INTRODUCTION

Motivation: study of the derived categories of coherent sheaves on algebraic varieties and similar algebraic invariants; what happens when the variety moves in a family? Do we get some kind of algebraic period map?

History: Beilinson/Mukai discovered that 1) derived category of coh. sheaves can be "affine", i.e., described by the endomorphism algebra of an object; 2) derived categories of nonisomorphic varieties can be equivalent.

Structure: it was realized early on that the triangulated structure axiomatics is somewhat deficient, e.g., no functorial cones. Bondal-Kapranov, then Toen: need to consider enhancements, i.e., dg-categories.

Homological mirror symmetry (Kontsevich): when W and M are mirror dual CY-varieties then $D^b\operatorname{Coh}(W)$ is equivalent to the Fukaya category of M. The latter is an A_∞-category.

What is an A_∞-algebra (aka strong homotopy algebra)? Have m_1 differential, m_2 double product and higher products m_n of degree $2-n$, obeying some axioms, such that the product on H^* induced by m_2 is associative.

A_∞-algebras are closely related to dg-algebras. At least over a field, one can pass from an A_∞-algebra to an equivalent dg-algebra. There is also a construction called homological perturbation lemma that constructs an A_∞-structure on cohomology of a dg-algebra.

Remark: there are results about uniqueness of enhancements of derived categories of interest but they require to consider the entire derived category, i.e., all objects and morphisms between them. A_∞-enhancements allow to consider only endomorphisms of a generator.

Bondal-Van den Bergh, Kontsevich: existence of generator. E.g., for a quasiprojective variety, there exists a vector bundle E, such that the entire derived category is recovered from the dg-algebra of endomorphisms of E. For example, on a curve can take $\mathcal{O} \oplus L$, where L is a line bundle of positive degree.

Idea: start with a nice generator of $D^b\operatorname{Coh}(X)$, then compute the associated A_∞-algebra, then study the corresponding moduli of A_∞-algebras. The hope is that there will be only finitely many data on which the A_∞-algebra depend, so typically will get some affine scheme of finite type with a reductive group action, so that the corresponding GIT-picture will provide notions of stability and the modular compactifications.

Applications: homological mirror symmetry; derived equivalences on coherent side (involving noncommutative orders); geometric realization of solutions of the Associative Yang-Baxter equation.

Plan: 0. Background. A-infinity algebras, homological perturbation, Massey products, twisted complexes, derived categories and their enhancements.

1. Computations for elliptic curves
2. General results on moduli spaces of A-infinity structures.
3. Moduli of curves and A-infinity structures.
4. Homological mirror symmetry for punctured tori.
5. Moduli of A-infinity structures, Yang-Baxter equation and noncommutative orders on curves.

1. HOMOLOGICAL BACKGROUND

1.1. A_∞ **and** A_n**-structures.** For a graded associative S-algebra A (where S is a commutative ring and A is flat as S-module), we denote the terms of the Hochschild cochain complex of A over S as follows: $CH^{s+t}(A/S)_t$ denotes the space of S-multilinear maps $A^{\otimes s} \to A$ of degree t (where tensoring is over S). We have the induced bigrading $HH^{s+t}(A/S)_t$ of the Hochschild cohomology. The corresponding grading by the upper index is compatible with the definition of the Hochschild cohomology for A_∞-algebras.

Below we use the notion of A_n-structure which is a truncated version of an A_∞-structure defined by Stasheff (see [51, Def. 2.1]). For a moment let A be a graded S-module. Recall that an S-linear A_n-structure is given by a collection of S-multilinear maps

$$(m_1, \dots, m_n) \in CH^2(A/S)_1 \times \dots \times CH^2(A/S)_{2-n}$$

satisfying the standard A_∞-identities involving only $m_1, \dots, m_n$ (see below). Following [51, (2.4)], A_n-structures can be described conveniently in terms of truncated bar-construction

$$\mathrm{Bar}_{\le n}(A) = \bigoplus_{i=1}^{n} T_S^i(A[1]).$$

It has a natural structure of a graded coalgebra over S (without counit), such that it is a sub-coalgebra of the full bar-construction $\mathrm{Bar}(A) = \oplus_{i\ge 1} T_S^i(A[1])$. Here we take as a primary grading on $\mathrm{Bar}_{\le n}(A)$ the grading induced by the one on A. We also have a *bar-grading* for which $T_S^i(A[1])$ has degree i.

For each cochain $c \in CH^{s+t}(A/S)_t$, where $s \ge 1$, we denote by D_c the corresponding coderivation of $\mathrm{Bar}(A)$ of degree $s+t-1$, preserving each sub-coalgebra $\mathrm{Bar}_{\le n}(A)$ (we recover c from the component $\mathrm{Bar}_{\le s}(A) \to A[1]$ of D_c). Explicitly,

$$D_c(a_1 \otimes \dots \otimes a_n) = \sum_{i=1}^{n-s+1} \pm a_1 \otimes \dots \otimes a_{i-1} \otimes c(a_i, \dots, a_{i+s-1}) \otimes a_{i+s} \dots \otimes a_n \tag{1.1.1}$$

(for the signs, see [10, Prop. 1.4]).

Definition 1.1.1. We say that $m = (m_1, \dots, m_n) \in CH^2(A/S)_1 \times \dots \times CH^2(A/S)_{2-n}$ define an (S-linear) *A_n-algebra structure* on A if

$$D_m^2|_{\mathrm{Bar}_{\le n}(A)} = 0.$$

An A_n-algebra (resp., A_∞-algebra) is called *minimal* if it has $m_1 = 0$.

Considering components of this identity with respect to the bar-grading, we can rewrite this as a collection of identities

$$\sum_{i=1}^{r} D_{m_i} D_{m_{r+1-i}}|_{\mathrm{Bar}_{\le n}(A)} = 0, \tag{1.1.2}$$

where $r = 1, \dots, n$. Note that since $m \in \mathrm{CH}^2(A/S)$, the degree of D_m is 1.

We denote by $[D, D'] = DD' - (-1)^{\deg(D)\deg(D')} D'D$ the supercommutator of coderivations. By definition,

$$[D_c, D_{c'}] = D_{[c,c']},$$

where $[c, c']$ is the Gerstenhaber bracket. Also, if D_c has degree 1 then D_c^2 is still a coderivation, so it corresponds to some cochain ($[c, c]/2$ when 2 is invertible). Thus, we can view the identity (1.1.2) as the linear equation for coderivations associated with some Hochschild cochains in $CH^3(A/S)_{3-r}$. Since such cochains c are uniquely determined from the restriction $D_c|_{\mathrm{Bar}_{\le r}(A)}$, we see that rth identity (1.1.2) can be checked on $\mathrm{Bar}_{\le r}(A)$. Also, we deduce the following result.

Lemma 1.1.2. *The elements $(m_1, \ldots, m_n)$ as above define an S-linear A_n-structure on A if and only if*

$$\sum_{i=1}^{r} D_{m_i} D_{m_{r+1-i}} = 0,$$

for $r = 1, \ldots, n$. If 2 is invertible in S then this is equivalent to

$$\sum_{i=1}^{r} [m_i, m_{r+1-i}] = 0,$$

$r = 1, \ldots, n$.

The first few identities (1.1.2) are easy to interpret. First, $D_{m_1}^2 = D_{m_1^2}$, so we get $m_1^2 = 0$. The next identity $[m_1, m_2] = 0$ is simply the Leibnitz identity for the differential m_1. The next identity

$$D_{m_2}^2 + D_{[m_1, m_3]} = 0$$

implies that m_2 induces an associative product on the cohomology with respect to m_1.

Examples 1.1.3. 1. By definition, a dg-algebra is the same as A_2-algebra. It is also the same as an A_∞-algebra that has $m_i = 0$ for $i \ge 3$.
2. Here is a simple example of a minimal A_∞-algebra with nontrivial m_3. Let us consider the quiver with three vertices X_1, X_2, X_3 and three arrows $a_{12} : X_1 \to X_2$, $a_{23} : X_2 \to X_3$, and $a_{31} : X_3 \to X_1$. Let A be the quotient of the path-algebra of this quiver (over some field) by the relations stating that the product of any two composable arrows is zero (so the only nontrivial m_2 is given by the product with idempotents $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ corresponding to vertices). Equip A with grading by $\deg(a_{12}) = \deg(a_{23}) = 0$ and $\deg(a_{31}) = 1$. We define m_3 by

$$m_3(a_{i+2,i}, a_{i+1,i+2}, a_{i,i+1}) = \mathbf{e}_i$$

for $i \in \mathbb{Z}/3$, and set $m_i = 0$ for $i > 3$. It is easy to check that this is indeed an A_∞-algebra. In fact, it is a “model” A_∞-algebra for an exact triangle.

Definition 1.1.4. The *group of gauge transformations* $\mathfrak{G}$ is the group of degree-preserving coalgebra automorphisms $\alpha : \mathrm{Bar}(A) \to \mathrm{Bar}(A)$ such that the component $\mathrm{Bar}(A) \to A[1]$ is given by a collection

$$(f_1 = \mathrm{id}, f_2, \ldots) \in CH^1(A/S)_{-1} \times CH^1(A/S)_{-2} \times \ldots .$$

The group of *extended gauge transformations* is defined similarly by requiring f_1 just to be invertible. Note that any such automorphism automatically preserves any sub-coalgebra $\mathrm{Bar}_{\le n}(A)$ and the condition $f_1 = \mathrm{id}$ is equivalent to the condition that α acts as identity on every quotient $\mathrm{Bar}_{\le i}(A)/\mathrm{Bar}_{\le i-1}(A)$.

We usually identify elements of $\mathfrak{G}$ with the corresponding collections $f = (f_1 = \mathrm{id}, f_2 \dots)$ and denote by α_f the corresponding automorphism of $\mathrm{Bar}(A)$. Note that the group $\mathfrak{G}$ acts on the set of A_n-structures for every n: for $f \in \mathfrak{G}$ and an A_n-structure m, the new A_n-structure $f * m$ is determined by

$$D_{f*m} = \alpha_f D_m \alpha_f^{-1},$$

where in the right-hand side we restrict α_f to $\mathrm{Bar}_{\leq n}(A)$. This action is compatible with the projection from the set of A_{n+1}-structures to that of A_n-structures and preserves minimality.

Exercise. Define a notion of an A_∞-morphism between A_∞-algebras A and B as a sequence of maps $f_n : A^{\otimes n} \to B$, $n \geq 1$, where $\deg(f_n) = 1 - n$, satisfying natural axioms (use bar-constructions). Further, define the composition of A_∞-morphisms and show that $f = (f_n)$ is invertible if and only if f_1 is invertible (the identity A_∞-morphism has $f_1 = \mathrm{id}$ and $f_{>1} = 0$). Similarly, define A_n-morphisms between A_n-algebras looking at $\mathrm{Bar}_{\leq n}$.

Definition 1.1.5. An A_∞-morphism from A to B is called a *quasi-isomorphism* if f_1 (which commutes with m_1) is such.

In fact, there is a notion of homotopy between A_∞-morphisms and hence, the notion of a homotopy equivalence. By a theorem of Prouté [40], every quasi-isomorphism between A_∞-algebras over a field is a homotopy equivalence. In particular, whenever there is a quasi-isomorphism of A_∞-algebras, $A \to B$, there is also a quasi-isomorphism in the opposite direction, $B \to A$. This is very different from the situation with the dg-algebras where a similar statement is not true.

Definition 1.1.6. An A_∞-structure on A is called *(strictly) unital* if there is an element $1 \in A_0$ such $m_1(1) = 0$, $m_2(1, x) = m_2(x, 1) = x$, and $m_i(x_1, \dots, x_i) = 0$, for $i > 2$ whenever one of x_i is equal to 1.

In fact, it turns out that any A_∞-algebra for which there is a cohomological unit in H^0A is quasi-isomorphic to a strictly unital one (this is due to Fukaya).

A_∞-categories and A_∞-functors are discussed similarly. Quasi-equivalence: the induced cohomology functor should be an equivalence.

Convention: denote morphisms in an A_∞-category as $\hom^*(X, Y)$, and denote by $\mathrm{Hom}^*(X, Y)$ the cohomology with respect to m_1.

1.2. **Homological perturbation.** There is a general construction of the A_∞-structure on the cohomology of a dg-algebra (A, d) over a field k, equipped with a projector $\Pi : A \to B$ onto a subspace of $\ker(d)$ and a homotopy operator Q such that $1 - \Pi = dQ + Qd$.

This goes back to Kadeishvili's work [13]. Merkulov's formula for this A_∞-structure (see [26]) was rewritten in [16] as a sum over trees:

$$m_n(b_1, \dots, b_n) = -\sum_T \epsilon(T) m_T(b_1, \dots, b_n). \tag{1.2.1}$$

Here T runs over all oriented planar rooted 3-valent trees with n leaves (different from the root) marked by $b_1, \dots, b_n$ left to right, and the root marked by Π (we draw the tree in such a way that leaves are above, and every vertex has two edges coming from above and one from below). The expression $m_T(b_1, \dots, b_n)$ is obtained by going down from leaves to the root, applying the multiplication in A at every vertex and applying the operator Q at every inner edge (see [16, sec. 6.4] for details). The sign $\epsilon(T)$ has form

$$\epsilon(T) = \prod_v (-1)^{|e_1(v)|+(|e_2(v)|-1)\deg(e_1(v))},$$

where v runs through vertices of T (we do not count the root or leaves as vertices), $(e_1(v), e_2(v))$ is the pair of edges above v, for an edge e we denote by $|e|$ the total number of leaves above e and by $\deg(e)$ the sum of degrees of all leaves above e (recall that leaves are marked by b_i).

The standard way of choosing Q for a given subspace $B \subset \ker(d)$ of cohomology representatives is to pick a subspace $C \subset A$ complementary to $\ker(d)$ and define Q by

$$Q(x) = \begin{cases} (d|_C)^{-1}(x), & x \in \mathrm{im}(d), \\ 0, & x \in B, \\ 0, & x \in C. \end{cases}$$

Example.

$$m_3(b_1, b_2, b_3) = \Pi[Q(b_1 b_2) b_3 - (-1)^{\deg(b_1)} b_1 Q(b_2 b_3)].$$

Next, construct a quasi-isomorphism of A_∞-algebras $H^*B \to B$, using similar tree-formula: the only difference, is Q replace Π at the end. E.g., $f_2(a_1, a_2) = \pm Q(a_1 a_2)$, etc. Thus, B is quasi-isomorphic to the original A_∞-algebra. This implies that the obtained A_∞-structure does not depend on choices up to a gauge equivalence (using Prouté's theorem).

Lemma 1.2.1. *Assume in addition that* $\Pi Q = Q\Pi = Q^2 = 0$. *Then the* A_∞*-structure on* B *given by the homological perturbation is unital.*

Proof. It is convenient to use Merkulov's original formula (equivalent to (1.2.1))

$$m_n(b_1, \dots, b_n) = \Pi\lambda_n(b_1, \dots, b_n),$$

where $\lambda_n : A^{\otimes n} \to A$ are defined for $n \geq 2$ by the following recursion: $\lambda_2(a_1, a_2) = a_1 a_2$,

$$\lambda_n(a_1, \dots, a_n) = \pm Q(\lambda_{n-1}(a_1, \dots, a_{n-1})) \cdot a_n \pm a_1 \cdot Q(\lambda_{n-1}(a_2, \dots, a_n)) +$$
$$\sum_{k+l=n; k,l \geq 2} \pm Q(\lambda_k(a_1, \dots, a_k)) \cdot Q(\lambda_l(a_{k+1}, \dots, a_n)).$$

Since, $\Pi Q = 0$, it is enough to prove that $\lambda_n(b_1, \dots, b_n) \in Q(A)$. Let us use induction in n. In the case $n = 3$ we have

$$\lambda_3(b_1, b_2, b_3) = Q(b_1 b_2) b_3 \pm b_1 Q(b_2 b_3)$$

and the assertion follows immediately from the fact that $Q(B) = 0$. Suppose now that $n \geq 4$ and the assertion holds for all $n' < n$. Since $Q^2 = 0$, the induction assumption easily implies that the first two terms in the recursive formula for λ_n belong to $Q(A)$. Similarly, all the remaining terms vanish if $n \geq 5$. In the case $n = 4$ the term $Q(b_1 b_2) \cdot Q(b_3 b_4)$ also vanishes because either $b_1 b_2 \in B$ or $b_3 b_4 \in B$ and $Q(B) = 0$. $\square$

Note that the assumptions of Lemma 1.2.1 hold for the standard choice of Q associated with the choice of a complementary subspace to $\ker(d)$.

Remark 1.2.2. The homological perturbation construction can be generalized in several ways. First, it works for dg-categories and the output becomes an A_∞-caegory. Secondly, one can start with an A_∞-algebra A, and B can be any homotopically equivalent complex to A. Then B can be equipped with an A_∞-algebra structure, such that the obtained A_∞-algebra is A_∞-equivalent to A.

We will need a version of the homological perturbation lemma for a dg-algebra (A, d) over a commutative ring R. It is straightforward to see that the construction still works once we have a homotopy Q. Thus, it is enough to know that embeddings $\operatorname{im}(d) \hookrightarrow \ker(d)$ and $\ker(d) \hookrightarrow A$ are splittable, or more generally that A is homotopy equivalent to a complex of R-modules with the trivial differential. To this end we will use the following simple observation.

