2 Splitting.

Decompositions of rings and modules play a very important role in the study of these objects, and so in this Section we shall review briefly what’s involved in direct decompositions of modules. More on the topic can be found in the exercises. Also, en route to our study of decompositions we will give a brief treatment of products and coproducts in the setting of module theory — a more general treatment can be found in the exercises.

Throughout this Section $R$ will denote an arbitrary ring. Let $(M_\alpha)_{\alpha \in \Omega}$ be an indexed set of $R$-modules; we will assume here that they are left modules. Let $P$ and $C$ be $R$-modules and let $(p_\alpha)_{\alpha \in \Omega}$ and $(i_\alpha)_{\alpha \in \Omega}$ be $R$ homomorphisms $p_\alpha : P \to M_\alpha$ and $i_\alpha : M_\alpha \to C$ for each $\alpha \in \Omega$. We say that the pair $(P; (p_\alpha)_{\alpha \in \Omega})$ is a product of $(M_\alpha)_{\alpha \in \Omega}$ if for every module $M$ and all homomorphisms $q_\alpha : M \to M_\alpha$ (\(\alpha \in \Omega\)), there is a unique homomorphism $f : M \to P$ so that the left diagram below commutes. Dually, we say that the pair $(C; (i_\alpha)_{\alpha \in \Omega})$ is a coproduct of $(M_\alpha)_{\alpha \in \Omega}$ if for every module $M$ and all homomorphisms $q_\alpha : M_\alpha \to M$ (\(\alpha \in \Omega\)), there is a unique homomorphism $f : C \to M$ so that the right diagram below commutes.

Clearly, if such a product or coproduct exists, then it is unique to within isomorphism. We now construct both a product and a coproduct for the modules $(M_\alpha)_{\alpha \in \Omega}$.

First, let

$$P = \prod_{\alpha \in \Omega} M_\alpha$$

be the Cartesian product of the modules $M_\alpha$ equipped with coordinatewise addition and scalar multiplication. Then clearly, $P$ is an $R$-module, and the coordinate projections $\pi_\alpha : P \to M_\alpha$ are module epimorphisms. Moreover, for each $\alpha \in \Omega$ we define $\iota_\alpha : M_\alpha \to P$ by

$$\pi_\beta \iota_\alpha(x) = \begin{cases} x, & \text{if } \beta = \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Then each $\iota_\alpha$ is an $R$-monomorphism with the property that

$$\pi_\alpha \iota_\alpha = 1_{M_\alpha}$$

---

1 The symbol $\iota$ is the lower case Greek iota with no dot as opposed to the Latin letter $i$ with a dot.
for each \( \alpha \in \Omega \). Next, we set
\[
\prod_{\Omega} M_\alpha = \sum_{\Omega} \iota_\alpha(M_\alpha) \leq \prod_{\Omega} M_\alpha.
\]
We leave the easy proof of the following Theorem to the reader.

2.1. Theorem. The pair \( (\prod_{\Omega} M_\alpha, (\pi_\alpha)_{\alpha \in \Omega}) \) is a product and the pair \( (\prod_{\Omega} M_\alpha, (\iota_\alpha)_{\alpha \in \Omega}) \) is a coproduct of \( (M_\alpha)_{\alpha \in \Omega} \). \hfill \blacksquare

We tend to refer to \( \prod_{\Omega} M_\alpha \) and \( \prod_{\Omega} M_\alpha \) as the product and the coproduct of \( (M_\alpha)_{\alpha \in \Omega} \) suppressing the use of the coordinate maps. Also, if \( M \) is a module and if \( M_\alpha = M \) for all \( \alpha \in \Omega \), then we may abbreviate
\[
M^\Omega = \prod_{\Omega} M_\alpha \quad \text{and} \quad M^{(\Omega)} = \prod_{\Omega} M_\alpha.
\]
You should note that \( M^\Omega \) is the \( R \) module of all functions from \( \Omega \) to \( M \) with coordinatewise operations, and that \( M^{(\Omega)} \) the submodule of \( M^\Omega \) of all functions \( \Omega \rightarrow M \) with finite support.

