmapsto g(y).$$

So each of these bimodule homomorphisms generates a pair of annihilator maps. For example, for the first, if $X \subseteq M$ and $A \subseteq M^*$, then

$$\mathbf{r}_{M^*}(X) = \{f \in M^* \ : \ X \subseteq \operatorname{Ker} f\} \qquad \text{and} \qquad \mathbf{l}_M(A) = \cap\{\operatorname{Ker} f \ : f \in A\}.$$

A problem of some significance is to determine those submodules of ${}_RM$ (M^*_S) that are annihilators. We'll look at this some more in the Exercises. On the other hand the special case of this set up that we're probably most familiar with is when $R = S$ is a field and $U = R$ is the regular bimodule. Then for each ${}_RM$, an R-vector space, M^* is just its usual dual of all linear functionals $f : M \longrightarrow R$ on M. In that special case every submodule of ${}_RM$ is a left annihilator.

As we mentioned in another earlier Example a particularly common example of all of this arises with a bimodule ${}_RM_S$ and the scalar multiplication map

$$\mu : R \otimes_R M \longrightarrow M.$$

In this case for each $X \subseteq M$ and $A \subseteq R$,

$$\mathbf{l}_R(X) = \{r \in R \ : \ rx = 0 \ \forall x \in X\} \qquad \text{and} \qquad \mathbf{r}_M(A) = \{x \in M \ : \ ax = 0 \ \forall a \in A\}.$$

So $\mathbf{l}_R(X)$ is a left ideal of R and $\mathbf{r}_M(A)$ is a right S submodule of M. Of course, if X is actually a submodule of M, then $\mathbf{l}_R(X)$ is an ideal of R, and if A is a right ideal of R, then $\mathbf{r}_M(A)$ is an (R, S) sub-bimodule of M. The left ideals in $\mathcal{L}_R(M)$ are often called M-**annihilator left ideals** and the S-submodules in $\mathcal{R}_M(R)$ are **annihilator** S-**submodules** of M.

For this last example of a bimodule ${}_RM_S$, there is a related notion and notation that is extremely useful. Thus, let X and Y be subsets of M. We then set

$$(X : Y) = \{r \in R \ : \ rY \subseteq X\}.$$

So one has that

$$(X : Y) \subseteq R \qquad \text{and} \qquad (X : Y)Y \subseteq X.$$

The case of most interest is when $N \leq M$ is an R-submodule of M, so that $(N : X)$ is a left ideal of R. In fact $(N : X)$ is the annihilator

$$(N : X) = \mathbf{l}_R(X + N/N)$$

in R of the subset $(N + X)/N$ of the R-module M/N. Thus, one often refers to $(N : X)$ as a **relative annihilator**. Finally, we note that for each $X \subseteq M$, the left ideal $(0 : X)$ is just the left annihilator of X and is also equal to the intersection of the annihilators of the $x \in X$. So for each $x \in M$ and $X \subseteq M$,

$$(0 : x) = \mathbf{l}_R(x) \qquad \text{and} \qquad (0 : X) = \mathbf{l}_R(X) = \bigcap_{x \in X} \mathbf{l}_R(x).$$

There is one particularly important special case of this last special case, the one where $M = R$. Here $\mu : R \otimes R \longrightarrow R$ is just multiplication in R. And in this case, for each $X \subseteq R$

$$\mathbf{l}_R(X) = \{r \in R \ : \ rX = 0\} \qquad \text{and} \qquad \mathbf{r}_R(X) = \{r \in R \ : \ Xr = 0\}.$$

The left ideals in $\mathcal{L}_R(R)$ are called the **annihilator left ideals** of R and the right ideals in $\mathcal{R}_R(R)$ are the **annihilator right ideals** of R.

We conclude with one simple instance in which we refer to annihilator ideals. Recall that a one-sided ideal I of R is **nilpotent** in there is some $n \in \mathbb{N}$ with $I^n = 0$. The one sided ideal I is **nil** if each element of I is nilpotent. Of course, every nilpotent one sided ideal is nil, but the converse fails. But here is one case in which they are the same.

11.7. Theorem. *Let R be a ring with no non-zero nilpotent ideals. If $\mathcal{L}_R(R)$, the lattice of annihilator left ideals, satisfies the Maximum Condition, then R has no non-zero nil one-sided ideals.*

Proof. If I is a non-zero nil left ideal, then for each $x \in I$ the right ideal xR is nil. So it will suffice to show that there is no non-zero nil principal right ideal xR. Suppose, on the contrary, a non-zero right ideal xR is nil. By hypothesis the set

$$\{\mathbf{l}_R(xa) \ : \ 0 \neq xa \in xR\}$$

has a maximal element, say $\mathbf{l}_R(xa)$. Let $y = xa$. We claim that RyR is nilpotent, so that $y = 0$ contrary to $y \neq 0$. Suppose then that $b \notin \mathbf{r}_R(y)$, so that $yb \neq 0$. Now $y \in xR$, so yb is nilpotent, so there is a least $n \in \mathbb{N}$ with $(yb)^n = 0$. Of course, $n > 1$, so $(yb)^{n-1} \neq 0$. But clearly, $\mathbf{l}_R(y) \leq \mathbf{l}_R((yb)^{n-1})$, so by the maximality of $\mathbf{l}_R(y)$, we have $\mathbf{l}_R(y) = \mathbf{l}_R((yb)^{n-1})$. So $yb \in \mathbf{l}_R((yb)^{n-1}) = \mathbf{l}_R(y)$, and $yby = 0$. But that means that $yRy = 0$, so that RyR is nilpotent, and $y = 0$. ∎

Exercises 11.