Lemma 1.2.3. *(i) Let $(C^\bullet, d)$ be a bounded above complex of projective R-modules, where R is a commutative ring. Assume in addition that every cohomology $H^i(C^\bullet)$ is a projective R-module. Then for each i the embeddings $\operatorname{im}(d^{i-1}) \subset \ker(d^i)$ and $\ker(d^i) \subset C^i$ are splittable, where $d^i : C^i \to C^{i+1}$.*
(ii) Let $C^{\bullet,\bullet}$ be a bicomplex of R-modules such that $C^{i,\bullet} = 0$ for $i \notin [-N, N]$ for some $N > 0$ (i.e., bounded in horizontal direction). Assume that each complex $C^{i,\bullet}$ is homotopy equivalent to a bounded above complex of projective R-modules. Assume also that the cohomology modules of the total complex $\operatorname{tot}(C)$ are projective. Then $\operatorname{tot}(C)$ is homotopy equivalent to a complex of R-modules with the trivial differential.

Proof. (i) The exact sequences

$$0 \to \operatorname{im}(d^i) \to \ker(d^{i+1}) \to H^{i+1} \to 0, \quad 0 \to \ker(d^i) \to C^i \to \operatorname{im}(d^i) \to 0 \tag{1.2.2}$$

show that it is enough to prove that $\ker(d^i)$ and $\operatorname{im}(d^i)$ are projective R-modules. We can check this by the descending induction on i. The base of induction holds since $C^i = 0$ for sufficiently large i. Assuming that $\ker(d^{i+1})$ is projective, we use exact sequences (1.2.2) to deduce first that $\operatorname{im}(d^i)$ is projective and then that $\ker(d^i)$ is projective. This gives the induction step.
(ii) Without loss of generality we can assume that $C^{i,\bullet} = 0$ for $i \notin [0, N]$. We can represent the total complex (up to a shift) as an iterated cone

$$\operatorname{Cone}(\dots \operatorname{Cone}(\operatorname{Cone}(C^{0,\bullet} \to C^{1,\bullet}) \to C^{2,\bullet}) \dots \to C^{N,\bullet}),$$

where we view the horizontal differentials as chain maps $d^i : C^{i,\bullet} \to C^{i+1,\bullet}$. Note that the above maps between iterated cones are well defined since $d^i \circ d^{i-1} = 0$ as maps of complexes. By assumption, for every i we have a homotopy equivalence of $C^{i,\bullet}$ with a bounded above complex $P^{i,\bullet}$ whose terms are projective R-modules. We can define chain maps $\overline{d}^i : P^{i,\bullet} \to P^{i+1,\bullet}$ (uniquely up to a homotopy) so that they correspond to d^i under these homotopy equivalences. Note that for a chain map $f : A \to B$, the homotopy class of $\operatorname{Cone}(F)$ depends only on an isomorphism class of the arrow $f : \mathbb{A} \to B$ in the homotopy category of complexes. This implies, that $\operatorname{tot}(C)$ is homotopy equivalent to some iterated cone

$$\operatorname{tot}(P) := \operatorname{Cone}(\dots \operatorname{Cone}(\operatorname{Cone}(P^{0,\bullet} \to P^{1,\bullet}) \to P^{2,\bullet}) \dots \to P^{N,\bullet}).$$

(Note that $P^{\bullet,\bullet}$ acquires a structure of a twisted complex in the sense of [3].) Note $\mathrm{tot}(P)$ is bounded above and has projective terms. Its cohomology groups are also projective, since they are the same as for $\mathrm{tot}(C)$. Therefore, by part (i), $\mathrm{tot}(P)$ is homotopy equivalent to a complex with the trivial differential. Hence, the same is true for $\mathrm{tot}(C)$. □

Remark 1.2.4. The situation of Lemma 1.2.3(ii) sometimes occurs when trying to run the homological perturbation for the functor $R\Gamma(X,\cdot)$ (derived global sections) applied to a sheaf of dg-algebras $\mathcal{A}^\bullet$ on X. The standard way of calculating $R\Gamma(X,\mathcal{A}^\bullet)$ leads to a bicomplex, and Lemma 1.2.3(ii) amounts to imposing suitable assumptions on the complexes $R\Gamma(\mathcal{A}^i)$ and on the hypercohomology of $\mathcal{A}^\bullet$. For an example, see Lemma 4.3.1.

1.3. **Relation of A_∞-structures to Hochschild cohomology.** When we discuss minimal A_∞-structures, i.e., A_∞-structures with $m_1 = 0$, the product m_{n+1} plays no role in the identity (1.1.2) with $r = n+1$, so it makes sense to make the following shift in numbering [1].

Definition 1.3.1. A *minimal A'_n-structure* is an A_{n+1}-structure with $m_1 = m_{n+1} = 0$. Equivalently, this is a minimal A_n-structure which extends to an A_{n+1}-structure, i.e., satisfies one extra equation $[m_2, m_n] + \ldots = 0$.

When we talk about minimal S-linear A'_n-structures on a graded associative S-algebra A, unless otherwise specified, we always assume that m_2 is the given product on A. Note that for a Hochschild cochain $c \in CH^{s+t}(A/S)_t$ we have

$$[D_{m_2}, D_c] = D_{m_2}D_c + (-1)^{s+t}D_cD_{m_2} = D_{\delta(c)},$$

where $\delta(c) = [m_2, c]$ is the Hochschild differential.

For example, a minimal A'_3-algebra is an associative algebra together with any $m_3 \in \mathrm{CH}^2(A)_{-1}$ satisfying $[m_2, m_3] = 0$. In other words, m_3 should be a Hochschild cocycle.

For a graded associative algebra A let us denote by $\mathcal{A}_n(A)$ (resp., $\mathcal{A}'_n(A)$) the set of all minimal S-linear A_n-structures (resp., A'_n-structures) on E. Note that $(m_2, \ldots, m_n, m_{n+1})$ is in $\mathcal{A}_{n+1}(A)$ if and only if $(m_2, \ldots, m_n)$ is in $\mathcal{A}'_n(A)$, so we have the natural projection

$$\mathcal{A}_{n+1}(A) \to \mathcal{A}'_n(A)$$

which realizes $\mathcal{A}'_n(A)$ as the quotient of $\mathcal{A}_{n+1}(A)$ by the free action of $CH^2(A/S)_{1-n}$ (by addition in the last component).

Lemma 1.3.2. *Let m and m' be two minimal A'_n-structures on the same graded associative algebra (A, m_2), such that $m_i = m'_i$ for $i < n$, where $n \geq 3$.*
(i) $\delta(m'_n - m_n) = 0$, i.e., $m'_n - m_n$ is a Hochschild cocycle for (A, m_2), so it defines a cohomology class in $HH^2(A)_{2-m}$.
(ii) Suppose $m' = f * m$, where f is a gauge equivalence with $f_i = 0$ for $1 < i < n-1$. Then $m'_n - m_n = \pm\delta(f_{n-1})$. Thus, there exists a gauge equivalence f with $m'_{\leq n} = f * m_{\leq n}$ and $f_{<n-1} = \mathrm{id}$ if and only if the cohomology class $[m'_n - m_n]$ in $HH^2(A)_{2-n}$ is trivial.

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[1] In [35, Sec. 4] there is an error in this respect: wherever minimal A_n-structures are mentioned, they should be replaced by minimal A'_n-structures in the sense of Definition 1.3.1

Proof. (i) The A_∞-axiom gives an expression of both $[m_2, m_n]$ and $[m_2, m'_n]$ in terms of m_i and m'_i with $i < n$.
(ii) This is easily obtained by evaluating both sides of the equation $D_{m'}\alpha_f = \alpha_f D_m$ on $a_1 \otimes \ldots \otimes a_n$ and using the explict form of α_f:
(1.3.1)

$$\alpha_f(a_1 \otimes \ldots \otimes a_n) = \sum_{i_1+\ldots+i_k=n} \pm f_{i_1}(a_1, \ldots, a_{i_1}) \otimes f_{i_2}(a_{i_1+1}, \ldots, a_{i_1+i_2}) \otimes \ldots \otimes f_{i_k}(a_{n-i_k+1}, \ldots, a_n).$$

□

Definition 1.3.3. For each n let us denote by $\mathfrak{G}_{\geq n} \subset \mathfrak{G}$ the subgroup of $f = (f_1 = \mathrm{id}, f_2, \ldots)$ with $f_i = 0$ for $2 \leq i < n$. To see that this is a subgroup we observe that this vanishing condition is equivalent to the condition that α_f acts as identity on all the quotients $\mathrm{Bar}_{\leq i}(A)/\mathrm{Bar}_{\leq i-n+1}$. In particular, $\mathfrak{G}_{\geq 2} = \mathfrak{G}$.

Let A be a graded associative algebra. Lemma 1.3.2(ii) implies that the subgroup $\mathfrak{G}_{\geq n}$ acts trivially on the set $\mathcal{A}_n(A)$ of minimal A_n-structures, while an element $f = (\mathrm{id}, 0, \ldots, 0, f_{n-1})$ of $\mathfrak{G}_{\geq n-1}$ acts on $m = (m_2, \ldots, m_n) \in \mathcal{A}_n(A)$ by

$$f * m = (m_2, \ldots, m_{n-1}, m_n + \delta(f_{n-1})).$$

Thus, the action of $\mathfrak{G}$ on $\mathcal{A}_n(A)$ factors through an action of $\mathfrak{G}/\mathfrak{G}_{\geq n}$. Furthermore, since the projection $\mathcal{A}_{n+1}(A) \to \mathcal{A}_n(A)$ is $\mathfrak{G}$-equivariant, we get a well defined action of $\mathfrak{G}/\mathfrak{G}_{\geq n}$ on the closed subscheme $\mathcal{A}'_n(A) \subset \mathcal{A}_n(A)$ (which is the image of the projection from $\mathcal{A}_{n+1}(A)$).

Any minimal A'_{n+1}-structure induces a minimal A'_n-structure by forgetting m_{n+1}. The following well-known result states that an obstacle to extending an A'_n-structure to an A'_{n+1}-structure lies in $HH^3(A)_{2-n}$ (it is stated without proof as [1, Lem. 2.3]).

Lemma 1.3.4. *(i) Let $\mathcal{A}$ be an associative algebra with generators $D_1, \ldots, D_n$ and defining relations*

$$\sum_{i=1}^{r} D_i D_{r+1-i} = 0,$$

for $r = 1, \ldots, n$. Set $S = \sum_{i=2}^{n} D_i D_{n+2-i}$. Then

$$D_1 S - S D_1 = 0.$$

(ii) For a minimal A'_n-structure $m = (m_2, \ldots, m_n)$ on A there exists a Hochschild cocycle $\phi_n(m) \in CH^3(A)_{1-n}$ (so $\delta(\phi_n(m)) = 0$) such that

$$D_{\phi_n(m)} = \sum_{i=3}^{n} D_{m_i} D_{m_{n+3-i}}. \tag{1.3.2}$$

The A'_n-structure m is extendable to an A'_{n+1}-structure $(m_2, \ldots, m_n, m_{n+1})$ if and only if $\phi_n(m)$ is a coboundary.

Proof. (i) Let us give $\mathcal{A}$ the grading by $\deg D_i = 1$ and use the corresponding supercommutators. Then we have

$$[D_1, S] = \sum_{i=2}^{n} [D_1, D_i] D_{n+2-i} - \sum_{i=2}^{n} D_i [D_1, D_{n+2-i}].$$

Applying the relations we can rewrite the sums in the right-hand side as

$$\sum_{i=2}^{n}[D_1, D_i]D_{n+2-i} = -\sum_{i\ge 2, j\ge 2, i+j\le n+1} D_iD_jD_{n+3-i-j},$$

$$\sum_{i=2}^{n} D_i[D_1, D_{n+2-i}] = \sum_{i\ge 2, j\ge 2, i+j\le n+1} D_{n+3-i-j}D_iD_j.$$

Thus, both sums are equal to

$$\sum_{i\ge 2, j\ge 2, k\ge 2, i+j+k=n+3} D_iD_jD_k,$$

so they cancel out.
(ii) The existence of the Hochschild cochain $\phi_n(m)$ follows from the fact that the expression in the right-hand side of (1.3.2) is a coderivation. The fact that $\phi_n(m)$ is δ-closed follows from (i). By Lemma 1.1.2, the condition on m_{n+1} to extend $m = (m_2, \dots, m_n)$ to an A_{n+1}-structure is

$$[D_{m_2}, D_{m_{n+1}}] = -\sum_{i=3}^{n} D_{m_i}D_{m_{n+3-i}},$$

i.e., $\delta(m_{n+1}) = -\phi_n(m)$, which implies the assertion. □

Due to the above results, $HH^2(A)$ and $HH^3(A)$ play an important role in classifying minimal A_∞-structures on A. The cohomology space $HH^1(A)$ also shows up in connection with the notion of a homotopy between gauge transformations (see [14] and [32, Sec. 2.1], where these are called homotopies between strict A_∞-isomorphisms).

Definition 1.3.5. (i) Let $f, f' : A \to B$ be a pair of A_∞-morphisms between A_∞-algebras. A *homotopy* h from f to f' is given by a collection of maps $h_i : A^{\otimes i} \to B$ of degree $-i$, where $i \ge 1$, satisfying some equations. These equations are written as follows: there exists a unique linear map $H : \mathrm{Bar}(A) \to \mathrm{Bar}(B)$ of degree -1 with the component $\mathrm{Bar}(A) \to B$ given by (h_i), such that

$$\Delta \circ H = (\alpha_f \otimes H + H \otimes \alpha_{f'}) \circ \Delta, \tag{1.3.3}$$

where $\alpha_f, \alpha_{f'} : \mathrm{Bar}(A) \to \mathrm{Bar}(B)$ are coalgebra homomorphisms corresponding to f and f', Δ denotes the comultiplication. Then the equation connecting h, f and f' is

$$\alpha_f - \alpha_{f'} = D_B \circ H + H \circ D_A, \tag{1.3.4}$$

where D_A (resp., D_B) is the coderivation of $\mathrm{Bar}(A)$ (resp., $\mathrm{Bar}(B)$) corresponding to the A_∞-structure on A (resp., B).
(ii) If A and B are only A_n-algebras and $f, f' : A \to B$ are A_n-morphism, then we can consider homotopies h from f to f' defined by a collection of maps (h_i) as above for $i \le n$ and satisfying euqations (1.3.3), (1.3.4) for the corresponding maps $\mathrm{Bar}_{\le n}(A) \to \mathrm{Bar}_{\le n}(B)$.

Note that if we only know that $f : A \to B$ is an A_∞-morphism, i.e., α_f is a homomorphism of dg-coalgebras, then one can easily check that equations (1.3.3) and (1.3.4) imply that $\alpha_{f'}$ is a homomorphism of dg-coalgebras, i.e., f' is an A_∞-morphism from A to B.

Lemma 1.3.6. *Let A and B be A_∞-algebras and $f = (f_i)$ be an A_∞-morphism from A to B. For every collection $(h_i)_{i\geq 1}$, where $h_i : A^{\otimes n} \to B$ has degree $-i$, there exists a unique A_∞-morphism f' from A to B such that h is a homotopy from f to f'. The similar assertion holds for homotopies $(h_i)_{1\leq i\leq n}$ between A_n-morphisms of A_n-algebras.*

Proof. We are going to construct the maps $H|_{\mathrm{Bar}(A)_{\leq n}}$ and $\alpha_{f'}|_{\mathrm{Bar}(A)_{\leq n}}$ recursively, so that at each step the equations (1.3.4) and (1.3.5) are satisfied when restricted to $\mathrm{Bar}(A)_{\leq n}$. Also, we want H to have (h_i) as components. Then the uniqueness will be clear.

It is easy to see that equation (1.3.3) is equivalent to the following formula

(1.3.5)

$$H[a_1|\dots|a_n] = \sum_{i_1<\dots<i_k<m<j_1<\dots<j_l=n} \pm[f_{i_1}(a_1,\dots,a_{i_1})|\dots|f_{i_k-i_{k-1}}(a_{i_{k-1}+1},\dots,a_{i_k})|$$
$$h_{m-i_k}(a_{i_k+1},\dots a_m)|f'_{j_1-m}(a_{m+1},\dots,a_{j_1})|\dots|f'_{j_l-j_{l-1}}(a_{j_{l-1}+1},\dots,a_{j_l})],$$

where $a_1,\dots,a_n \in A$, $n \geq 1$ (one can have $k = 0$ or $l = 0$ in this sum). Note that $H|_{A[1]}$ is given by h_1 and $\alpha_{f'}|_{A[1]}$ is given by $f'_1 = f_1 - m_1 \circ h_1 - h_1 \circ m_1$. Now assume that the restrictions of H and $\alpha_{f'}$ to $\mathrm{Bar}(A)_{\leq n-1}$ are already constructed, in particular, the maps $f'_i : A^{\otimes i} \to B$ are defined for $i \leq n-1$. Then the formula (1.3.5) defines uniquely the extension of H to $\mathrm{Bar}(A)_{\leq n}$ (note that in the RHS of this formula only f'_i with $i \leq n-1$ appear). It remains to apply formula (1.3.4) to define $\alpha_{f'}|_{\mathrm{Bar}(A)_{\leq n}}$. □

There is an analog of Lemma 1.3.2 for gauge equivalences and homotopies between them.