Suppose that \( M \) is a module and that there are homomorphisms
\[
f_\alpha : M \rightarrow M_\alpha \quad \text{and} \quad g_\alpha : M_\alpha \rightarrow M.
\]
Then there exists unique homomorphisms \( f : M \rightarrow \prod_{\Omega} M_\alpha \) and \( g : \prod_{\Omega} M_\alpha \rightarrow M \) such that
\[
f \circ \pi_\alpha = f_\alpha \quad \text{and} \quad \iota_\alpha \circ g = g_\alpha
\]
for all \( \alpha \in \Omega \). We call the map \( f \) the product of the maps \( (f_\alpha) \) and the map \( g \) the coproduct of the maps \( (g_\alpha) \). Moreover, these will be denoted by
\[
f = \prod_{\Omega} f_\alpha \quad \text{and} \quad g = \prod_{\Omega} g_\alpha.
\]

2.2. Corollary.
\[
\operatorname{Ker} \prod_{\Omega} f_\alpha = \bigcap_{\Omega} \operatorname{Ker} f_\alpha \quad \text{and} \quad \operatorname{Im} \prod_{\Omega} f_\alpha = \sum_{\Omega} \operatorname{Im} f_\alpha. \hfill \blacksquare
\]

Now for each \( x \in \prod_{\Omega} M_\alpha \) we have \( \pi_\alpha(x) = 0 \) for almost all values of \( \alpha \). Thus,
\[
\sum_{\Omega} \iota_\alpha \pi_\alpha : \prod_{\Omega} M_\alpha \rightarrow \prod_{\Omega} M_\alpha
\]
is an endomorphism of \( \prod_{\Omega} M_\alpha \). In fact, we have

2.3. Proposition. The coordinate maps of the coproduct \( \prod_{\Omega} M_\alpha \) satisfy for all \( \alpha, \beta \in \Omega \)
\[
\pi_\alpha \iota_\beta = \begin{cases} 1_{M_\alpha}, & \text{if } \beta = \alpha; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{\Omega} \iota_\alpha \pi_\alpha = 1_{\prod_{\Omega} M_\alpha}. \hfill \blacksquare
A particularly important case occurs with the coproducts of the regular modules $R_R$ and $R_R$. These are the free $R$-modules. For example, the left $R$-module $R^{(\Omega)}$ is the free left $R$-module on $\Omega$. These free modules generate all modules in the following sense. Let $R^M$ have generating set $X$, so that $M = RX$. For each $x \in X$ right multiplication $\rho_x : R \rightarrow M$ defined by $\rho_x : a \mapsto ax$ is an $R$-homomorphism with $\text{Im} \rho_x = Rx$. Thus, the coproduct

$$\prod_x \rho_x : R^{(X)} \rightarrow M$$

is an $R$-homomorphism from the free module $R^{(X)}$ to $M$ with image $\sum_X Rx = M$. Thus, we have proved

2.4. Corollary. If $M$ is an $R$-module, then $M$ is a factor of a free module.

Next, suppose that each $M_\alpha$ is a submodule of some module $M$. Then for each $\alpha$ let $i_\alpha : M_\alpha \rightarrow M$ be the inclusion map. Then the coproduct of the inclusion maps is an $R$-homomorphism

$$\prod_\Omega i_\alpha : \prod_\Omega M_\alpha \rightarrow M.$$ 

Note that an element of the coproduct $\prod_\Omega M_\alpha$ is just a vector $x = (x_\alpha)_{\alpha \in \Omega}$ with $x_\alpha \in M_\alpha$ equal to zero for almost all $\alpha$. So the coproduct of the inclusion maps is characterized by

$$\prod_\Omega i_\alpha : x \mapsto \sum_\Omega x_\alpha.$$

Our immediate goal is to determine conditions under which we can guarantee that $(M, (i_\alpha)_{\alpha \in \Omega})$ is a coproduct of the submodules $(M_\alpha)_{\alpha \in \Omega}$. The solution is easy and, we assume, well known.