11.1. Prove that the inclusion in Lemma 11.2 part (2) can be strict. [Hint: Consider the ring $R = \mathbb{Z}[x, y]/(xy)$. Let ${}_RM = {}_RR$ be the regular module, and consider the submodules Rx and Ry of M. Finally compute annihilators w.r.t. the multiplication map $\mu : R \otimes_R M \longrightarrow M$.]

11.2. Given a bimodule homomorphism $\mu : {}_RP \otimes_S Q_T \longrightarrow {}_RU_T$ we know that $\mathcal{L}_P(Q)$ and $\mathcal{R}_Q(P)$ are complete lattices. (See Proposition 11.3.) Moreover, we know, for example, that in each case the greatest lower bound of any set of submodules is the intersection of the set.

(a) Let $\mathcal{A} \subseteq \mathcal{L}_P(Q)$. Find a formula for the least upper bound of $\mathcal{A}$.

(b) Show that $\mathcal{L}_P(Q)$ need not be a sublattice of the lattice of submodules of P.

11.3. Consider a bimodule ${}_RU_S$ and a module ${}_RM$. Compute annihilators w.r.t. the bimodule homomorphism $\mu : M \otimes_{\mathbb{Z}} M^* \longrightarrow U$ where $M^* = \mathrm{Hom}_R(M, U)$ is the U-dual of M. Prove that a submodule N of ${}_RM$ is in the lattice $\mathcal{L}_M(M^*)$ iff U cogenerates M/N.

11.4. Prove that if R is a ring for which the regular module ${}_RR$ cogenerates all left R modules, then ${}_RR$ is injective. [Hint: Let $E = E({}_RR)$ be the injective envelope of ${}_RR$. Since ${}_RR$ cogenerates E, there is a monomorphism $\varphi : E \longrightarrow R^{\Omega}$ for some set Ω. For each $\alpha \in \Omega$ let $e_\alpha = \pi_\alpha(1)$ where the $\pi_\alpha : R^{\Omega} \longrightarrow R$ are the coordinate projections. Let I be the right ideal $I = \sum e_\alpha R$. Then $\mathbf{l}_R(I) = 0$ so $I = R$, and $1 = \sum e_\alpha a_\alpha$. Then the map $x \longmapsto \sum \pi_\alpha(x)e_\alpha$ is a split epi, and ${}_RR$ is injective.]

11.5. A ring R is a **<u>cogenerator</u> <u>ring</u>** in case ${}_RR$ cogenerates all left R-modules and R_R cogenerates all right R-modules. By Exercise 11.4 if R is a cogenerator ring, then ${}_RR$ and R_R are both injective. Prove that R is a cogenerator ring iff ${}_RR$ and R_R are both injective and R satisfies the double annihilator property in the sense that for every left ideal I and every right ideal J,

$$\mathbf{l}_R\mathbf{r}_R(I) = I \qquad \text{and} \qquad \mathbf{r}_R\mathbf{l}_R(J) = J.$$

11.6. The type of duality that is given by pairs of annihilator maps $\mathbf{l}_P$ and $\mathbf{r}_Q$ are almost densely distributed in mathematics albeit in many different guises. Such pairings are at the heart of Galois theory and algebraic geometry. Here is another one that occurs in elementary analysis. Let X be a compact Hausdorff space and let $C = C(X, \mathbb{R})$ be the ring of all continuous functions $f : X \longrightarrow \mathbb{R}$. For each set $A \subseteq C$ and each set $Y \subseteq X$ let

$$\mathbf{r}_X(A) = \{x \in X \;:\; f(x) = 0 \;\forall f \in A\} \qquad \text{and} \qquad \mathbf{l}_C(Y) = \{f \in C \;:\; f(x) = 0 \;\forall x \in Y\}.$$

You might check that these "annihilators" satisfy the properties of Lemma 11.1. Then prove that

(a) A subset $Y \subseteq X$ is closed iff it is an annihilator; that is, iff $Y = \mathbf{r}_X(A)$ for some $A \subseteq C$;

(b) A subset $I \subseteq C$ is an ideal iff it is an annihilator; that is iff $I = \mathbf{l}_C(Y)$ for some $Y \subseteq X$.

(c) For every $Y \subseteq X$, its closure is $\mathbf{r}_X\mathbf{l}_C(Y)$ and for each $A \subseteq C$ the ideal generated by A is $\mathbf{l}_C\mathbf{r}_X(A)$.

(d) Let $\mathcal{M}$ be the set of all maximal ideals of C. For each subset $\mathcal{I} \subseteq \mathcal{M}$ let

$$\overline{\mathcal{I}} = \{M \in \mathcal{M} \;:\; \cap\mathcal{I} \subseteq M\}.$$

Prove that $\mathcal{I} \longmapsto \overline{\mathcal{I}}$ defines a closure operator on $\mathcal{M}$ and that in the resulting topology $\mathcal{M}$ is homeomorphic to X.