Lemma 1.3.7. *Let m and m' be minimal A_N-structures on the associative algebra A, f, f' be a pair of gauge equivalences from m to m'. Assume that $f_i = f'_i$ for $i < n$, where $2 \leq n < N$.*
(i) Set $c(a_1,\dots,a_n) = (f'_n - f_n)(a_1,\dots,a_n)$. Then c is a Hochschild n-cocycle (defining a cohomology class in $HH^1(A)_{1-n}$).
(ii) If $h : f \to f'$ is a homotopy such that $h_i = 0$ for $i < n-1$, then one has $f'_n - f_n = \pm\delta h_{n-1}$. Thus, $f_{\leq n}$ and $f'_{\leq n}$ are homotopic if and only if the class $[f'_n - f_n]$ in $HH^1(A)_{1-n}$ is trivial.

1.4. **Triple Massey products for A_∞-structures.** Note that for a minimal A_∞-structure, m_3 is Hochschild cocycle, and a gauge equivalence can change it by a coboundary (see Lemma 1.3.2). Thus, to any gauge-equivalence class of a minimal A_∞-structure one can associate a Hochschild cohomology class $[m_3]$ of the corresponding associative algebra. However, this class is often hard to compute. Massey products provide invariants which are easier to compute.

Let us consider a more general notion for a not necessarily minimal A_∞-category $\mathcal{A}$.

We start with a triple of composable morphisms $[a_1] \in \mathrm{Hom}(Z,T)$, $[a_2] \in \mathrm{Hom}(Y,Z)$, $[a_3] \in \mathrm{Hom}(X,Y)$ represented by some m_1-closed elements a_1, a_2, a_3. Assume that the compositions $[a_1][a_2]$ and $[a_2][a_3]$ vanish in $H^0(\mathcal{A})$. Then we can find $a_{12} \in \mathrm{hom}^{-1}(Y,T)$, $a_{23} \in \mathrm{hom}^{-1}(X,Z)$ such that

$$m_1(a_{12}) = m_2(a_1,a_2), \quad m_1(a_{23}) = m_2(a_2,a_3).$$

Now let us set

$$MP([a_1],[a_2],[a_3]) = [m_3(a_1,a_2,a_3) + m_2(a_{12},a_3) - m_2(a_1,a_{23})]. \tag{1.4.1}$$

It is easy to check that the expression in the right-hand side is m_1-closed: this follows from

$$m_1m_3(a_1,a_2,a_3)=m_2(a_1,m_2(a_2,a_3))-m_2(m_2(a_1,a_2),a_3)=m_2(a_1,m_1a_{23})-m_2(m_1a_{12},a_3)$$

and from the Leibnitz identity.

Furthermore, if we change a_{12} or a_{23} or representatives for $[a_i]$, then the right-hand side of (1.4.1) would change by adding summands of the form $[a_1][b_{23}]$ and $[b_{13}][a_3]$ for some $b_{12}\in \mathrm{Hom}^{-1}(Y,T)$, $b_{23}\in \mathrm{Hom}^{-1}(X,Z)$. Thus, the triple Massey product $MP([[a_1],[a_2],[a_3])$ is a well defined element in

$$\operatorname{coker}(\mathrm{Hom}^{-1}(X,Z)\oplus\mathrm{Hom}^{-1}(Y,T)\xrightarrow{[a_1]\circ ?,?\circ[a_3]}\mathrm{Hom}^{-1}(X,T).$$

When $m_3=0$ this definition coincides with the usual definition given in the dg-context. On the other hand, if $m_1=0$ then the triple Massey product is the class represented by $[m_3(a_1,a_2,a_3)]$. Finally, we claim that this Massey product is preserved under any equivalence of A_∞-categories. This is a consequence of the following result (proved in [31, Prop. 1.1] with different sign conventions).

Proposition 1.4.1. *Let $F:\mathcal{C}\to\mathcal{C}'$ be an A_∞-functor between A_∞-categories, let $[F_1]: H^*\mathcal{C}\to H^*\mathcal{C}'$ be the functor between the corresponding graded categories inducced by F_1. Then for a triple of composable arrows as above, one has*

$$[F_1]MP([a_1],[a_2],[a_3])=MP([F_1][a_1],[F_1][a_2],[F_1][a_3]).$$

Proof. We have by the A_∞-functor axioms,

$$\begin{aligned}m_2(F_1a_1,F_1a_2)&=F_1(m_2(a_1,a_2))-m_1F_2(a_1,a_2)=F_1m_1a_{12}-m_1F_2(a_1,a_2)\\&=m_1(F_1a_{12}-F_2(a_1,a_2)),\end{aligned}$$

and similarly

$$m_2(F_1a_2,F_1a_3)=m_1(F_1a_{23}-F_2(a_2,a_3)).$$

Thus, $MP([F_1a_1],[F_1a_2],[F_1a_3])$ is represented by the element

$$m_3(F_1a_1,F_1a_2,F_1a_3)+m_2(F_1a_{12}-F_2(a_1,a_2),F_1a_3)-m_2(F_1a_1,F_1a_{23}-F_2(a_2,a_3)).$$

Now from the A_∞-functor axioms we get the congruence modulo $\mathrm{im}(m_1)$,

$$\begin{aligned}&m_3(F_1a_1,F_1a_2,F_1a_3)+m_2(F_1a_1,F_2(a_2,a_3))-m_2(F_2(a_1,a_2),F_1a_3)\\&\equiv F_1m_3(a_1,a_2,a_3)+F_2(m_2(a_1,a_2),a_3)-F_2(a_1,m_2(a_2,a_3)).\end{aligned}$$

Thus, we can rewrite the above representative as

$$\begin{aligned}&F_1m_3(a_1,a_2,a_3)+m_2(F_1a_{12},F_1a_3)-m_2(F_1a_1,F_1a_{23})+F_2(m_1a_{12},a_3)-F_2(a_1,m_1a_{23})\\&\equiv F_1\big(m_3(a_1,a_2,a_3)+m_2(a_{12},a_3)-m_2(a_1,a_{23})\big),\end{aligned}$$

where we used again the A_∞-functor axioms. □

The above definition of triple Massey products can be slightly generalized: instead of considering a decomposable tensor $f\otimes g\otimes h$ one can take any tensor in the appropriate subspace of $\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z)\otimes\mathrm{Hom}(Z,T)$.

1.5. **Triangulated structure and generators.**

1.5.1. *Enhancements.* Standard dg-enhancement of the derived category: dg-category of complexes.

For perfect complexes easier to use

$$\hom(E^\bullet, F^\bullet) = \mathcal{R}(\underline{\mathrm{Hom}}(E^\bullet, F^\bullet)),$$

where $\mathcal{R}$ is some functorial multiplicative chain model for computing cohomology, e.g., one can use Cech or Dolbeault resolutions.

Lunts-Orlov: uniqueness of enhancements for $D^b(\mathrm{Coh}\, X)$ and for perfect complexes, where X is a projective scheme over a field.

1.5.2. *Triangulated A_∞-categories and twisted complexes.* Definition of the A_∞-category of twisted objects over an A_∞-category $\mathcal{A}$.

FIrst take closure under shifts and under direct sums. Then consider pairs (X, δ), where X is an object of $\mathcal{A}$, $\delta \in \hom^1(X, X)$ is strictly upper-triangular with respect to a finite split filtration, such that

$$\sum_t (-1)^{\binom{t}{2}} m_t(\delta, \dots, \delta) = 0.$$

Note that here the left-hand side is well defined and takes values in $\hom^2(X, X)$. The hom-space between two such objects (X, δ_X) and (Y, δ_Y) is simply $\hom(X, Y)$. There are natural A_∞-products (m^t_n) for the twisted objects (in particular, the differential m^t_1 on $\hom(X, Y)$), which are obtained by inserting the twisting elements δ in any number wherever possible.

Definition 1.5.1. An A_∞-category is called *triangulated* if $\mathcal{A} \to Tw\mathcal{A}$ is a quasi-equivalence.

The category $Tw\mathcal{A}$ is always triangulated and can be characterized as a universal triangulated envelope of $\mathcal{A}$.

1.5.3. *Exact triangles in an A_∞-category.* Start with $a : X \to Y$ a closed morphism of degree 0. Then we can view a as a morphism $X[1] \to Y$ of degree 1. We define $\mathrm{Cone}(f)$ to be the twisted complex $(X[1] \oplus Y, a)$. We have a natural closed maps $b : Y \to \mathrm{Cone}(f)$ and $c : \mathrm{Cone}(f) \to X[1]$. Note that $c \circ b = 0$ while $b \circ a = d(\alpha)$, where $\alpha \in \hom^{-1}(X[1], \mathrm{Cone}(f))$ with the component id_X. We have $m_3(c, b, a) = 0$. However, the triple Massey product $MP(c, b, a)$ is the class of $m_2(c, \alpha) = \mathrm{id}_X$ (it is not necessarily univalued).

This construction of the cones shows that if $\mathcal{A}$ is a triangulated A_∞-category then $H^0(\mathcal{A})$ is triangulated in the usual sense. Any A_∞-functor between triangulated A_∞-categories gives an exact functor.

Example 1.5.2. Note that triple Massey products can be computed using the triangulated structure on $H^0(Tw\mathcal{A})$. Consider the cone of the middle arrow (which is a twisted complex),

$$C_2 := \mathrm{Cone}(a_2) = (Y[1] \oplus Z, a_2).$$

Then the pair $\widetilde{a}_3 := (a_3, a_{23})$ gives a closed morphism in $\mathrm{Hom}^{-1}(X, C_2)$, while the pair $\widetilde{a}_1 := (a_{13}, a_1)$ gives a closed morphism in $\mathrm{Hom}^0(C_2, T)$. Now the Massey product is the class of the composition $\widetilde{a}_1 \circ \widetilde{a}_3$.

1.5.4. *Generators.* Recall that an object G of a triangulated category $\mathcal{T}$ is called a *classical generator* (aka *split-generator*) if the minimal triangulated subcategory of $\mathcal{T}$ closed under passing to direct summands and containing G is the entire category.

Sometimes, one also uses a more restricted notion of G generating $\mathcal{T}$ as a triangulated category (i.e., not allowing passing to direct summands).

Proposition 1.5.3. *(see* [43, Lemma 3.34] *If the image of cohomological full and faithful A_∞-functor $F : \mathcal{A} \to \mathcal{B}$ generates $\mathcal{B}$ (as a triangulated A_∞-category) then F extends to a quasi-equivalence $F : Tw\mathcal{A} \to \mathcal{B}$.*

An additive category is called *split-closed* if every idempotent in it splits. A triangulated A_∞-category $\mathcal{A}$ is called split-closed if $H^0\mathcal{A}$ is such. Every triangulated A_∞-category has a canonical *split-closure*, which can be constructed using the Yoneda embedding into the A_∞-category of A_∞-modules. Typically, A_∞-categories arising in geometry, such as derived categories of coherent sheaves (or their perfect subcategories), are split-closed.

We say that an object G generates a triangulated A_∞-category $\mathcal{A}$ if the split-closed triangulated A_∞-subcategory containing G is the entire $\mathcal{A}$.

Proposition 1.5.4. *(see* [43, Cor. 4.9]*) if $\mathcal{A} \subset \mathcal{B}$ full subcategory in a split-closed triangulated A_∞-category, such that $\mathcal{A}$ split-generates $\mathcal{B}$, then $\mathcal{B}$ is equivalent to the split closure of $Tw\mathcal{A}$.*

1.6. Cyclic A_∞-structures.

Definition 1.6.1. Let $(A, m_\bullet)$ be an A_∞-structure over a field k, equipped with a bilinear form $\langle \cdot, \cdot \rangle \to k$. We say that $(A, m_\bullet)$ is cyclic with respect to this bilinear form if

$$\langle m_n(a_1, \dots, a_n), a_{n+1} \rangle = (-1)^{n(\deg(a_1)+1)} \langle a_1, m_n(a_2, \dots, a_{n+1}) \rangle.$$

We say that a bilinear form has degree N if $\langle x, y \rangle = 0$ for homogeneous elements x, y such that $\deg(x) + \deg(y) \neq N$.

It is often convenient to have a cyclic structure since it cuts down the number of higher products to be considered. On a conceptual level these cyclic symmetries should be viewed as an algebraic version of a Calabi-Yau condition. Cyclic A_∞-structures play a central role in Costello's construction of a Gromov-Witten type potential (see [6]).

It is a natural question what data should be given on a dg-algebra, so that the corresponding minimal A_∞-algebra given by a homological perturbation is cyclic. Kontsevich-Soibelman in [17] give the following sufficient criterion which uses cyclic homology HC_*.

Theorem 1.6.2. *(*[17, Thm. 10.2.2] *Let A be an A_∞-algebra over a field k of characteristic zero with finite dimensional cohomology $H^*(A)$ (with respect to m_1). Suppose $\theta : HC_N(A) \to k$ is a functional such that the induced degree N pairing on $H^*(A)$ given by*

$$\langle x, y \rangle = \theta(\iota(xy)),$$

where $\iota : H^*(A) \to HC_*(A)$ is the natural map, is perfect. Then the corresponding minimal A_∞-structure on $H^*(A)$ is gauge equivalent to a cyclic A_∞-structure with respect to the above pairing.

Corollary 1.6.3. *Let* (B, d) *be a dg-algebra over a field* k *of characteristic zero such that* $H^*(B)$ *is finite-dimensional. Suppose we are given a functional* $\theta : B_N \to k$ *such that* $\theta(dB_{N-1}) = 0$, $\theta(xy) = (-1)^{\deg(x)\deg(y)}\theta(yx)$, *and the induced pairing* $\theta(xy)$ *(of degree* N*) on* $H^*(B)$ *is perfect. Then the minimal* A_∞*-structure on* $H^*(B)$ *obtained by homological perturbation is cyclic with respect to* $\theta(xy)$.

Proof. We just have to observe that θ extends (trivially) to a functional on $HC_N(B)$. □

Here is an example of a geometric setup where the above result can be applied.

Corollary 1.6.4. *Let* $\mathcal{A}$ *be a bounded complex of coherent sheaves on a reduced connected projective curve* C *over a field* k *of characteristic zero, equipped with a structure of dg-*$\mathcal{O}$*-algebras. Suppose we are given a morphism of coherent sheaves* $\tau : \mathcal{A}_0 \to \omega_C$ *satisfying* $\tau \circ d = 0$, $\tau(xy) = (-1)^{\deg(x)\deg(y)}\tau(yx)$ *and such that inducing pairing*

$$\mathcal{A} \otimes \mathcal{A} \to \omega_C \tag{1.6.1}$$

is perfect in derived category, i.e., the induced morphisms $\mathcal{A} \to \underline{\mathrm{Hom}}(\mathcal{A}, \omega_C) \to R\underline{\mathrm{Hom}}(\mathcal{A}, \omega_C)$ *are quasi-isomorphisms. Then there is a minimal* A_∞*-structure on* $H^*(C, \mathcal{A})$ *compatible with the pairing* $\theta(xy)$*, where* $\theta : H^1(C, \mathcal{A}) \to H^1(C, \omega_C) \simeq k$ *is induced by* τ.

Proof. Let $B = K^\bullet(\mathcal{A})$ be the Cech complex associated with a covering of C by two open subsets U_1 and U_2. We equip B with multiplicative structures using the following products of Cech 0-cochains and Cech 1-chains:

$$(a_1, a_2) \cdot b_{12} = \frac{1}{2}(a_1 + a_2)b_{12}, \quad b_{12} \cdot (a_1, a_2) = \frac{1}{2}b_{12}(a_1 + a_2). \tag{1.6.2}$$

The map τ induces a morphism of Cech complexes

$$K^\bullet(\theta) : K^\bullet(\mathcal{A})[1] \to K^\bullet(\omega_C).$$

The functional $H^1(C, \omega_C) \xrightarrow{\sim} k$ can be viewed as a functional on $K^1(\omega_C)$. Composing it with $K^\bullet(\theta)$ we get a functional

$$\theta : B_1 \to k$$

which vanishes on the image of d. Also, the condition that τ vanishes on supercommutators implies the same condition for θ. Finally, the assumption that we get a perfect pairing (1.6.1) together with Serre duality implies that $\theta(xy)$ induces a perfect pairing on $H^*(B)$. Thus, all the conditions for applying Corollary 1.6.3 are satisfied. □

Let us point out the following higher-dimensional version of Corollary 1.6.4.