2.5. Theorem. For submodules $M_\alpha \leq M$ with inclusion maps $i_\alpha : M_\alpha \rightarrow M$ for all $\alpha \in \Omega$, the following statements are equivalent:

(a) $(M, (i_\alpha)_{\alpha \in \Omega})$ is a coproduct of the submodules $(M_\alpha)_{\alpha \in \Omega}$;

(b) The coproduct $\prod_\Omega i_\alpha$ of the inclusion maps is an isomorphism;

(c) For each $x \in M$ there exist unique $x_\alpha \in M_\alpha$ with $x_\alpha = 0$ for almost all $\alpha \in \Omega$ such that $x = \sum_\Omega x_\alpha$;

(d) $M = \sum_\Omega M_\alpha$ and for each $\alpha_0, \alpha_1, \ldots, \alpha_n$ distinct in $\Omega$

$$M_{\alpha_0} \cap (M_{\alpha_1} + \cdots + M_{\alpha_n}) = 0.$$

We say that the submodules $(M_\alpha)_{\alpha \in \Omega}$ of $M$ are independent if they satisfy the second condition of part (d) in Theorem 2.5. If the submodules $(M_\alpha)_{\alpha \in \Omega}$ of $M$ satisfy any (and hence all) of the conditions
of Theorem 2.5, then we say that $M$ is the **direct sum** of its submodules $(M_{\alpha})_{\alpha \in \Omega}$, and we denote this by

$$\bigoplus_{\Omega} M_{\alpha},$$

and in the finite case if $\Omega = \{1, 2, \ldots, n\}$, then by

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

It is this finite case that will be of greatest interest to us in this course. Using Proposition 2.3 and Theorem 2.5 it is easy to obtain a valuable characterization of such decompositions in terms of idempotent endomorphisms.

Recall that an element $e \in R$ is **idempotent** in case $e^2 = e$. A set $\{e_{\alpha} \mid \alpha \in \Omega\}$ of idempotents is **(pairwise) orthogonal** in case $e_{\alpha}e_{\beta} = 0$ for all $\alpha \neq \beta$ in $\Omega$.

### 2.6. Theorem

Let $M_1, \ldots, M_n$ be submodules of the module $RM$. Then

$$M = M_1 \oplus \cdots \oplus M_n$$

iff there exist pairwise orthogonal idempotents $e_1, \ldots, e_n \in \text{End}(RM)$ such that $M_i = Me_i$ for all $i = 1, \ldots, n$ and

$$e_1 + \cdots + e_n = 1.$$

**Proof.** ($\Rightarrow$) Let $i_1, \ldots, i_n$ be the inclusion maps for $M_1, \ldots, M_n$ in $M$, and let

$$f = \prod_{k=1}^{n} i_k : \prod_{k=1}^{n} M_k \rightarrow M.$$

By Theorem 2.5, $f$ is an isomorphism. Then by Proposition 2.3 for each $k = 1, \ldots, n$ the map

$$e_k = f i_k \pi_k f^{-1} : M \rightarrow M$$

satisfies for all $j$, $\text{Im} e_k = M_k$, $e_k \circ e_j = \delta_{jk} e_k$ and $e_1 + \cdots + e_n = 1_M$. So letting each $e_k$ operate on the right, the $e_1, \ldots, e_n$ are idempotent endomorphisms satisfying the conditions of the Theorem.

($\Leftarrow$) Since $e_1 + \cdots + e_n = 1_M$ we have that $M = M_1 + \cdots + M_n$. Since the $e_k$ are orthogonal, we have that $Me_j e_k = \delta_{jk} M_k$, whence $M_1, \ldots, M_n$ are independent. Thus, an appeal to Theorem 2.5 completes the proof. $lacksquare$

### 2.7. Corollary

Let $I_1, \ldots, I_n$ be left ideals of $R$. Then

$$RR = I_1 \oplus \cdots \oplus I_n$$

iff there exist pairwise orthogonal idempotents $e_1, \ldots, e_n \in R$ with $e_1 + \cdots + e_n = 1$ and $I_k = Re_k$ for $k = 1, \ldots, n$. 


Proof. This follows from Theorem 2.6 using the fact that right multiplication is a ring isomorphism \( \rho: R \rightarrow \text{End}(R R) \).