Proposition 1.6.5. *Let* X *be a projective equidimensional CM-scheme of dimension* N *over a field* k *of characteristic zero,* $(\mathcal{A}_\bullet, d)$ *a complex of coherent sheaves over* X*, equipped with a dg-algebra structure (with unit). Assume that we have a morphism* $\tau : \mathcal{A}_0 \to \omega_X$ *such that* $\tau \circ d = 0$ *and* $\tau(xy) = (-1)^{\deg(x)\deg(y)}\tau(yx)$*. Then* τ *gives rise to a morphism in derived category*

$$\mathcal{A}_\bullet \otimes \mathcal{A}_\bullet \to \omega_X.$$

Assume that the induced pairings

$$H^i(C, \mathcal{A}_\bullet) \otimes H^{n-i}(C, \mathcal{A}_\bullet) \to H^n(C, \omega_X) \to k$$

are perfect (where $H^i(C, \mathcal{A}_\bullet)$ *are hypercohomology). Then the minimal* A_∞*-structure on* $H^*(C, \mathcal{A}_\bullet)$ *obtained by homological perturbation is gauge equivalent to a one, cyclic with respect to the pairing* $\theta(xy)$*, where* $\theta : H^n(C, \mathcal{A}_\bullet) \to H^n(C, \omega_C) \to k$ *is induced by* τ*.*

Proof. The Cech complex of $\mathcal{A}$ with respect to a finite covering $\mathcal{U}$ of X is obtained from the corresponding cosimplicial dg-algebra $C = C_{\mathcal{U}}(\mathcal{A})$. Applying instead Thom-Sullivan normalization $N(\cdot)^{TS}$ (see [12, Sec. 5.2], [52, App. A,B]), we get a dg-algebra $B = N(C)^{TS}$ computing $H^*(C, \mathcal{A}_\bullet)$. Furthermore, by functoriality of the construction, from the morphism τ, viewed as a chain map of complexes $\mathcal{A} \to \omega_X$, we get a chain map

$$N(\tau) : B \to N(C_{\mathcal{U}}(\omega_X))^{TS},$$

where the latter complex represents $R\Gamma(X, \omega_X)$. Note that the product structure on $N(C)^{TS}$ is induced by the natural morphisms $N(C)^{TS} \otimes N(C)^{TS} \to N(C \otimes C)^{TS}$, together with the product maps on C. Thus, we derive that $N(\tau)$ satisfies the same identity as τ, i.e., it vanishes on supercommutators. Composing $N(\tau)$ with a chain map $T : N(C_{\mathcal{U}}(\omega_X))^{TS} \to k[-N]$ representing the canonical trace map $H^n(X, \omega_C) \to k$, we get a chain map $\theta : B \to k[-N]$ vanishing on supercommutators. By assumption, the induced pairing on $H^*(B)$ is perfect, so we can apply Corollary 1.6.3 again. □

There is a particular case when the cyclic A_∞-structure can be obtained directly by homological perturbation (i.e., without using Theorem 1.6.2). Namely, suppose we are given a dg-algebra (B, d) and a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on B (the symmetry means $\langle x, y \rangle = (-1)^{\deg(x)\deg(y)} \langle y, x \rangle$).

Proposition 1.6.6. *(*[18]*) Suppose*

$$\langle dx, y \rangle + (-1)^{\deg(x)} \langle x, dy \rangle = 0 \tag{1.6.3}$$

and the homotopy $Q : B \to B$ *in the data for the homological perturbation satisfies*

$$\langle Qx, y \rangle = (-1)^{\deg(x)} \langle x, Qy \rangle. \tag{1.6.4}$$

Then the minimal A_∞*-structure on* $H^*(B)$ *given by the tree formula is cyclic with respect to the pairing induced by* $\langle \cdot, \cdot \rangle$*.*

To get the homotopy operator of this kind one can use complements to $\ker(d)$ of a special kind.

Lemma 1.6.7. *Assume that* $C \subset B$ *is a subspace such that* $\ker(d) \oplus B = C$*, and let* $A \subset \ker(d)$ *be a subspace of cohomology representatives. Assume* $\langle C, C \rangle = 0$ *and* $\langle C, A \rangle = 0$*. Then the homotopy operator* Q *associated with* A *and* C *satisfies* (1.6.4) *and hence, we obtain a cyclic* A_∞*-structure on* $H^*(B)$*.*

Using this we get the following direct construction of cyclic A_∞-structures.

Proposition 1.6.8. *Let* $B = B_0 \oplus B_1$ *be a dg-algebra concentrated in degrees* $[0, 1]$*,* $\langle \cdot, \cdot \rangle$ *a symmetric pairing of degree* 1 *on* B *satisfying* (1.6.3)*, such that* $H^*(B)$ *is finite-dimensional and the induced pairing on* $H^*(B)$ *is perfect. Let also* $A \subset \ker(d) \subset B$ *be a subspace of cohomology representatives. Then there exists a subspace* $C \subset B_0$*, complementary to* $\ker(d)$*, such that the corresponding* A_∞*-structure on* $H^*(B)$*, obtained by the homological perturbation, is cyclic with respect to the pairing induced by* $\langle \cdot, \cdot \rangle$*.*

Proof. The condition $\langle C, C\rangle = 0$ is automatic since $C \subset B_0$. The pairing $C \otimes A_1 \to k$ can be interpreted as a map $C \to A_1^* \simeq A_0$ (where the latter isomorphism is given by the pairing between A_0 and A_1). Correcting C by this map, we get a new subspace in $C \oplus A_0$, which is still complementary to $\ker(d)$, and which is orthogonal to A_1. Then we can apply Lemma 1.6.7 and Proposition 1.6.6. □

Corollary 1.6.9. *Let C be a projective connected reduced curve over a field k of characteristic $\neq 2$, and let $\mathcal{A}$ be a coherent sheaf of $\mathcal{O}_C$-algebras, equipped with a moprhism $\tau : \mathcal{A} \to \omega_X$. Assume that we have a morphism $\tau : \mathcal{A} \to \omega_C$ such that $\tau(xy) = \tau(yx)$. Assume that the induced pairing*

$$\mathcal{A} \otimes \mathcal{A} \to \omega_C$$

is perfect in the derived category. Then the minimal A_∞-structure on $H^*(C, \mathcal{A})$ obtained by homological perturbation is gauge equivalent to a one cyclic with respect to the pairing $\theta(xy)$, where $\theta : H^1(C, \mathcal{A}) \to H^n(C, \omega_C) \to k$ is induced by τ.

Proof. The corresponding Cech complex $K^\bullet(\mathcal{A})$, equipped with the product (1.6.2), satisfies assumptions of Proposition 1.6.8. □

2. Examples of calculations for elliptic curves

2.1. **Some triple Massey products.** Let C be an elliptic curve. Consider composable arrows

$$\mathcal{O} \to \mathcal{O}_{x'} \xrightarrow{[1]} P \to \mathcal{O}_x \tag{2.1.1}$$

where $x, x' \in C$ and P is a line bundle of degree 0. Assume $x \neq x'$ and $P \neq \mathcal{O}$, then have $\mathrm{Hom}^*(\mathcal{O}, P) = 0$ and $\mathrm{Hom}^*(\mathcal{O}_{x'}, \mathcal{O}_x) = 0$. So this is a perfect setup for triple Massey products: the double compositions are automatically zero and there is no ambiguity.

By applying translation, can assume that $x' = e$, the neutral element of the group law. To compute the Massey product we include the second arrow in the exact triangle

$$P \to P(e) \to \mathcal{O}_e \to P[1]$$

Then we have to find a section of $P(e)$ with residue 1 at e and then evaluate at y.

Let L be a fixed line bundle of degree 1 on C and $\theta \in H^0(C, L)$ is a generator, such that θ vanishes at a point $e \in C$, which we can take as the neutral element of the group law. We can realize P uniquely as $t_y^* L \otimes L^{-1}$. Thus, $\theta(z + y)/\theta(z)$ is a section of P with a pole of order 1 at e, i.e., a section of $P(e)$. We should normalize it by the value of the residue at e, so we should consider $s(z) = \frac{\theta'(0)\cdot\theta(z+y)}{\theta(y)\theta(z)}$ and then evaluate it at x which gives

$$s(x) = \frac{\theta'(0) \cdot \theta(x + y)}{\theta(y)\theta(x)} =: F(x, y),$$

the Kronecker function (also studied by Zagier).

There is a generalization of this picture to higher rank vector bundles, which we sketch following [31, Sec. 1]. Recall that by the classical result of Atiyah, a vector bundle V on an elliptic curve C is stable if and only if it is simple and for a given pair $(r > 0, d)$, with $\gcd(r, d) = 1$, the moduli space $\mathcal{M} = \mathcal{M}_{r,d}$ of stable vector bundles of rank r and degree d is isomorphic to C. Note that distinct $V_1, V_2 \in \mathcal{M}$ one has $\mathrm{Hom}(V_1, V_2) = 0$ (by stability). Hence, using Serre duality, we see that $\mathrm{Ext}^1(V_1, V_2) = 0$. Thus, for a pair of points $x_1 \neq x_2$ of C, we again obtain a well defined univalued triple Massey product by looking at composable arrows

$$V_1 \to \mathcal{O}_{x_1} \xrightarrow{[1]} V_2 \to \mathcal{O}_{x_2}.$$

Note that this Massey product is a map

$$\begin{aligned}&(V_1|_{x_1})^* \otimes V_2|_{x_1} \otimes (V_2|_{x_1})^* \simeq \\ &\mathrm{Hom}(V_1, \mathcal{O}_{x_1}) \otimes \mathrm{Ext}^1(\mathcal{O}_{x_1}, V_2) \otimes \mathrm{Hom}(V_2, \mathcal{O}_{x_2}) \to \mathrm{Hom}(V_1, \mathcal{O}_{x_2}) \simeq (V_1|_{x_2})^*\end{aligned}$$

(here we use some trivialization of ω_C and the Serre duality for the identification $V_2|_{x_1} \simeq \mathrm{Ext}^1(\mathcal{O}_{x_1}, V_2)$). Replacing the middle arrow by a universal map $V_2|_{x_1} \otimes \mathcal{O}_{x_1} \xrightarrow{[1]} V_2$, we can compute the Massey product in the same way as before by using the exact triangle

$$V_2 \to V_2(x_1) \to V_2|_{x_1} \xrightarrow{[1]} V_2$$

Thus, our Massey product is determined from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(V_1|_{x_1}, V_2|_{x_1}) \arrow[rd, "MP"] & \mathrm{Hom}(V_1, V_2(x_1)) \arrow[l, "\sim"'] \arrow[d, "\mathrm{ev}_{x_2}"] \\
 & \mathrm{Hom}(V_1|_{x_2}, V_2|_{x_2})
\end{tikzcd}$$

where the horizontal arrow is an isomorphism due to the condition $\mathrm{Ext}^*(V_1, V_2) = 0$. Dualizing, we can view this Massey product as an element

$$r^{V_1V_2}_{x_1x_2} \in \mathrm{Hom}(V_2|_{x_1}, V_1|_{x_1}) \otimes \mathrm{Hom}(V_1|_{x_2}, V_2|_{x_2}).$$

One can apply the A_∞-axiom of the form $m_3(m_3(f_1, g_1, f_2), g_2, f_3) + \ldots = 0$ to sequences of composable arrows

$$V_1 \xrightarrow{f_1} \mathcal{O}_{x_1} \xrightarrow{g_1} V_2 \xrightarrow{f_2} \mathcal{O}_{x_2} \xrightarrow{g_2} V_3 \xrightarrow{f_3} \mathcal{O}_{x_3},$$

where $\deg(f_i) = 0$, $\deg(g_i) = 1$, and (V_i) and (x_j) are distinct. Using results of Sec. 1.6 one check in addition the following cyclic symmetry:

$$\langle f', m_3(g_1, f_2, g_2) \rangle = -\langle m_3(f', g_1, f_2), g_2 \rangle$$

where $f' \in \mathrm{Hom}(V_3, \mathcal{O}_{x_1})$ and $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathrm{Hom}(V_i, \mathcal{O}_{x_j})$ and $\mathrm{Ext}^1(\mathcal{O}_{x_j}, V_i)$. This allows to express all terms of the A_∞-axiom via $r^{V_i,V_{i'}}_{x_j,x_{j'}}$ and leads to the following equation

$$(2.1.2) \qquad (r^{V_3V_2}_{x_1x_2})^{12}(r^{V_1V_3}_{x_1x_3})^{13} - (r^{V_1V_3}_{x_2x_3})^{23}(r^{V_1V_2}_{x_1x_2})^{12} + (r^{V_1V_2}_{x_1x_3})^{13}(r^{V_2V_3}_{x_2x_3})^{23} = 0$$

in $\mathrm{Hom}(V_2|_{x_1}, V_1|_{x_1}) \otimes \mathrm{Hom}(V_3|_{x_2}, V_2|_{x_1}) \otimes \mathrm{Hom}(V_1|_{x_3}, V_3|_{x_3})$, which is called a *set-theoretical Associative Yang-Baxter Equation (AYBE)*. We will return to this equation later in Sec. 5.2.

Lifting the points x_i to a universal covering $\mathbb{C} \to C$ one can choose trivializations of all vector spaces $V_i|_{x_j}$ and express the tensors $r^{V_i,V_j}_{x_i,x_j}$ in terms of the Kronecker function $F(x, y)$ (with C replaced by a finite étale covering), see [31, Sec. 2.2].

Remark 2.1.1. There is a partial generalization of this picture to higher genus curves studied in [33].

2.2. **Line bundles of degree 0 and 1: transcendental computation.** Let $C = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, be an elliptic curve. We want to compute (following [39]) the A_∞-structure, obtained by homological perturbation, on the algebra

$$E = \mathrm{Ext}^*(G, G), \text{ where } G = \bigoplus_{i=1}^{r} P_i \oplus \bigoplus_{j=1}^{s} L_j,$$

where (P_i) are distinct line bundles of degree 0, and (L_j) are distinct line bundles of degree 1 on C.

Let L be the standard line bundle of degree 1 on C such that the theta-function

$$\theta = \theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i \tau n^2 + 2\pi i n z)$$

descends to a global section of L. Then the line bundles (P_i), (L_j) can be written in the form

$$P_i = P(w_i), \quad L_j = t_{z_j}^* L,$$

for some complex numbers (w_i), (z_j) (unique modulo Λ), where $P(w) := t_w^* L \otimes L^{-1}$.

Note that E is obtained as the cohomology of the Dolbeault dg-algebra

$$\Omega G := (\Omega^{0,*}(\underline{\mathrm{End}}(G)), \overline{\partial}).$$

To construct the cohomology representatives and the homotopy operator Q on ΩG we use the flat metric on C and on the relevant line bundles. Namely, the hermitian metric on L is given by

$$(f, g) = \int_C f(z)\overline{g(z)} \exp\big(-2\pi \frac{y^2}{\mathrm{Im}(\tau)}\big) dx dy,$$

where $z = x + iy$. To get metrics on L_j we use the translation $t_{z_j}^*$. Also, we get the induced metrics on $P_i = t_{w_i}^* L \otimes L^{-1}$.

Then we get the required complements to $\ker(\overline{\partial})$ and to $\mathrm{im}(\overline{\partial})$ in $\ker(\overline{\partial})$ as orthogonals with respect to the metric. In particular, E will be embedded into ΩG as the subspace of harmonic forms.

We fix a generator $\xi \in H^1(C, \mathcal{O})$ which is represented by the $(0,1)$-form $d\overline{z}$. Let $\eta \in H^1(C, L^{-1})$ denote the unique generator such that $\eta \circ \theta = \xi$. Then the space $\mathrm{Ext}^*(G, G)$ has the following natural basis:
(i) identity elements in $\mathrm{Hom}(P_i, P_i)$, $\mathrm{Hom}(L_j, L_j)$;
(ii) the elements $\xi_i \in \mathrm{Ext}^1(P_i, P_i)$ and $\xi_j \in \mathrm{Ext}^1(L_j, L_j)$, corresponding to the canonical generator $\xi \in H^1(C, \mathcal{O})$;
(iii) $\theta_{ij} := t_{z_j - w_i}^* \theta \in H^0(t_{z_j - w_i}^* L) \simeq \mathrm{Hom}(P_i, L_j)$;
(iv) $\eta_{ji} := t_{z_j - w_i}^* \eta \in H^1(t_{z_j - w_i}^* L^{-1}) \simeq \mathrm{Ext}^1(L_j, P_i)$.

Note that θ_{ij} are holomorphic functions, so they are harmonic. The $(0,1)$-form $d\overline{z}$ representing ξ is also harmonic. The harmonic $(0,1)$-form with values in L^{-1} representing

$\eta \in H^1(C, L^{-1})$ is

$$\eta := \sqrt{2\operatorname{Im}(\tau)} \cdot \overline{\theta(z,\tau)} \exp\bigl(-2\pi \frac{\operatorname{Im}(z)^2}{\operatorname{Im}(\tau)}\bigr) d\overline{z}.$$

Aside from multiplications with the identity elements, the only nontrivial compositions in E are

$$\eta_{ji} \circ \theta_{ij} = \xi_i, \quad \theta_{ij} \circ \eta_{ji} = \xi_j.$$

The *Eisenstein-Kronecker-Lerch series* (see [53, ch. VIII]) are given by

$$K_a^*(z,w,s;\Lambda) = \sum_{\lambda \in \Lambda \setminus \{-z\}} \frac{(\overline{z}+\overline{\lambda})^a}{|z+\lambda|^{2s}} \langle \lambda, w \rangle_\Lambda,$$

where $a \in \mathbb{Z}_{\geq 0}$, $z, w \in \mathbb{C}$, s is a real number,

$$\langle z, w \rangle_\Lambda := \exp[A^{-1}(z\overline{w} - w\overline{z})],$$

where $A = \operatorname{Im}(\tau)/\pi$. This series converges absolutely for $\operatorname{Re} s > a/2 + 1$. It is known that $K_a^*(z,w,s;\Lambda)$ analytically extends (for fixed z, w) to a meromorphic function on the entire s-plane, with possible poles only at $s = 0$ (for $a = 0$, $z \in \Lambda$) and at $s = 1$ (for $a = 0$, $w \in \Lambda$). Using this analytical continuation the *Eisenstein-Kronecker numbers* $e^*_{a,b}(z,w;\Lambda)$, for integers $a \geq 0$, $b > 0$, are defined as the following special values:

$$e^*_{a,b}(z,w) = K^*_{a+b}(z,w,b;\Lambda).$$

Note that these values are not continuous in z and w (the discontinuity occurs when either $z \in \Lambda$ or $w \in \Lambda$).

Due to different conventions in [39], in the theorem below we actually compute the A_∞-structure (m_n) on E^{op}, obtained as the cohomology of the dg-algebra $(\Omega G)^{op}$. The opposite A_∞-structure on E differs from this by some signs.

Theorem 2.2.1. *([39, Thm. A]) For $a, b, c, d \geq 0$ one has*

$$m_n((\xi_i)^a, \theta_{ij}, (\xi_j)^b, \eta_{ji'}, (\xi_{i'})^c, \theta_{i'j'}, (\xi_{j'})^d) =$$
$$(-1)^{\binom{n+1}{2}+1} \frac{A^{b+d+1}(b+d)!}{a!b!c!d!} \cdot e^*_{a+c,b+d+1}(z_{j'} - z_j, w_i - w_{i'}) \cdot \frac{f_{i'j} f_{ij'}}{f_{ij} f_{i'j'}} \cdot \theta_{ij'},$$

where $f_{ij} = \exp(A^{-1}(z_j - w_i + \overline{w_i})^2/2)$. Note that here the indices in the pairs (i, i') and (j, j') are not necessarily distinct.

The remaining m_n are determined by the condition that our A_∞-structure on $E = \operatorname{Ext}^*(G,G)$ is *cyclic* with respect to a natural pairing (see Sec. 1.6). Note also that one can get rid of the exponential factor (depending on f_{ij}) by rescaling the basis of E.

For example,

$$e^*_{0,2k}(0,0) = e_{2k} = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^{2k}}$$

for $k \geq 2$, $e^*_{2k+1}(0,0) = 0$, and

$$e^*_{0,2}(0,0) = e^*_2 = \sum_m \sum_{n; n \neq 0 \text{ if } m=0} \frac{1}{(m\tau+n)^2} - \frac{\pi}{\operatorname{Im}(\tau)}.$$

We will not give a full proof of Theorem 2.2.1, only a sample calculation. Namely, let us calculate $m_3(\theta_{ij}, \eta_{ji'}, \theta_{i'j'})$. Note that this is a univalued triple Massey product

$$P_i \to L_j \xrightarrow{[1]} P_{i'} \to L_{j'},$$

similar to (2.1.1) (in fact it can be obtained from (2.1.1) by an autoequivalence of the derived category). This time we calculate using the homological perturbation:

$$m_3(\theta_{ij}, \eta_{ji'}, \theta_{i'j'}) = \Pi[Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'})\theta_{i'j'} - \theta_{ij}Q_{P(z_{j'}-z_j)}(\eta_{ji'}\theta_{i'j'})].$$

Here for a holomorphic line bundle M we denote by $Q_M : \Omega^{(0,1)}(M) \to \Omega^{(0,0)}(M)$ our homotopy operator defined using hermitian metrics. In the case when $M = P(w)$ for $w \notin \Lambda$, Q_M is simply the inverse of $\overline{\partial}$.

For $w \notin \Lambda$, the line bundle $P(w)$ is trivialized as an C^∞-line bundle by a nowhere vanishing section $\exp(-2\pi i w \cdot v)$, where we use the real coordinates (u, v) such that $z = u + v\tau$. Then the sections

$$\varphi_{w,\lambda}(z) := \langle \lambda, z \rangle \cdot \exp(-2\pi i w \cdot v)$$

form an orthonormal basis of L^2-sections of $P(w)$. Furthermore, one has

$$\overline{\partial}\varphi_{w,\lambda} = A^{-1}(\lambda + w) \cdot \varphi_{w,\lambda} d\overline{z},$$

and $Q_{P(w)}$ is just the inverse to $\overline{\partial}$, so

$$Q_{P(w)}(\varphi_{w,\lambda} d\overline{z}) = \frac{A}{\lambda + w}\varphi_{w,\lambda}.$$

We need to decompose $\theta_{ij}\eta_{ji'}$ (resp., $\theta_{i'j'}\eta_{ji'}$) in the orthonormal bases of sections of $P(w_{i'} - w_i)$ (resp., $P(z_{j'} - z_j)$). For this we use the identity

$$\begin{aligned}&\sqrt{2\operatorname{Im}(\tau)} \cdot \theta(z) \cdot \overline{\theta(z + z_0)} \exp(-2\pi \operatorname{Im}(\tau)(v + v_0)^2) = \\ &\exp(2\pi i z_0 v_0) \cdot \textstyle\sum_{\lambda \in \Lambda} c_\lambda(-z_0) \cdot \langle \lambda, z_0 \rangle \cdot \varphi_{-z_0,\lambda}(z),\end{aligned} \tag{2.2.1}$$

where $z_0 = u_0 + v_0\tau$,

$$c_{m\tau-n}(z) = (-1)^{mn} \exp(-A^{-1}(|\lambda|^2 + 2\overline{\lambda}z + z^2)/2).$$

This identity is proved by interpreting the Fourier coefficients of the above product as integrals; then by rewriting them as hermitian pairings of the form

$$(\theta(z + w), \theta(z) \cdot \exp(-2\pi w \cdot v))$$

of sections of the line bundle $t_w^* L \simeq L \otimes P(w)$; and finally using the expansion of the theta-function to rewrite as a Gaussian integral over $\mathbb{R}$ (see [29, Sec. 2]; the above identity also follows from [47, Prop. 4.1]). Applying (2.2.1) we get

$$\theta_{ij}(z)\eta_{ji'}(z) = \sum_{\lambda \in \Lambda} \varphi_{w_{i'}-w_i,\lambda}(z) \cdot c_\lambda(w_{i'} - w_i)\langle \lambda, z_j - w_{i'} \rangle \exp(2\pi i(w_i - w_{i'})\frac{\operatorname{Im}(z_j - w_{i'})}{\operatorname{Im}\tau}) \cdot d\overline{z},$$

and hence,

$$\begin{aligned}&Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'}) = \\ &A \cdot \sum_{\lambda \in \Lambda} \varphi_{w_{i'}-w_i,\lambda}(z) \cdot \frac{c_\lambda(w_{i'} - w_i)\langle \lambda, z_j - w_{i'} \rangle}{\lambda + w_{i'} - w_i} \exp(2\pi i(w_i - w_{i'})\frac{\operatorname{Im}(z_j - w_{i'})}{\operatorname{Im}\tau}).\end{aligned}$$

Now computing $\Pi[Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'})\theta_{i'j'}]$ is equivalent to calculating the pairings

$$(Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'})\theta_{i'j'},\theta_{ij'}) = \int_C Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'})\cdot\theta_{i'j'}\overline{\theta_{ij'}}\exp(-2\pi\frac{\mathrm{Im}(z+z_{j'}-w_i)^2}{\mathrm{Im}\,\tau})dxdy.$$

Applying (2.2.1) again we get the expansion

$$\theta_{i'j'}\overline{\theta_{ij'}}\exp(-2\pi\frac{\mathrm{Im}(z+z_{j'}-w_i)^2}{\mathrm{Im}\,\tau}) =$$
$$\frac{1}{\sqrt{2\,\mathrm{Im}\,\tau}}\cdot\sum_{\lambda\in\Lambda}\varphi_{w_i-w_{i'},\lambda}(z)\cdot c_\lambda(w_i-w_{i'})\cdot\langle\lambda,z_{j'}-w_i\rangle\exp(2\pi i(w_{i'}-w_i)\frac{\mathrm{Im}(z_{j'}-w_i)}{\mathrm{Im}(\tau)}).$$

Now we observe that

$$\int_C \varphi_{w,\lambda}\varphi_{-w,\lambda'}dxdy = \begin{cases}\mathrm{Im}(\tau) & \lambda+\lambda'=0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, the above pairing is equal to

$$(Q_{P(w_{i'}-w_i)}(\theta_{ij}\eta_{ji'})\theta_{i'j'},\theta_{ij'}) = A\cdot\sqrt{\frac{\mathrm{Im}\,\tau}{2}}\cdot\exp(2\pi i(w_i-w_{i'})\frac{\mathrm{Im}(z_j-z_{j'}+w_i-w_{i'})}{\mathrm{Im}\,\tau})\cdot$$
$$\sum_{\lambda\in\Lambda}\frac{c_\lambda(w_{i'}-w_i)c_{-\lambda}(w_i-w_{i'})\langle\lambda,z_j-z_{j'}+w_i-w_{i'}\rangle}{\lambda+w_{i'}-w_i},$$

whereas $(\theta_{ij'},\theta_{ij'}) = 1/\sqrt{2\,\mathrm{Im}\,\tau}$.

Computing $\Pi[\theta_{ij}Q_{P(z_{j'}-z_j)}(\eta_{ji'}\theta_{i'j'})$ in a similar way, we get the following answer

$$m_3(\theta_{ij},\eta_{ji'},\theta_{i'j'}) = A\cdot\big(\Phi(w_{i'}-w_i,z_{j'}-z_j)-\Phi(z_{j'}-z_j,w_{i'}-w_i)\big)\cdot\theta_{ij'},\quad\text{where}$$

$$\Phi(z_0,w_0) = \exp(A^{-1}z_0(w_0-\overline{w}_0))\cdot\sum_{\lambda\in\Lambda}\frac{1}{\lambda+z_0}\exp(-A^{-1}|\lambda+z_0|^2)\langle-\lambda,w_0\rangle.$$

To get the statement of Theorem 2.2.1 in this case one has to use in addition the identity

$$e^*_{01}(z,w) = \exp(A^{-1}z(w-\overline{w}))[\Phi(z,-w)-\Phi(-w,z)].$$

On the other hand, our answer is compatible with the computation of the triple Massey product in Sec. 2.1 because of the identity

$$\Phi(z,-w)-\Phi(-w,z) = 2\pi iF(z,w), \tag{2.2.2}$$

where $F(z,w)$ is the Kronecker function. The relation between $e^*_{01}(z,w)$ and the Kronecker function is classical (see [53, VIII.2, Eq.(3), p.70]). On the other hand, the identity (2.2.2) was discovered in [30]. The main part of the proof is checking that $\Phi(z,-w)-\Phi(-w,z)$ is meromorphic in z and w (this uses the Poisson summation formula).

Note that if we just take the generator $G=\mathcal{O}\oplus L$ then the A_∞-structure on $\mathrm{Ext}^*(G,G)$ is expressed in terms of $e^*_{a,b}(0,0)$. Using the A_∞-constraints one can show that these are expressed as some polynomials with $\mathbb{Q}$-coefficients in (e^*_n) (and hence, as polynomials in e^*_2, e_4 and e_6). Namely, let us set for $a,b\ge 0$,

$$g_{a,b} = \frac{b!}{A^a}e^*_{a,b+1}.$$

Then $g_{a,b} = g_{b,a}$, and $g_{a,b} = 0$ if $a + b$ is even. Applying the A_∞-constraint to the string

$$(\xi)^a, \theta, \eta, \theta, \eta, \theta, (\xi)^b$$

one gets the identity

$$\sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_1,0} g_{a_2,b} - \frac{a+2+\delta_{b,0}}{a+1} g_{a+1,b} = \sum_{b=b_1+b_2} \binom{b}{b_1} g_{0,b_1} g_{a,b_2} - \frac{b+2+\delta_{a,0}}{b+1} g_{a,b+1},$$

which gives a recursive formula for $g_{a+1,b}$ in terms of all $g_{a',b'}$ with $a' \leq a$. Since $g_{0,n} = n! e^*_{n+1}$, we get a procedure to express all $g_{a,b}$ as polynomials in (e^*_n) (with rational coefficients).

Kaneko-Zagier theory states that the ring $\mathbb{C}[e^*_2, e_4, e_6]$ is isomorphic to the ring of quasimodular forms $\mathbb{C}[E_2, E_4, E_6]$, where

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where $\sigma_p(n)$ is the sum of the pth powers of the divisors of n. More precisely, this isomorphism sends E_4 and E_6 to themselves viewed as modular forms (recall that $E_{2k}(\exp(2\pi i\tau)) = e_{2k}(\tau)/(2\zeta(2k))$), and it sends E_2 to the holomorphic part of $\frac{3}{\pi^2} e_2$, where $e^*_2 = e_2 - \pi/\operatorname{Im}\tau$.

Remark 2.2.2. Caldararu and Tu [4] use this to get a purely holomorphic model for the A_∞-structure on $\operatorname{Ext}^*(G, G)$. More precisely, one can view the A_∞-algebra given by Theorem 2.2.1 (with $w_i = z_j = 0$) as an A_∞-algebra E_τ over the ring of *almost holomorphic forms*, i.e., polynomials in e^*_2, e_4, e_6. On the other hand, applying the Kaneko-Zagier isomorphism, one gets an A_∞-algebra E^{hol}_τ over the ring of quasimodular forms. Extending scalars, we can view both E_τ and E^{hol}_τ as minimal A_∞-algebras over the ring of C^∞-functions on the upper-half plane (with the same underlying associative algebra). Caldararu and Tu show in [4, Thm. 5.14] that there is a gauge equivalence between these two structures.

Remark 2.2.3. Our formulas also show that in the case $G = \mathcal{O} \oplus L$ one has $m_n = 0$ for all odd n. In fact, any minimal A_∞-algebra structure on the corresponding algebra E is gauge equivalent to the one with $m_3 = 0$ since $HH^2(E)_{-1} = 0$ (see ???). In fact it turns out that $HH^2(E)_{2-i} = 0$ for $i = 3, 4, 5, 7$ and $i \geq 9$, while $HH^2(E)_{-4}$ and $HH^2(E)_{-6}$ are 1-dimensional. Thus, by a gauge equivalence we can turn any A_∞-structure into the one with $m_3 = m_4 = m_5 = 0$, so that m_6 and m_8 will be Hochschild cocycles. It is not a priori clear that the classes of $[m_6]$ and $[m_8]$ in the 1-dimensional spaces $HH^2(E)_{-4}$ and $HH^2(E)_{-6}$ are well defined functions of a gauge equivalence orbit of m. Later we will see that this is indeed the case, and the relevant moduli space of A_∞-structures is equivalent to the affine plane (which can be though of as the space of cubics in the Weierstrass normal form).

3. MODULI SPACES OF A_∞-STRUCTURES

3.1. **The moduli problem.** We start with a given graded sheaf of $\mathcal{O}$-algebras $\mathcal{E}$ over a scheme S and would like to define the corresponding moduli problem for A_∞-structures on $\mathcal{E}$.

Note that all the notions related to A_∞-structures over a commutative ring generalize readily to the case of sheaves of $\mathcal{O}$-modules over a scheme. Namely, for a graded sheaf $\mathcal{F}$ of locally free $\mathcal{O}$-modules over a scheme S we denote by $\underline{CH}^{s+t}(\mathcal{F}/S)_t$ the sheaf of homomorphisms of $\mathcal{O}$-modules $\mathcal{F}^{\otimes s} \to \mathcal{F}$ of degree t, and by $CH^{s+t}(\mathcal{F}/S)_t$ its space of global sections over S. We have a natural notion of an A_n-structure (resp., A_∞-structure) on $\mathcal{F}$, given by a collection of global sections

$$m = (m_1, \ldots, m_n) \in CH^2(\mathcal{F}/S)_1 \times \ldots \times CH^2(\mathcal{F}/S)_{2-n})$$

(resp., $m = (m_1, m_2, \ldots)$ with $m_n \in CH^2(\mathcal{F}/S)_{2-n}$), satisfying the standard A_∞-identities involving only $m_1, \ldots, m_n$ (resp., all A_∞-identities). Similarly, the definitions of A_∞-morphisms and homotopies between them and the results of Sec. 1.3 immediately generalize to this context.

Since we are interested in *minimal* A_n-structures (resp., A_∞-structure), i.e., those with $m_1 = 0$, we consider A'_n-structures, i.e., A_n-structures satisfying one addiitional A_∞-identity involving $[m_2, m_n]$ (see Definition 1.3.1). The action of the group of gauge transformations on the set of minimal A'_n-structures also immediately generalizes to the relative context: we have a sheaf of groups $\mathfrak{G}$ over S, where an element of $\mathfrak{G}(U)$ is a collection of sections

$$f = (f_1 = \mathrm{id}, f_2, \ldots) \in H^0(U, \underline{CH}^1(\mathcal{F}/S)_{-1} \times \underline{CH}^1(\mathcal{F}/S)_{-2} \times \ldots),$$

with the product rule obtained by interpreting f as a coalgebra automorphism of the bar-coalgebra of $\mathcal{F}$ (see Sec. 1.3). We use the notation $\mathfrak{G}[2, n-1] := \mathfrak{G}/\mathfrak{G}_{\geq n}$ for the quotient of $\mathfrak{G}$ acting on the set of minimal A'_n-structures on $\mathcal{F}$. We denote the projection $\mathfrak{G} \to \mathfrak{G}[2, n-1]$ by $f \mapsto f_{\leq n-1}$.

Remark 3.1.1. The above definition of an A_n-algebra over a scheme is a bit naive. A more flexible notion should involve defining m_i's only over an open covering U_i of S, and the gluing should be given by a collection of higher homotopies defined on intersections $U_{i_1} \cap \ldots \cap U_{i_r}$. We do not need the most general definition since we only aim at constructing the usual space as a moduli space of A_∞-structures (in good situations), not an ∞-stack. Even at this level we will need a certain gluing procedure, but a much simpler one.