Another particularly useful corollary is

2.8. Corollary. Let \( R \) and \( S \) be rings and let \( F: R \text{Mod} \rightarrow S \text{Mod} \) be an additive functor. If \( _RM \) is a module and \( e_1, \ldots, e_n \in \text{End}(R M) \) are orthogonal idempotents with \( e_1 + \cdots + e_n = 1_M \), then \( F(e_1), \ldots, F(e_n) \in \text{End}(SF(M)) \) are orthogonal idempotents with \( F(e_1) + \cdots + F(e_n) = 1_{F(M)} \). In particular,

\[
F(M) = F(M)F(e_1) \oplus \cdots \oplus F(M)F(e_n).
\]

Let \( M \) be a module and let \( N \leq M \) be a submodule. Then \( N \) is a **direct summand** of \( M \) in case there exists some submodule \( K \leq M \) with

\[
M = N \oplus K.
\]

If such a submodule \( K \) exists, it is called a **direct complement** of \( N \) in \( M \).

If \( N \) is a direct summand of \( M \) with direct complement \( K \), then there exists an epimorphism \( f : M \rightarrow N \) with kernel \( K \) and a homomorphism \( g : N \rightarrow M \) with \( fg = 1_N \). This latter condition alone characterizes direct summands.

2.9. Lemma. Let \( f : M \rightarrow N \) and \( g : N \rightarrow M \) be homomorphisms with

\[
fg = 1_N.
\]

Then \( f \) is an epimorphism, \( g \) is a monomorphism, and

\[
M = \text{Ker} f \oplus \text{Im} g.
\]

Proof. Trivially \( f \) is epic and \( g \) is monic. Finally, \( gf \) is idempotent and \( M = (gf)M \oplus (1 - gf)M = \text{Im} g \oplus \text{Ker} f \).

If, as in Lemma 2.9, \( fg = 1_N \), then we say that \( f \) is a **split epimorphism** and \( g \) is a **split monomorphism**. More generally, a short exact sequence

\[
0 \rightarrow K \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0
\]

is **split** or **split exact** in case \( f \) is a split mono and \( g \) is a split epi.
2.10. Lemma. For a short exact sequence

\[ 0 \rightarrow K \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \]

of \( R \)-modules, the following are equivalent:

(a) The sequence is split;

(b) The monomorphism \( f \) is split;

(c) The epimorphism \( g \) is split;

(d) \( \text{Im } f = \text{Ker } g \) is a direct summand of \( M \).

Proof. We’ll prove that (d) implies (c), and leave the rest of the easy proof as an exercise. So assume (d), say \( M = \text{Im } f \oplus H \). If both \( m = f(k) + h \) and \( m' = f(k') + h' \) satisfy \( g(m) = g(m') \), then \( h - h' \in \text{Im } f \cap H \), so that \( h = h' \). Thus, there is an \( R \)-isomorphism \( g' : N \rightarrow H \) with \( g'g = 1_N \).

A direct summand of a module \( M \) is determined by an idempotent endomorphism of \( M \). That same idempotent determines the endomorphisms of the direct summand from the endomorphisms of \( M \).

2.11. Proposition. Let \( RM \) be a module and let \( e \in \text{End}(RM) \) is an idempotent. Then \( \text{End}(Re) \cong e \text{End}(RM)e \).

Proof. For each \( \varphi \in \text{End}(Me) \) let \( \overline{\varphi} : x \mapsto xe \varphi \). Then \( \overline{\varphi} \in \text{End}(M) \) and \( \overline{\varphi} = e\overline{\varphi}e \). Then \( \varphi \mapsto \overline{\varphi} \) is an injective ring homomorphism. Finally, if \( \psi \in \text{End}(M) \), then \( e\overline{\psi}e = e\psi e \).

2.12. Corollary. Let \( e \in R \) be a nonzero idempotent. Then right multiplication

\[ \rho : eRe \rightarrow \text{End}(Re) \]

defines a ring isomorphism \( eRe \cong \text{End}(Re) \).

Exercises 2.

2.1. An abelian group \( M \) is divisible in case for each \( a \neq 0 \) in \( \mathbb{Z} \), we have \( aM = M \). Let \( (M_\alpha)_{\alpha \in \Omega} \) be an indexed set of abelian groups.