Now let us fix a scheme S and a sheaf $\mathcal{E}$ of graded associative $\mathcal{O}_S$-algebras over S. We assume also that $\mathcal{E}$ is locally free of finite rank over $\mathcal{O}_S$. We denote by $\mathcal{E}|_s$ the fiber of $\mathcal{E}$ over a point $s \in S$. Roughly speaking, we would like to classify families of minimal A_∞-algebras, up to gauge equivalence, such that the corresponding family of graded associative algebras is obtained from $\mathcal{E}$.

Definition 3.1.2. (i) For a sheaf of graded associative $\mathcal{O}_T$-algebras E over a scheme T we denote by $\mathcal{A}'_n(E/T)$, where $n \geq 2$ (resp., $\mathcal{A}_\infty(E/T)$), the set of minimal A'_n-structures (resp., A_∞-structures) on E.
(ii) Now for a fixed $(\mathcal{E}/S)$ as above, for $n \geq 2$, we have the presheaf $\mathcal{A}'_n = \mathcal{A}'_{n,\mathcal{E}}$ (resp., $\mathcal{A}_\infty = \mathcal{A}_{\infty,\mathcal{E}}$) on the category of S-schemes, which associates with $\varphi : T \to S$ the set $\mathcal{A}'_n(\varphi^*\mathcal{E}/T)$ (resp., $\mathcal{A}_\infty(\varphi^*\mathcal{E}/T)$). This functor is represented by an affine scheme of finite type over S, which we still denote by $\mathcal{A}'_{n,\mathcal{E}}$. Namely, $\mathcal{A}'_{n,\mathcal{E}}$ is the closed subscheme in the total space of the vector bundle $CH^2(\mathcal{E}/S)_{-1} \oplus \ldots \oplus CH^2(\mathcal{E}/S)_{2-n}$ given by the relevant A_∞-equations.

We have a natural projection

$$\pi_n : \mathcal{A}'_n \to \mathcal{A}'_{n-1} : m \mapsto m_{\leq n-1}. \tag{3.1.1}$$

Next, we have the sheaf of groups $\mathfrak{G}$ of gauge transformations acting on each functor $\mathcal{A}'_n$ through the quotient $\mathfrak{G}[2, n-1]$, and the first apporximation to our moduli functo is obtained by taking the quotient by this action.

Definition 3.1.3. For each $n \geq 2$, we define the functor

$$\widetilde{\mathcal{M}}_n : \mathrm{Sch}_S^{op} \to \mathrm{Sets}$$

where Sch_S is the category of S-schemes, as follows. For an S-scheme $f : T \to S$, we define

$$\widetilde{\mathcal{M}}_n(T) := \mathcal{A}'_n(T)/\mathfrak{G}[2, n-1](T).$$

Similarly, we set

$$\widetilde{\mathcal{M}}_\infty(T) := \mathcal{A}_\infty(T)/\mathfrak{G}(T).$$

In general, the quotient-functor $\widetilde{\mathcal{M}}_n$ is not representable and (at least) needs to be sheafified. Let us consider the topology on the category Sch_S, such that open coverings of $p : T \to S$ are pull-backs under p of Zariski open coverings of S. We call this *S-Zariski topology.*

Definition 3.1.4. Let us denote by $\mathcal{M}_n$ (resp., $\mathcal{M}_\infty$) the sheafification of the functor $\widetilde{\mathcal{M}}_n$ (resp., $\widetilde{\mathcal{M}}_\infty$) with respect to the S-Zariski topology.

3.2. **Nice quotients.**

Here we make a digression on a special situation when an action of a group scheme on a scheme admits a quotient. We work over a fixed base scheme S.

Definition 3.2.1. Let G be a group scheme, X be a G-scheme. We say that a G-invariant morphism $\pi : X \to Q$ is a *nice quotient* for the G-action on X if locally over S (in Zariski topology) there exists a section $\sigma : Q \to X$ of π and a morphism $\rho : X \to G$, such that

$$x = \rho(x)\sigma(\pi(x)) \quad \text{and} \quad \rho(\sigma(x)) = 1. \tag{3.2.1}$$

In this situation we call $\sigma(Q)$ a *nice section* for the action of G on X. We say that π is a *strict nice quotient* if ρ and σ can be defined globally over S.

In the case when S is a point we obtain precisely the situation of [35, Def. 4.2.2].

Note that a nice quotient is automatically a categorical quotient (in the category of S-schemes). Indeed, let $f : X \to Z$ be a G-invariant morphism, where Z has trivial G-action. Then $f(x) = f(\sigma(\pi(x)))$, so f is a composition of $f \circ \sigma : Q \to Z$ with π. This implies that the existence of a nice quotient is a local quesion in S. Namely, if $X_i \to Q_i$ are nice quotients for $X_i = p^{-1}(U_i)$, where (U_i) is an open covering of S, $p : X \to S$ is a projection, then we can glue them into a global morphism $\pi : X \to Q$.

Remark 3.2.2. If $\pi : X \to Q$ is a nice quotient for the G-action on X then π is a universal geometric quotient (see [27]). Indeed, any base change of π is still a nice quotient. The following properties are clear: π is surjective, $U \subset Q$ is open if and only if $\pi^{-1}(U)$ is open, geometric fibers are precisely the orbits of geometric points. Finally, we claim that $\mathcal{O}_Q$ coincides with G-invariants in $\pi_*\mathcal{O}_X$. Indeed, given a G-invariant function f on $\pi^{-1}(U)$ then $f(x) = f(\sigma(\pi(x)))$, so it descends to the function $f \circ \sigma$ on U.

Let us consider the following presheaf of sets on Sch_S:

$$T \mapsto X(T)/G(T).$$

Lemma 3.2.3. *Let $\pi : X \to Q$ is a nice (resp., strict nice) quotient for the G-action then the sheafification of the above presheaf with respect to the S-Zariski topology (resp., the presheaf itself) is naturally isomorphic to the functor represented by Q. Thus, a T-point of Q can be represented by a collection of V_i-points of X, where $V_i = f^{-1}(U_i)$ for some open covering (U_i) of S, such that for any i, j, the corresponding V_{ij}-points of X, where $V_{ij} = V_i \cap V_j$, differ by $G(V_{ij})$-action.*

Proof. We have a natural morphism from $X(T)/G(T)$ to the sheaf represented by Q, which becomes an isomorphism over an open affine covering of S (due to the existence of a decomposition (3.2.1)). This immediately implies the assertion. $\square$

The following lemma will help us to construct nice quotients inductively.

Lemma 3.2.4. *Let G be a group scheme over S acting on a scheme X over S. Assume that G fits into an exact sequence of group schemes*

$$1 \to H \to G \to G' \to 1$$

and that the projection $G \to G'$ admits a section $s : G' \to G$ which is a morphism of schemes (not necessarily compatible with the group structures). Suppose we have a scheme X' with an action of G' and a morphism $f : X \to X'$ compatible with the G-action via the homomorphism $G \to G'$. Assume that there exists a nice quotient $\pi_H : X \to Q_H$ for the H-action on X and a nice quotient $\pi' : X' \to Q'$ for the G'-action on X'. Finally, assume that the following condition holds: for any S-scheme T and any points $x \in X(T)$, $g \in G(T)$ such that $f(gx) = f(x)$ there exists an open covering $T = \cup T_i$ and a point $h_i \in H(T_i)$ for each i, such that $gx = h_i x$. Then there exists a nice quotient for the G-action on X. The same assertion holds for strict nice quotients.

Proof. It is enough to prove the assertion for strict nice quotients. Without loss of generality we can assume that the section $s : G' \to G$ satisfies $s(1) = 1$. By assumption, we have sections $\sigma_H : Q_H \to X$ and $\sigma' : Q' \to X'$ and the corresponding maps $\rho_H : X \to H$ and $\rho' : X' \to G'$ satisfying (3.2.1). Let us define morphisms $\rho_f : X \to G$ and $\pi_f : X \to X$ by

$$\rho_f = s \circ \rho' \circ f, \quad \pi_f(x) = \rho_f(x)^{-1}x.$$

One immediately checks that

$$f \circ \pi_f = \sigma' \circ \pi' \circ f.$$

In particular, $\pi_f(x) \in f^{-1}(\sigma'(Q'))$. Let us set $\widetilde{Q} = f^{-1}(\sigma'(Q')) \subset X$. Note that for $x \in \widetilde{Q}$ we have

$$\rho_f(x) = s(\rho'(f(x)) = s(1) = 1,$$

since $\rho'|_{\sigma'(Q')} = 1$. Hence, for $x \in \widetilde{Q}$ we have $\pi_f(x) = x$. Now we set

$$Q = \sigma_H^{-1}(\widetilde{Q}) \subset Q_H,$$

and define the maps $\pi : X \to Q$ and $\rho : X \to G$ required for the definition of a nice quotient by

$$\pi = \pi_H \circ \pi_f,$$

$$\rho(x) = \rho_f(x)\rho_H(\pi_f(x)).$$

Note that π is well-defined. Indeed, we need to show that $(\sigma_H\pi_H\pi_f)(x) \in \widetilde{Q}$. But $\pi_f(x) \in \widetilde{Q}$, so this follows from the identity

$$(\sigma_H\pi_H\pi_f)(x) = \rho_H(\pi_f(x))^{-1}\pi_f(x)$$

and the fact that $\widetilde{Q}$ is preserved by the action of H. We also have a section $\sigma : Q \to X$ of π given by $\sigma = \sigma_H|_Q$.

It remains to check that our data defines a strict nice quotient for the G-action on X. We have

$$x = \rho_f(x)\pi_f(x) = \rho_f(x)\rho_H(\pi_f(x))\sigma_H(\pi(x)) = \rho(x)\sigma(\pi(x)).$$

Also, by definition, we have $\sigma_H(Q) \subset \widetilde{Q}$, so for $y \in Q$ one has $\rho_f(\sigma_H(y)) = 1$ and $\pi_f(\sigma_H(y)) = \sigma_H(y)$. Hence,

$$\rho(\sigma(y)) = \rho_f(\sigma_H(y))\rho_H(\pi_f(\sigma_H(y))) = \rho_H(\sigma_H(y)) = 1.$$

It remains to prove that π is G-invariant. Given some $x \in X(T)$ and an element $g \in G(T)$, we observe that

$$f(\pi_f(gx)) = \sigma'(\pi'(f(gx))) = \sigma'(\pi'(f(x))) = f(\pi_f(x)).$$

Thus, our assumption implies that $\pi_f(gx)$ and $\pi_f(x)$ locally in T belong to the same H-orbit. Hence, locally in T we can find $h \in H(T)$ such that $\pi_f(gx) = h\pi_f(x)$. Therefore,

$$\pi(gx) = \pi_H(\pi_f(gx)) = \pi_H(h\pi_f(x)) = \pi_H(\pi_f(x)) = \pi(x).$$

□

3.3. **Representability theorem.** As before, we fix a graded sheaf of $\mathcal{O}_S$-algebras $\mathcal{E}$ over a scheme S, such that $\mathcal{E}$ is locally free of finite rank as an $\mathcal{O}_S$-module. The following theorem shows that under the assumption that certain graded components of $HH^1(\mathcal{E}|_s)$ vanish, the functor $\mathcal{M}_n$ (resp., $\mathcal{M}_\infty$) is representable by an affine S-scheme.

For each intervals of integers I and J let us consider the following vanishing condition:

(V^I_J): $HH^i(\mathcal{E}|_s)_{-j} = 0$ for $i \in I$ and $j \in J$, for every point $s \in S$.

Theorem 3.3.1. *(i) Assume that either* $(V^{\le 1}_{[1,n-3]})$ *holds, or* S *is a regular scheme of dimension* ≤ 1 *and* $(V^1_{[1,n-3]})$ *holds. Then there exists a nice quotient* $\mathcal{A}'_n(\mathcal{E})/\mathfrak{G}[2, n-1]$ *for the action of* $\mathfrak{G}[2, n-1]$ *on* $\mathcal{A}'_n(\mathcal{E})$*. This quotient* $\mathcal{A}'_n(\mathcal{E})/\mathfrak{G}[2, n-1]$*, which is affine of finite type over* S*, represents the functor* $\mathcal{M}_n$*. If in addition* S *is affine then there exists a strict nice quotient* $\mathcal{A}'_n(\mathcal{E})/\mathfrak{G}[2, n-1]$*, and the natural map of functors* $\widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ *is an isomorphism.*

(ii) Assume that the condition $(V^{\le 1}_{\ge 1})$ *holds (resp.,* S *is regular of dimension* ≤ 1 *and* $(V^1_{\ge 1})$ *holds). Then the scheme* $\varprojlim_n \mathcal{M}_n$*, affine over* S*, represents the functor* $\mathcal{M}_\infty$*. In the case when* S *is affine, the natural map* $\widetilde{\mathcal{M}}_\infty \to \mathcal{M}_\infty$ *is an isomorphism.*

(iii) Assume that there exists a nice quotient $\mathcal{M}_n$ *for the action of* $\mathfrak{G}[2, n-1]$ *on* $\mathcal{A}'_n(\mathcal{E})$*. Assume in addition that either* $(V^{\le 2}_{n-1})$ *holds, or* S *is regular of dimension* ≤ 1 *and* (V^2_{n-1}) *holds. Then there exists a nice quotient* $\mathcal{M}_{n+1}$ *for the action of* $\mathfrak{G}[2, n]$ *on* $\mathcal{A}'_{n+1}(\mathcal{E})$*, and the natural map* $\mathcal{M}_{n+1} \to \mathcal{M}_n$ *is a closed embedding.*

Assume in addition that either $(V^{\le 2}_{\ge n-1})$ *holds, or* S *is regular of dimension* ≤ 1 *and* $(V^{2}_{\ge n-1})$ *holds. Then the scheme* $\varprojlim_n \mathcal{M}_n$ *represents the functor* $\mathcal{M}_\infty$*, and the morphism* $\mathcal{M}_\infty \to \mathcal{M}_n$ *is a closed embedding. In the case when* S *is affine, the natural map* $\widetilde{\mathcal{M}}_\infty \to \mathcal{M}_\infty$ *is an isomorphism.*

(iii') Assume that there exists a nice quotient $\mathcal{M}_n$ *for the action of* $\mathfrak{G}[2, n-1]$ *on* $\mathcal{A}'_n(\mathcal{E})$*. Assume in addition that either* $(V^{\le 3}_{n-1})$ *holds, or* S *is regular of dimension* ≤ 1 *and* $(V^{[2,3]}_{n-1})$ *holds. Then the natural map* $\mathcal{M}_{n+1} \to \mathcal{M}_n$ *is an isomorphism.*

Assume in addition that either $(V^{\le 3}_{\ge n-1})$ *holds, or* S *is regular of dimension* ≤ 1 *and* $(V^{[2,3]}_{\ge n-1})$ *holds. Then the natural morphism* $\mathcal{M}_\infty \to \mathcal{M}_n$ *is an isomorphism.*

The statement of the above theorem is a bit long since we aimed at greater generality, so let us state a useful corollary from it.

Corollary 3.3.2. *Assume the for some* $n \ge 2$ *the conditions* $(V^{\le 1}_{[1,n-3]})$ *and* $(V^{\le 2}_{\ge n-1})$ *hold (resp.,* S *is regular of dimension* ≤ 1 *and the conditions* $(V^{1}_{[1,n-3]})$ *and* $(V^{2}_{\ge n-1})$ *hold). Then the functor* $\mathcal{M}_\infty$ *is representable be a scheme, which is affine of finite type over* S*.*

Lemma 3.3.3. *(i) Let* $(V^\bullet, d)$ *be a bounded below complex of vector bundles over a scheme* S *such that* $H^i(V^\bullet|_s) = 0$ *for* $i < p$ *for every point* $s \in S$*. Then for each* $i < p$*, the image* $\operatorname{im}(d^i)$ *of the differential* $d^i : V^i \to V^{i+1}$ *is a subbundle of* V^{i+1} *and* $\underline{H}^i(V^\bullet) = 0$*.*

(ii) Let $(V^\bullet, d)$ *be a complex of vector bundles over an affine scheme* S*. Assume that for some integer* i *one has* $\underline{H}^i(V^\bullet) = 0$ *and the image of* d^i *(resp.,* d^{i-1}*) is a subbundle of* V^{i+1} *(resp.,* V^i*). Then there exist decompositions of vector bundles*

$$V^i = B^i \oplus K^i, \quad V^{i+1} = B^{i+1} \oplus K^{i+1},$$

such that d^{i-1} *is a surjection* $V^{i-1} \to B^i$*, while* d^i *factors as*

$$d^i : V^i \to K^i \xrightarrow{\sim} B^{i+1} \to V^{i+1}.$$

In particular, for any $\varphi : T \to S$ *the complex* $H^0(T, \varphi^* V^\bullet)$ *is exact in degree* i*. For example, in the situation of (i) with affine* S *this is true for all* $i < p$*.*

Proof. (i) Without loss of generality we can assume that $V^i = 0$ for $i < 0$ and $p > 0$. Then the map $d_s : V^0|_s \to V^1|_s$ is injective for every $s \in S$. We claim that this implies that $d : V^0 \to V^1$ is the embedding of a subbundle. Indeed, it is enough to prove the similar assertion for a morphism $f : A^m \to A^n$ of free modules over a local ring A, such that $f \bmod \mathfrak{m}$ is injective, where $\mathfrak{m} \subset A$ is a maximal ideal. But in this case we can choose a projection $p : A^n \to A^m$ to a subset of m coordinates, such that $p \circ f \bmod \mathfrak{m}$ is an isomorphism. This implies that $\det(p \circ f)$ is nonzero $\bmod\, \mathfrak{m}$, hence it is invertible in A. Thus, the composition $p \circ f : A^m \to A^m$ is an isomorphism, so f is a split embedding. Also, we see that $\underline{H}^0(V^\bullet) = 0$.