(a) Prove that the product \( \prod_\Omega M_\alpha \) is divisible (torsionfree) iff each \( M_\alpha \) is divisible (torsionfree).

(b) Prove that the coproduct \( \bigsqcup_\Omega M_\alpha \) is torsion iff each \( M_\alpha \) is torsion.

(c) Show that even though each \( M_\alpha \) is torsion, then product \( \prod_\Omega M_\alpha \) need not be.
2.2. Prove that the abelian group $\mathbb{Z}^N/\mathbb{Z}^{(N)}$ is not torsionfree. Deduce that the inclusion monomorphism $0 \rightarrow \mathbb{Z}^{(N)} \rightarrow \mathbb{Z}^N$ does not split.

2.3. Let $e, f \in R$ be idempotent.

(a) Prove that as $(eRe, fRf)$ bimodules, $\text{Hom}_R(Re, Rf) \cong eRf$.

(b) Deduce that $Re \cong Rf$ iff there exists some $a \in eRf$ and $b \in fRe$ with $ab = e$ and $ba = f$.

(c) Show that $Re \cong Rf$ iff $eR \cong fR$.

(d) Suppose that $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are idempotents in $R$ with $1 = e_1 + \cdots + e_n = f_1 + \cdots + f_n$ and $Re_i \cong Rf_i$ for each $i = 1, \ldots, n$. Prove that there is an inner automorphism $\varphi$ of the ring $R$ with $\varphi(e_i) = f_i$ for all $i = 1, \ldots, n$.

2.4. Let $F \cong R^{(\Omega)}$ be a free right module on $\Omega$. Then there exist elements $(x_\alpha)_{\alpha \in \Omega}$ in $F$ with the property that for each $x \in F$ there exists a unique element $(a_\alpha)_{\alpha \in \Omega}$ in $R^{(\Omega)}$ such that $x = \sum_{\Omega} x_\alpha a_\alpha$. Such a set $(x_\alpha)_{\alpha \in \Omega}$ is a free basis for $F$. Let $\varphi \in \text{End}(F_R)$. Then for each $\beta \in \Omega$ there exists an element $(\varphi_{\alpha \beta})_{\beta \in \Omega} \in R^{(\Omega)}$ with

$$\varphi(x_\beta) = \sum_{\Omega} \varphi_{\alpha \beta} x_\alpha.$$ 

(a) Show that the matrix $[\varphi_{\alpha \beta}]$ is an $\Omega \times \Omega$ column finite matrix over $R$.

(b) Show that the map $\varphi \mapsto [\varphi_{\alpha \beta}]$ is a ring isomorphism from $\text{End}(F_R)$ onto the ring $\text{CFM}_\Omega(R)$ of all $\Omega \times \Omega$ column finite matrices over $R$.

2.5. A free module $F$ is said to have rank $\text{card}(\Omega)$ in case $F \cong R^{(\Omega)}$. If $F$ is free of rank $\text{card} \Omega$, prove that

(a) If $\Omega$ is infinite and $F \cong R^{(\Lambda)}$, then $\text{card} \Lambda = \text{card} \Omega$.

(b) If $F$ has a finite generating set, then $\Omega$ is finite.

2.6. A ring $R$ is said to be left SBN in case every non-zero free left module of finite rank is isomorphic to $R^R$.

(a) Prove that for a ring $R$ the following statements are equivalent:

i. $R$ is SBN;

ii. $R^{R(2)} \cong R^R$;

iii. There exist $p, p', i, i' \in R$ with $ip + i'p' = pi = p'i' = 1$ and $p'i = pi' = 0$.

(b) Prove that if $\Omega$ is infinite, then the ring $\text{CFM}_\Omega(R)$ of all $\Omega \times \Omega$ column finite matrices over $R$ is SBN.

(c) Prove that no commutative ring can be SBN.
2.7. Let \( R \) be a non-zero module. For every module \( RM \) the **trace of \( T \) in \( M \)** and the **co-trace of \( T \) in \( M \)** are the submodules \( \text{Tr}_M(T) \) and \( \text{coTr}_M(T) \) defined by

\[
\text{Tr}_M(T) = \sum_{f \in \text{Hom}(T,M)} \text{Im} f, \quad \text{and} \quad \text{coTr}_M(T) = \bigcap_{f \in \text{Hom}(M,T)} \text{Ker} f.
\]

We say that \( T \) **generates** the left \( R \)-module \( M \) in case \( \text{Tr}_M(T) = M \). Dually, we say that \( T \) **co-generates** \( M \) in case \( \text{coTr}_M(T) = 0 \).

(a) Prove that if \( f : M \rightarrow N \) is an \( R \)-homomorphism, then

\[
f(\text{Tr}_M(T)) \leq \text{Tr}_N(T), \quad \text{and} \quad f(\text{coTr}_M(T)) \leq \text{coTr}_N(T).
\]

(b) Prove that both \( \text{Tr}_R(T) \) and \( \text{coTr}_R(T) \) are ideals of \( R \).

(c) For a module \( RM \) prove that the following are equivalent:

i. \( T \) generates \( M \);

ii. \( M \) is isomorphic to a factor of some coproduct \( T^{(X)} \) of \( T \);

iii. For each \( f : M \rightarrow N \), if \( fh = 0 \) for all \( h \in \text{Hom}(T,M) \), then \( f = 0 \).

(d) For a module \( RM \) prove that the following are equivalent:

i. \( T \) cogenerates \( M \);

ii. \( M \) is isomorphic to a submodule of some product \( T^X \) of \( T \);

iii. For each \( f : N \rightarrow M \), if \( hf = 0 \) for all \( h \in \text{Hom}(M,T) \), then \( f = 0 \).

(e) In the category of all finitely generated abelian groups describe the class of groups that are

i. Generated by \( \mathbb{Z} \); \quad cogenerated by \( \mathbb{Z} \);

ii. Generated by \( \mathbb{Q} \); \quad cogenerated by \( \mathbb{Q} \);

iii. Generated by \( \mathbb{Z}_2 \); \quad cogenerated by \( \mathbb{Z}_2 \).

2.8. Let \( R_1, \ldots, R_n \) be a set of rings and let \( R = R_1 \times \cdots \times R_n \) be their cartesian product. For each \( i = 1, \ldots, n \), let \( \iota_i : R_i \rightarrow R \) be the \( i^{\text{th}} \) coordinate map. Then notice that \( \iota_i(R_i) \) is an ideal of \( R \), that \( u_i = \iota_i(1_i) \) is a central idempotent of \( R \), and that \( \iota_i \) is a ring isomorphism \( \iota_i : R_i \rightarrow \iota_i(R_i) \).

Finally, observe that the central idempotents \( u_1, \ldots, u_n \) are orthogonal and \( 1 = u_1 + \cdots + u_n \).

Next, let \( S \) be an arbitrary ring and let \( v_1, \ldots, v_n \) be central idempotents of \( S \). Then for each \( i \), \( Sv_i = v_iS = v_iSv_i \) is a ring with identity \( v_i \) and is an ideal of \( S \). We say that \( S \) is the **ring direct sum** of the ideals \( Sv_1, \ldots, Sv_n \) and we write

\[
S = Sv_1 \oplus \cdots \oplus Sv_n
\]

if the central idempotents \( v_1, \ldots, v_n \) are pairwise orthogonal and \( 1 = v_1 + \cdots + v_n \). In particular, \( R = \iota_1(R_1) \oplus \cdots \oplus \iota_n(R_n) \).
(a) Let \( R \) be a ring and let \( I_1, \ldots, I_n \) be left ideals with \( R = I_1 \oplus \cdots \oplus I_n \). Prove that each \( I_i \) is a ring and \( R = I_1 \oplus \cdots \oplus I_n \) iff each \( I_i \) is an ideal of \( R \).

(b) Show that it is possible to have an ideal \( I \) of a ring with \( _RR = I \oplus K \) for some left ideal \( K \) of \( R \) yet \( R \neq I \oplus K \). [Hint: Consider a ring of \( 2 \times 2 \) upper triangular matrices.]

(c) Let \( I \) be an ideal of a ring \( R \). Show that \( R = I \oplus K \) for some ideal \( K \) of \( R \) iff there is an idempotent \( e \in R \) with \( I = Re = eR \).