Now since $V^1/d(V^0)$ is a vector bundle, we can replace our complex with the quasi-isomorphic complex

$$\overline{V}^\bullet : 0 \to V^1/d(V^0) \to V^2 \to \dots$$

and iterate the same argument (note that $H^*(\overline{V}^\bullet|_s) = H^*(V^\bullet|_s)$).

(ii) Let us set $B^i := \operatorname{im}(d^{i-1}) = \ker(d^i)$, $B^{i+1} := \operatorname{im}(d^i)$, and let K^i (resp., K^{i+1}) be the image of any splitting of the projection $V^i \to V^i/B^i$ (resp., $V^{i+1} \to V^{i+1}/B^{i+1}$), which exists since S is affine. This gives the required decompositions of bundles over S. These decompositions carry over to the complex $H^0(T, f^*V^\bullet)$, which implies its exactness in degree i. □

We also have the following version for complexes over regular schemes of dimension ≤ 1.

Lemma 3.3.4. *Let $(V^\bullet, d)$ be a complex of vector bundles over a regular scheme S of dimension ≤ 1. Assume that for some i one has $H^i(V^\bullet|_s) = 0$ for all $s \in S$. Then $\underline{H}^i(V^\bullet) = 0$, $\underline{H}^{i+1}(V^\bullet)$ is locally free, and the image of the differential $d^i : V^i \to V^{i+1}$ (resp., d^{i-1}) is a subbundle of V^{i+1} (resp., V^i).*

Proof. The question is local, so we can assume that $S = \mathrm{Spec}(A)$, where A is a spectrum of a local ring. If A is a field then the assertion is clear, so we can assume that A is a dvr. Let $\mathfrak{m}$ denote the maximal ideal in A. Since $\mathfrak{m} = (t)$, where t is not a zero divisor, we have a short exact sequence of complexes

$$0 \to V^\bullet \xrightarrow{t} V^\bullet \to V^\bullet/\mathfrak{m}V^\bullet \to 0.$$

Let us consider the corresponding long exact sequence of cohomology,

$$\ldots \to H^i(V^\bullet) \xrightarrow{t} H^i(V^\bullet) \to 0 \to H^{i+1}(V^\bullet) \xrightarrow{t} H^{i+1}(V^\bullet) \to \ldots$$

By Nakayama lemma, we get $H^i(V^\bullet) = 0$ (note that $H^i(V^\bullet)$ is finitely generated since A is Noetherian). Also, multiplication by t is injective on $H^{i+1}(V^\bullet)$, so it is a free A-module. Note that $\mathrm{im}(d^j)$ is a free A-module of finite rank for any j, as a submodule of V^{j+1}. Now the exact sequence

$$0 \to \ker(d^{i+1})/\mathrm{im}(d^i) \to V^{i+1}/\mathrm{im}(d^i) \to \mathrm{im}(d^{i+1}) \to 0$$

shows that $V^{i+1}/\mathrm{im}(d^i)$ is free. Finally, $V^i/\mathrm{im}(d^{i-1}) = V^i/\ker(d^i) \simeq \mathrm{im}(d^i)$ is also free. □

Note that the sheaf of groups $\mathfrak{G}[2, n-1]$ is representable by a unipotent affine group scheme over S which we still denote as $\mathfrak{G}[2, n-1]$. Note also that the projection $\mathfrak{G} \to \mathfrak{G}[2, n-1]$ admits a section (not compatible with the group structures) and so is universally surjective.

Lemma 3.3.5. *Let E be a sheaf of graded associative $\mathcal{O}_T$-algebras over a scheme T.*
(i) Assume that $HH^1(E/T)_{-i} = 0$ for $i = r, \ldots, d-2$, where $d \geq 2$, $r \geq 1$. Suppose $m = (m_\bullet)$ and $m' = (m'_\bullet)$ are a pair of minimal A'_n-structures on E, where $n \geq d$, such that $m_{\leq d} = m'_{\leq d}$ and there exists a gauge transformation f with $f * m = m'$ and $f_{\leq r} = \mathrm{id}$. Then there exists a gauge transformation f', homotopic to f, such that $f' * m = m'$ and $f'_{\leq d-1} = \mathrm{id}$.
(ii) The natural map

$$\mathcal{A}_\infty(E/T)/\mathfrak{G}(T) \to \varprojlim_n \mathcal{A}'_n(E/T)/\mathfrak{G}[2, n-1](T)$$

is surjective (where $\mathfrak{G}$ is the group of gauge equivalences associated with E/T). Assume that either $HH^1(E/T)_{<0} = 0$ or for some integer $N > 0$, one has $HH^2(E/T)_{<-N} = 0$. Then the above map is an isomorphism.

Proof. (i) In the case $d < r+2$ the assertion holds with $f' = f$. Now we use induction on d (with the base case $d = r+1$). Assuming the assertion holds for $d-1$, we can find a gauge transformation f', homotopic to f such that $f' * m = m'$ and $f'_{<d-1} = \mathrm{id}$. We have to show that f' can be improved to make in addition $f'_{d-1} = 0$. By Lemma 1.3.2(ii), we have

$$0 = m'_d - m_d = \pm\delta(f'_{d-1}),$$

so f'_{d-1} is a Hochschild cocycle giving a class in $HH^1(E/T)_{1-j}$. Since this class is zero by our assumptions, there exists a Hochschild cochain ϕ in $CH^0(E/T)_{1-j}$ such that $f'_{d-1} = [m_2, \phi]$. By Lemma 1.3.7(ii), we can use ϕ to construct a homotopy from f' to a gauge transformation f'' with $f''_{<d-1} = f'_{<d-1}$ and $f''_{d-1} = 0$.
(ii) To prove the surjectivity, suppose we have a collection $(\alpha_n)_{n\geq 3}$ of minimal A_n-structures on E/T, and a set of gauge equivalences $(u_n \in \mathfrak{G}[2, n-1](T))$ such that $(\alpha_{n+1})_{\leq n} = u_n \cdot \alpha_n$. Then we can recursively construct minimal A_n-structures (α'_n), such that $(\alpha'_{n+1})_{\leq n} = \alpha'_n$, and gauge equivalences $(v_n \in \mathfrak{G}[2, n-1](T))$ such that $\alpha'_n = v_n \cdot \alpha_n$ and $(v_{n+1})_{\leq n-1} u_n = v_n$. Namely, if (α'_i), (v_i) for $i \leq n$ are already constructed, then we pick a gauge equivalence $v_{n+1} \in \mathfrak{G}[2, n](T)$, such that $(v_{n+1})_{\leq n-1} = v_n u_n^{-1}$, and set $\alpha'_{n+1} := v_{n+1} \cdot \alpha_{n+1}$. Then (α'_n) defines the required minimal A_∞-structure.

For the injectivity part, consider first the case $HH^1(E/T)_{<0} = 0$. Suppose α and β are minimal A_∞-structures such that $\alpha_{\leq n}$ is gauge equivalent to $\beta_{\leq n}$ for each n. We are going to construct recursively a sequence of gauge equivalences $u_1 = \mathrm{id}, u_2, u_3, \ldots$, such that $(u_n)_{\leq n-1} = \mathrm{id}$ and for every $n \geq 2$, one has

$$\alpha_{\leq n} = (u_{n-1} u_{n-2} \ldots u_1)\beta_{\leq n}.$$

Indeed, the induction base $n = 2$ is clear since $\alpha_{\leq 2} = \beta_{\leq 2}$. Assume that $n \geq 2$ and (u_i) for $i \leq n-1$ are already constructed and satisfy the above property. Then the A'_{n+1}-structures $\alpha_{\leq n+1}$ and $(u_{n-1} u_{n-2} \ldots u_1)\beta_{\leq n+1}$ agree up to n, and are gauge equivalent. Hence, by part (i), there exists a gauge equivalence u_n such that $(u_n)_{\leq n-1} = \mathrm{id}$ and

$$\alpha_{\leq n+1} = (u_n u_{n-1} \ldots u_1)\beta_{\leq n+1}.$$

It remains to note that the infinite product $\ldots u_3 u_2 u_1$ converges in $\mathfrak{G}$ to some element u, such that $\alpha = u\beta$.

In the case $HH^2(E/T)_{2-n} = 0$ for all $n \geq N$, the proof of injectivity is easier: for any pair of A_∞-structures α and β such that $\alpha_{\leq N}$ is gauge equivalent to $\beta_{\leq N}$, we claim that α is gauge equivalent to β. Indeed, this follows by iteratively applying Lemma 1.3.2(ii). □

Lemma 3.3.6. *Let $\mathcal{E}/S$ be a sheaf of graded associative $\mathcal{O}_S$ algebras over a scheme S, such that $\mathcal{E}$ is locally free of finite rank over S. Assume in addition that the scheme S is affine.*
(i) Let us fix an integer $d \geq 2$. Assume that either $(V^{\leq 1}_{[1,d-2]})$ holds, or S is regular of dimension ≤ 1 and $(V^{1}_{[1,d-2]})$ holds. Let $\varphi : T \to S$ be a morphism of schemes, and let m and m' be a pair of minimal A'_n-structures on $\varphi^*\mathcal{E}$ for some $n \geq d$, such that m is gauge equivalent to m' (over T) and $m_{\leq d} = m'_{\leq d}$. Then there exists a gauge equivalence u over T, such that $u_{\leq d-1} = \mathrm{id}$ and $m' = u \cdot m$.

(ii) Assume that either $(V^{\le 1}_{\ge 1})$ *holds, or for some* $N > 0$, $(V^{\le 2}_{\ge N})$ *holds (resp.,* S *is regular of dimension* ≤ 1, *and either* $(V^{1}_{\ge 1})$ *or* $(V^{2}_{\ge N})$ *holds). Then the natural map*

$$\widetilde{\mathcal{M}}_\infty(T) \to \varprojlim_n \widetilde{\mathcal{M}}_n(T)$$

is an isomorphism for every S*-scheme* T.

Proof. (i) Set $E = \varphi^*\mathcal{E}$. Note that for each j, the Hochschild complex

$$(CH^*(E/T)_{-j}, \delta) = (H^0(T, \underline{CH}^*(E/T)_{-j}, \delta)$$

is obtained by taking global sections of the pull-back $\varphi^*\underline{CH}^*(\mathcal{E}/S)_{-j}$. Thus, applying Lemmas 3.3.3 and 3.3.4 to the complexes $(\underline{CH}^*(\mathcal{E}/S)_{-j}, \delta)$ (which are bounded below), we obtain that $HH^1(\varphi^*\mathcal{E}/T)_{-j} = 0$ for $j = 1, \dots, d-2$. Note that here the assumption that $\mathcal{E}$ is a vector bundle over S implies that the same is true for the terms of these complexes. Now the assertion follows from Lemma 3.3.5(i).
(ii) As in part (i), we get one of the vanishings $HH^1(\varphi^*\mathcal{E}/T)_{<0} = 0$ or $HH^2(\varphi^*\mathcal{E}/T)_{<-N} = 0$. Hence, the assertion follows from Lemma 3.3.5(ii). $\square$

Let us denote the graded components of the Hochschild differential as

$$\delta^i_t : \underline{CH}^i(\mathcal{E}/S)_t \to \underline{CH}^{i+1}(\mathcal{E}/S)_t.$$

Proof of Theorem 3.3.1.
(i) It is enough to prove the existence of a strict nice quotient for the $\mathfrak{G}[2, n-1]$-action on $\mathcal{A}'_n = \mathcal{A}'_n(\mathcal{E})$ in the case when S is affine. Indeed, then it would follow that $\widetilde{\mathcal{M}}_n$ is represented by this quotient (see Lemma 3.2.3), and hence, the map $\widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ is an isomorphism.

The existence of a strict nice quotient is proved by the induction on n, using Lemma 3.2.4. Assume that $n > 2$ and we already have a section S_{n-1} for the $\mathfrak{G}[2, n-2]$-action on $\mathcal{A}'_{n-1}$. We have an exact sequence of sheaves of groups over S,

$$0 \to \underline{CH}^1(\mathcal{E})_{2-n} \to \mathfrak{G}[2, n-1] \to \mathfrak{G}[2, n-2] \to 0.$$

We want to find a section for the $\underline{CH}^1(\mathcal{E})_{2-n}$-action on $\mathcal{A}'_n$. By Lemma 3.3.3(i), there exists a complement $\mathcal{K}^2_{2-n} \subset \underline{CH}^2(\mathcal{E})_{2-n}$ to the subbundle $\operatorname{im}\delta^1_{2-n}$. Let $S\mathcal{A}'_n$ denote the closed subset of $\mathcal{A}'_n$ given by the condition $m_n \in \mathcal{K}^2_{2-n}$. Since the action of $x \in \underline{CH}^1(\mathcal{E})_{2-n}$ on $(m_2, \dots, m_n) \in \mathcal{A}'_n$ changes m_n to $m_n + \delta^1(x)$ and does not change $(m_2, \dots, m_{n-1})$, we see that $S\mathcal{A}'_n$ is a section for the $\underline{CH}^1(\mathcal{E})_{2-n}$-action on $\mathcal{A}'_n$. Furthermore, we claim that the projection $\mathcal{A}'_n \to S\mathcal{A}'_n$ induced by the projection $\underline{CH}^2(\mathcal{E})_{2-n} \to \mathcal{K}^2_{2-n} : m \mapsto m_{\mathcal{K}}$ is a strictly nice quotient for the action of $\underline{CH}^1(\mathcal{E})_{2-n}$ on $\mathcal{A}'_n$. Indeed, let us choosing any splitting Q of a surjective map of bundles $\underline{CH}^1(\mathcal{E})_{2-n} \overset{\delta^1_{2-n}}{\twoheadrightarrow} \operatorname{im}\delta^1_{2-n}$. Then starting from any $(m_2, \dots, m_n) \in \mathcal{A}'_n$ we will have

$$m_n = (m_n)_{\mathcal{K}} + \delta^1_{2-n} Q(m - (m_n)_{\mathcal{K}}),$$

so that

$$(m_2, \dots, m_n) = Q(m - (m_n)_{\mathcal{K}}) * (m_2, \dots, m_{n-1}, (m_n)_{\mathcal{K}}),$$

as required for a strictly nice quotient.

Now we can apply Lemma 3.2.4 to the projection (3.1.1) and the compatible actions of $\mathfrak{G}[2, n-1] \to \mathfrak{G}[2, n-2]$. Note that to apply this Lemma we need to check that the intersection of an $\mathfrak{G}[2, n-1]$-orbit with a fiber of π_n is a $\underline{CH}^1(\mathcal{E})_{2-n}$-orbit. But this follows from Lemma 3.3.6(i). Thus, we deduce that

$$S_n := S\mathcal{A}'_n \cap \pi_n^{-1}(S_{n-1})$$

is a section for the $\mathfrak{G}[2, n-1]$-action on $\mathcal{A}'_n$.

(ii) First, assume that S is affine. Then, combining part (i) with Lemma 3.3.6(ii), we derive that the functor $\widetilde{\mathcal{M}}_\infty$ is represented by the scheme $\varprojlim_n \mathcal{M}_n$, affine over S. Hence, in this case the map $\widetilde{\mathcal{M}}_\infty \to \mathcal{M}_\infty$ is an isomorphism. Thus, in the case of general S the map of sheaves $\mathcal{M}_\infty \to \varprojlim_n \mathcal{M}_n$ becomes an isomorphism over an affine open covering of S, hence, it is an isomorphism.

(iii) We can assume that S is affine and we have a nice section S_n for the action of $\mathfrak{G}[2, n-1]$ on $\mathcal{A}'_n(\mathcal{E})$. We claim that there exists a nice section S_{n+1} for the action of $\mathfrak{G}[2, n]$ on $\mathcal{A}'_{n+1}(\mathcal{E})$ and the projection $S_{n+1} \to S_n$ is a closed embedding.

First, recall that $\mathcal{A}'_{n+1}$ is a closed subset of $\mathcal{A}'_n \times \text{tot}(CH^2_{1-n})$ given by the equation

$$\delta^2(m_{n+1}) = -\phi_n(m_{\le n}),$$

where $\phi_n : \mathcal{A}'_n \to \text{tot}(CH^3_{1-n})$ is a certain morphism (see Lemma 1.3.4(ii)) such that $\delta^3 \circ \phi_n = 0$. Now by Lemmas 3.3.3(i) and 3.3.4, there exist decompositions of vector bundles

$$\underline{CH}^2(\mathcal{E})_{1-n} = \mathcal{B}^2 \oplus \mathcal{K}^2, \quad \underline{CH}^3(\mathcal{E})_{1-n} = \mathcal{B}^3 \oplus \mathcal{K}^3$$

such that $\mathcal{B}^3$ is the image of δ^2 and the restriction $\delta^2|_{\mathcal{K}^2} : \mathcal{K}^2 \to \mathcal{B}^3$ is an isomorphism. Let $(\phi_{\mathcal{B}}, \phi_{\mathcal{K}})$ be the components of ϕ_n with respect to the decomposition of $\underline{CH}^3_{1-n}$. Then the equations defining $\mathcal{A}'_{n+1}$ become

$$\delta^2(m_{n+1}) = \phi_{\mathcal{B}}(m_{\le n}), \quad 0 = \phi_{\mathcal{K}}(m_{\le n}).$$

Thus, on the subscheme $S\mathcal{A}'_{n+1}$ cut out by the condition $m_{n+1} \in \mathcal{K}^2$, we can solve the first equation for m_{n+1}, which shows that the projection $S\mathcal{A}'_{n+1} \to \mathcal{A}_n$ is a closed embedding.