2.9. Recall that a **poset** (or **partially ordered set**) is a pair \((X, \leq)\) consisting of a set \( X \) and a partial order \( \leq \) on \( X \) where a **partial order** is a reflexive, transitive, and anti-symmetric relation. Usually, for a poset \((X, \leq)\) we ignore the relation and call \( X \) itself the poset. Each poset \( X \) can be completely characterized by means of a category \( \mathcal{X}(X) = \mathcal{X} \). The objects of this category \( \mathcal{X} \) are the elements of \( X \) and the morphisms are the pairs \((x, y)\) with \( x \leq y \). For each \( x, y \in X \) there is a unique morphism, namely \((x, y)\), from \( x \) to \( y \) if \( x \leq y \) and no morphism from \( x \) to \( y \) otherwise. Finally, composition is given by \((x, y)(y, z) = (x, z)\).

(a) Prove that if \( X \) is a poset, then \( \mathcal{X}(X) \) is a category with the property that if \( x, y \) are objects of \( \mathcal{X}(X) \), then \( \text{Mor}_\mathcal{X}(x, y) \) has at most one element.

(b) Let \( X \) and \( Y \) be two posets. Prove that \( X \cong Y \) iff \( \mathcal{X}(X) \cong \mathcal{X}(Y) \). Thanks to this we usually identify each poset \( X \) with its category \( \mathcal{X}(X) \).

(c) Let \( \mathcal{C} \) be a small category (i.e., its object class is a set). Prove that \( \mathcal{C} \) is isomorphic to the category \( \mathcal{X}(X) \) of a poset \( X \) iff for each pair \( x, y \) of objects of \( \mathcal{C} \), the set \( \text{Mor}_\mathcal{C}(x, y) \cup \text{Mor}_\mathcal{C}(y, x) \) of morphisms between \( x \) and \( y \) has at most one element.

2.10. In this Exercise we define products and coproducts in an arbitrary category. Let \( \mathcal{C} \) be a category and let \( \mathcal{C} = (C_\alpha)_{\alpha \in \Omega} \) be an indexed set of objects. (That is, \( \mathcal{C} = (C_\alpha)_{\alpha \in \Omega} \) is a function on the set \( \Omega \) to the object class of \( \mathcal{C} \).) We build a category \( \text{Mor}(\mathcal{C}) \) whose objects are all pairs \((A, (a_\alpha)_{\alpha \in \Omega})\) with \( A \) an object of \( \mathcal{C} \), and \( a_\alpha : A \to C_\alpha \) a morphism in \( \mathcal{C} \) for each \( \alpha \in \Omega \). A morphism \( g : (A, (a_\alpha)_{\alpha \in \Omega}) \to (B, (b_\alpha)_{\alpha \in \Omega}) \) is a morphism \( g : A \to B \) in \( \mathcal{C} \) such that for all \( \alpha \in \Omega \) we have that \( b_\alpha \circ g = a_\alpha \) or that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{a_\alpha} & & \downarrow{b_\alpha} \\
C_\alpha & & \\
\end{array}
\]

commutes. A terminal object \((P, (p_\alpha)_{\alpha \in \Omega})\) of \( \text{Mor}(\mathcal{C}) \), if it exists, is a **product** in \( \mathcal{C} \) of \( \mathcal{C} = (C_\alpha)_{\alpha \in \Omega} \). So if a product of \( \mathcal{C} \) exists in \( \mathcal{C} \), then it is unique to within isomorphism. Note that if \((P, (p_\alpha)_{\alpha \in \Omega})\) is a product for \( \mathcal{C} \), then for each \((A, (a_\alpha)_{\alpha \in \Omega})\) in \( \text{Mor}(\mathcal{C}) \) there is a unique
morphism \( a : A \rightarrow P \) in \( \mathcal{C} \) such that

\[
\begin{array}{c}
A \xrightarrow{a} P \\
\downarrow a\alpha \quad \downarrow p\alpha \\
C\alpha \quad \mathcal{O}
\end{array}
\]

commutes for all \( \alpha \in \Omega \). If a product exists, choose one \((P,(p\alpha)_{\alpha \in \Omega})\). The morphism \( a : A \rightarrow P \) is the **product** of the morphisms \((a\alpha)_{\alpha \in \Omega}\). We often write

\[
P = \prod_{\alpha \in \Omega} C\alpha
\]

and refer to this object as the product. This is slightly dangerous because the morphisms \((p\alpha)_{\alpha \in \Omega}\), called the **coordinate projections**, are definitely part of the package.

Next, we define the dual notion of a “coproduct”. Although we could define this very efficiently as a product in the opposite category \( \mathcal{C}^{\text{op}} \), we’ll start it from scratch. So let \( \mathcal{C} \) be a category and let \( \mathcal{C} = (C\alpha)_{\alpha \in \Omega} \) be an indexed set of objects. This time we build a category \( \text{Mor}(\mathcal{C},-) \) whose objects are all pairs \((A,(a\alpha)_{\alpha \in \Omega})\) with \( A \) an object of \( \mathcal{C} \), and \( a\alpha : C\alpha \rightarrow A \) a morphism in \( \mathcal{C} \) for each \( \alpha \in \Omega \). A morphism \( g : (A,(a\alpha)_{\alpha \in \Omega}) \rightarrow (B,(b\alpha)_{\alpha \in \Omega}) \) is a morphism \( g : A \rightarrow B \) in \( \mathcal{C} \) such that for all \( \alpha \in \Omega \) we have that \( f \circ a\alpha = b\alpha \) or that the diagram

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow a\alpha \quad \downarrow b\alpha \\
C\alpha \quad \mathcal{O}
\end{array}
\]

commutes. Now an initial object \((Q,(q\alpha)_{\alpha \in \Omega})\) of \( \text{Mor}(\mathcal{C},-) \), if it exists, is a **coproduct** in \( \mathcal{C} \) of \( \mathcal{C} = (C\alpha)_{\alpha \in \Omega} \). So if a coproduct of \( \mathcal{C} \) exists in \( \mathcal{C} \), then it is unique to within isomorphism. Here if \((Q,(q\alpha)_{\alpha \in \Omega})\) is a coproduct for \( \mathcal{C} \), then for each \((A,(a\alpha)_{\alpha \in \Omega})\) in \( \text{Mor}(\mathcal{C},-) \) there is a unique morphism \( a : Q \rightarrow A \) in \( \mathcal{C} \) such that

\[
\begin{array}{c}
Q \xrightarrow{a} A \\
\downarrow q\alpha \quad \downarrow a\alpha \\
C\alpha \quad \mathcal{O}
\end{array}
\]

commutes for all \( \alpha \in \Omega \). The morphism \( a : Q \rightarrow A \) is the **coproduct** of the morphisms \((a\alpha)_{\alpha \in \Omega}\).

If a product exists, choose one \((Q,(q\alpha)_{\alpha \in \Omega})\). We often write

\[
Q = \prod_{\alpha \in \Omega} C\alpha
\]

and refer to this object as the coproduct. Again, this is slightly dangerous because the morphisms \((q\alpha)_{\alpha \in \Omega}\), also called the **coordinate morphisms**, are part of the coproduct.

(a) Let \( \mathcal{C} \) be a poset. (See the previous exercise.) Prove that if \( \mathcal{C} = (C\alpha)_{\alpha \in \Omega} \) has a product \((P,(p\alpha)_{\alpha \in \Omega})\), then \( P \) is the meet of the set \( \{C\alpha\}_{\alpha \in \Omega} \) and hence is unique.
(b) Let $\mathcal{C}$ be a poset. Prove that if $\mathcal{C} = (C_\alpha)_{\alpha \in \Omega}$ has a coproduct $(Q, (q_\alpha)_{\alpha \in \Omega})$, then $Q$ is the join of the set $\{C_\alpha\}_{\alpha \in \Omega}$ and hence is unique.

(c) Again let $\mathcal{C}$ be a poset. Deduce that $\mathcal{C}$ is a lattice iff every finite set of objects has a product and coproduct and that $\mathcal{C}$ is complete iff every set has such a product and coproduct.

(d) Let $G$ be a non-trivial group; so $G$ is a category with a single object and for which every morphism is an automorphism. Show that it fails to have products and coproducts even for finite sets.