Furthermore, $S\mathcal{A}'_{n+1}$ is a nice section for the action of $\underline{CH}^1(\mathcal{E})_{1-n}$ on $\mathcal{A}'_{n+1}$, so we can apply Lemma 3.2.4 to deduce that the preimage $S_{n+1} \subset S\mathcal{A}'_{n+1}$ is a nice section for the action of $\mathfrak{G}[2, n]$ on $\mathcal{A}'_{n+1}(\mathcal{E})$. Note that here we use the fact that any two A'_{n+1}-structures m' and m over $\varphi : T \to S$, with $m'_{\le n} = m_{\le n}$, are in one orbit of $CH^1(\mathcal{E}/T)_{1-n}$ by the triviality of $HH^2(\mathcal{E}/T)_{1-n}$.

Hence, the composition $S_{n+1} \hookrightarrow S\mathcal{A}'_{n+1} \to S_n$ is still a closed embedding .

The remaining assertions follow from this and from Lemma 3.3.6(ii).

(iii') Again it is enough to consider the case when S is affine and there is a nice section S_n for the action of gauge transformations on $\mathcal{A}'_n(\mathcal{E})$. As in part (iii), we have a nice section $S_{n+1} \subset S\mathcal{A}'_{n+1}$ given as the preimage of S_n. Now, we observe that the additional vanishing assumption we imposed give the vanishing of the component $\phi_{\mathcal{K}}$ (by Lemmas 3.3.3(i) and 3.3.4). Thus, in this case the projection $S\mathcal{A}'_{n+1} \to \mathcal{A}'_n$ is an isomorphism, and hence the same is true for the projection $S_{n+1} \to S_n$.

The last assertion follows from this and from Lemma 3.3.6(ii). □

3.4. A_∞**-structures with a segment of defining higher products.** In the case when E is a graded associative algebra with $HH^2(E)_{2-i} \neq 0$ only for i in some interval $[q, q+p]$, any A_∞-structure on E is equivalent to the one with $m_i = 0$ for $2 < i < q$ and is determined by $(m_q, \ldots, m_{q+p})$. The following result (valid in characteristic zero) gives a sufficient criterion for the corresponding moduli functor to be representable. Note that it does not follow from Theorem 3.3.1, as we impose weaker vanishing assumptions on $HH^1(E)_i$. In the case $p < q - 2$ we get a criterion for the moduli space to be a closed subscheme of the affine space $\prod_{n=q}^{q+p} HH^2(E)_{2-n}$.

Theorem 3.4.1. *Let E be a finite-dimensional graded associative algebra over a field k of characteristic zero. Assume that for some integers $p \geq 0$ and $q \geq 3$, one has*

$$HH^1(E)_{1-i} = 0 \text{ for } i \in [2, p+1].$$

$$HH^2(E)_{2-i} = 0 \text{ for } i > 2,\ i \notin [q, q+p]$$

Then for each $n \geq 3$, there exists a strict nice quotient $\mathcal{A}'_n/\mathfrak{G}[2, n-1]$, so $\widetilde{\mathcal{M}}_n = \mathcal{M}_n$ is representable by an affine scheme over k. Furthermore, if $p < q - 2$ then $\mathcal{M}_{p+q}$ is isomorphic to the affine space $\prod_{n=q}^{p+q} HH^2(E)_{2-n}$, and we have a natural closed embedding

$$\widetilde{\mathcal{M}}_\infty = \mathcal{M}_\infty \hookrightarrow \mathcal{M}_{p+q}.$$

Lemma 3.4.2. *(i) Let c be a Hochschild cochain in $CH^2(E)_{2-n}$ and let f be a gauge transformation, such that $f_{<k} = \mathrm{id}$ for some $k \geq 2$. Then $\alpha_f D_c \alpha_f^{-1} = D_{c+c'}$, where $c' \in CH^2(E)$ has zero components in $CH^2(E)_i$ for $i > 3 - k - n$.*
(ii) Assume that we are working over a field of characteristic zero. Let m and m' be A'_n-structures on E (with given m_2) such that $m'_r = m_r$ for some r, $3 \leq r < n$, and $m_i = 0$ for $3 \leq i < q$, for some $q \geq 3$. Assume also that $n \leq q + r - 3$, and there exists a gauge transformation f with $f * m = m'$ and $f_{\leq r-2} = \mathrm{id}$. Then there exists a gauge transformation $\widetilde{f}$ with $\widetilde{f} * m = m'$ and $\widetilde{f}_{\leq r-1} = \mathrm{id}$.

Proof. (i) This is a straightforward check using the explicit form of D_c and α_f (see (1.1.1) and (1.3.1)).
(ii) We have

$$0 = m'_r - m_r = \pm\delta(f_{r-1}),$$

so $[m_2, f_{r-1}] = 0$. Hence, $D_{f_{r-1}}$ commutes with D_{m_2}. Let us consider the automorphism $\exp(D_{f_{r-1}})$ of $\mathrm{Bar}(E)$ (which is defined since the characteristic is zero). Then it commutes with D_{m_2} and its component mapping from $(E[1])^{\otimes r-1}$ to E is f_{r-1}. Thus, we have

$$\alpha_f = \alpha_{\widetilde{f}} \circ \exp(D_{f_{r-1}})$$

for some gauge equivalence $\widetilde{f}$ such that $\widetilde{f}_{\leq r-1} = \mathrm{id}$. Let us define $\widetilde{m}$ from

$$D_{\widetilde{m}} = \exp(D_{f_{r-1}}) D_m \exp(-D_{f_{r-1}}),$$

so that $m' = \widetilde{f} * \widetilde{m}$. Note that since $D_{f_{r-1}}$ commutes with D_{m_2}, we have

$$\exp(D_{f_{r-1}}) D_{m_2} \exp(-D_{f_{r-1}}) = D_{m_2}.$$

On the other hand, viewing $\exp(D_{f_{r-1}})$ as a gauge transformation, from part (i) we get that for every i, $q \leq i \geq n$, one has

$$\exp(D_{f_{r-1}})D_{m_i}\exp(-D_{f_{r-1}}) = D_{m_i+c(i)},$$

with $c(i)$ having zero components in $\mathrm{CH}^2(E)_{>4-i-r}$. Hence, we get $\widetilde{m} = m$ (since $i \geq q$ and $n \leq q+r-3$), and the assertion follows. □

Proof of Theorem 3.4.1. As in Theorem 3.3.1, we prove the existence of a strict nice quotient by induction on n. Assuming that such a quotient exists for the action of $\mathfrak{G}[2, n-2]$ on $\mathcal{A}'_{n-1}$, and arguing as in Theorem 3.3.1, we reduce ourselves to proving the following assertion for every $n \geq 3$. Given a k-scheme T, A'_n-structures m and m' on E_T, and a gauge equivalence f such that $f * m = m'$, we need to check that $m'_n - m_n$ is in the image of δ^1_{2-n}.

First, we note that without loss of generality we can apply the same element of $\mathfrak{G}[2, n-1]$ to both m and m' to make them simpler (since CH^1_{2-n} is a normal subgroup in $\mathfrak{G}[2, n-1]$). Thus, we can use the vanishing of $HH^2(E)_{2-i}$ for $3 \leq i < q$, to assume that $m'_{<q} = m_{<q} = 0$.

Further, note that in the case $n > p+q$ the assertion is automatic, due to the vanishing of $HH^2(E)_{2-n}$ (see Lemma 1.3.2(ii)). Thus, we can assume that $n \leq p+q$.

Next, applying Lemma 1.3.7(ii), we can modify f by a homotopy, so that we have $f_{\leq p+1} = \mathrm{id}$. At this point, if $n \leq p+3$ then $m'_n - m_n = \pm\delta(f_{n-1})$, and we are done. Thus, we can assume that $n > p+3$. In this case we can apply Lemma 3.4.2(ii) with $r = p+2$ and replace f by $\widetilde{f}$, such that $\widetilde{f} * m = m'$ and $\widetilde{f}_{\leq p+2} = \mathrm{id}$. We can iterate this procedure until we get $f_{\leq n-2} = 0$, in which case $m'_n - m_n = \pm\delta(f_{n-1})$.

Now assume that $p < q-2$. Let us choose for each $n \in [q, p+q]$ a subspace $R_{2-n} \subset ZH^2(E)_{2-n}$ of closed Hochschild cochains projecting isomorphically onto $HH^2(E)_{2-n}$. For each $n \in [q, p+q]$ we have a natural closed embedding

$$\prod_{i=q}^{n} R_{2-i} \hookrightarrow \mathcal{A}'_n,$$

extending $(m_q, \ldots, m_n)$ to an A'_n-structure with $m_i = 0$ for $3 \leq i < q$. Indeed, we note that due to the assumption $p < q-2$, the A'_n-identities in this case reduce to $\delta(m_i) = 0$ for $i = q, \ldots, n$. Now we can prove by induction on $n \in [q, p+q]$ that $\prod_{i=q}^{n} R_{2-i}$ is a nice section for the action of $\mathfrak{G}[2, n-1]$ on $\mathcal{A}'_n$. Indeed, this follows easily from the inductive construction of this section used before.

The last assertion follows from Theorem 3.3.1(iii). □

Corollary 3.4.3. *Let E be a finite-dimensional graded associative algebra over a field k of characteristic zero. If for some $q \geq 3$ one has $HH^2(E)_{2-i} = 0$ for $i > 3$, $i \neq q$, then $\mathcal{M}_\infty$ is representable by a closed subscheme of the affine space $HH^2(E)_{2-q}$.*

Example 3.4.4. There is an interesting example showing that the characteristic of the field is important in the above Corollary. Namely, one can consider E to be the algebra over a field k associated with the following quiver with relations. ??? Seidel in [45] considers A_∞-structures on E for $k = \mathbb{C}$ and proves that they are classified by the space

$HH^2(E)_{-2}$ which can be identified with the space of binary quartics over k. It turns out that in the case when k is a field of characteristic 2, one still has $HH^2(E)_{-1} = HH^2(E)_{<-2} = 0$, while $HH^2(E)_{-2}$ is the space of binary quartics. However, the space $HH^1(E)_{-1}$, which is identified with the space of binary quadrics, acts on $HH^2(E)_{-2}$ by $q * f = f + q^2$ (where $f \in HH^2(E)_{-2}$ and $q \in HH^1(E)_{-1}$. Geometric meaning???

4. A_∞-STRUCTURES ASSOCIATED TO CURVES

4.1. **Moduli of curves with nonspecial divisors.** We are going to consider A_∞-structures arising on certain special generators of perfect derived categories of projective curves. Here we describe precisely which curves we consider and study the corresponding moduli problem.

Let C be a reduced connected projective curve over a field k, and let $p_1, \ldots, p_n$ be distinct smooth k-points of C (*marked points*). We assume that there is at least one marked point on each irreducible component of C, and consider the following generator of the perfect derived category $\mathrm{Perf}(C)$:

$$G := \mathcal{O}_C \oplus \bigoplus_{i=1}^{n} \mathcal{O}_{p_i}. \tag{4.1.1}$$

The fact that it is a generator is proved in a standard way: it is enough to show that a sequence of line bundles $\mathcal{O}(-nD)$, where $D = p_1 + \ldots + p_n$ is contained in the thick subcategory generated by G (see [42, Prop. 7.9]). But this follows easily from the exact sequences

$$0 \to \mathcal{O}_C(-(n+1)D) \to \mathcal{O}_C(-nD) \to \mathcal{O}_D \to 0.$$

In addition, we impose the condition $H^1(C, \mathcal{O}(p_1 + \ldots + p_n)) = 0$, i.e., we require the divisor $p_1 + \ldots + p_n$ to be nonspecial. This assumption may seem a bit unmotivated at the moment but it is needed in order to have a nice moduli space, as well as to guarantee for the algebras $\mathrm{Ext}^*(G, G)$ to give a nice moduli space of A_∞-structures.

Definition 4.1.1. We define by $\mathcal{U}^{ns}_{g,n}$ the moduli stack of pointed curves $(C, p_1, \ldots, p_n)$ as above (we leave to the reader to define the corresponding groupoids-valued functor). We also consider the $\mathbb{G}_m^n$-torsor $\widetilde{\mathcal{U}}^{ns}_{g,n}$ over $\mathcal{U}^{ns}_{g,n}$, obtained by considering the data $(C, p_1, \ldots, p_n, v_1, \ldots, v_n)$ where v_i is a nonzero tangent vector to C at p_i.

The choices of nonzero tangent vectors rigidify our moduli problem. We will show that under mild restrictions on the characteristic, $\widetilde{\mathcal{U}}^{ns}_{g,n}$ is equivalent to a quasiprojective scheme. Rescaling $(v_1, \ldots, v_n)$ we get an action of $\mathbb{G}_m^n$ on $\widetilde{\mathcal{U}}^{ns}_{g,n}$. The action of the diagonal subgroup $\mathbb{G}_m \subset \mathbb{G}_m^n$ will play a special role in our considerations.

We observe that there is a natural morphism

$$\pi : \widetilde{\mathcal{U}}^{ns}_{g,n} \to G(n-g, n) \tag{4.1.2}$$

to the Grassmannian of $(n-g)$-dimensional subspaces in the n-dimensional space, associating with $(C, p_1, \ldots, p_n, v_1, \ldots, v_n)$ the kernel of the coboundary homomorphism

$$H^0(C, \mathcal{O}(p_1 + \ldots + p_n)/\mathcal{O}) \to H^1(C, \mathcal{O}).$$

Note that this homomorphism is surjective since $H^1(\mathcal{O}(p_1+\ldots+p_n))=0$ and that the tangent vectors give a trivialization of the space $H^0(C,\mathcal{O}(p_1+\ldots+p_n)/\mathcal{O})=\bigoplus_i \mathcal{O}(p_i)|_{p_i}$.

Recall that the Grassmannian $G(n-g,n)$ is covered by the open cells U_S, isomorphic to the affine spaces, indexed by subsets $S\subset[1,n]$ of cardinality g: by definition, W is in U_S if it is a graph of a linear map $\langle u_j \mid j\notin S\rangle \to \langle u_i \mid i\in S\rangle$. Equivalently, $W\in U_S$ means that the elements $(u_i)_{i\in S}$ project to a basis of k^n/W. This immediately imples that the preimage $\pi^{-1}(U_S)$ is the open substack corresponding to $(C,p_\bullet,v_\bullet)$ such that $H^1(C,\mathcal{O}(\sum_{i\in S}p_i))=0$.

Theorem 4.1.2. *(*[38, Thm. 1.2.2]*) Assume that either $n\geq g\geq 1$, $n\geq 2$ and we work over* $\mathrm{Spec}(\mathbb{Z}[1/2])$*, or $n=g=1$ and we work over* $\mathrm{Spec}(\mathbb{Z}[1/6])$*, or $g=0$, $n\geq 2$ and the base is* $\mathrm{Spec}(\mathbb{Z})$*. Then the stack $\widetilde{\mathcal{U}}^{ns}_{g,n}$ is equivalent to a scheme, affine over $G(n-g,n)$ with respect to the morphism π (see* (4.1.2)*). Furthermore, the push-forward of $\mathcal{O}$ under the projection $C\setminus D\to\widetilde{\mathcal{U}}^{ns}_{g,n}$, where C is the universal curve, is locally free (of infinite rank). Let $\mathbb{G}_m\subset\mathbb{G}_m^n$ be the diagonal subgroup. Then π is $\mathbb{G}_m$-invariant and the action of $\mathbb{G}_m$ on the sheaf of algebras $\pi_*\mathcal{O}$ has nonnegative weights. The subscheme of $\mathbb{G}_m$-invariants in $\widetilde{\mathcal{U}}^{ns}_{g,n}$ gives a section of the morphism π.*

In fact, we can describe explicitly the curves corresponding to $\mathbb{G}_m$-invariant points in $\widetilde{\mathcal{U}}^{ns}_{g,n}$ as follows. They are parametrized by the subspaces $W\subset k^n$ of dimension $n-g$. We view such a subspace as a subspace of linear forms in independent variables $u_1,\ldots,u_n$, identifying k^n with $\bigoplus_{i=1}^n k\cdot u_i$. Then we define a subalgebra $A_W\subset\bigoplus_{i=1}^n k[u_i]$ by

$$A_W:=k\cdot 1\oplus W\oplus\bigoplus_{i=1}^n u_i^2k[u_i].$$

This algebra has a natural increasing filtration coming from the grading, and we define the curve C_W as

$$C_W:=\mathrm{Proj}(\mathcal{R}(A_W)), \tag{4.1.3}$$

where $\mathcal{R}(A_W)$ is the Rees algebra of A_W. The Proj of a Rees algebra always comes with a natural divisor, $\mathrm{Proj}(\mathrm{gr}_F A_W)$, which in our case can be identified with the collection of n distinct smooth points $p_1,\ldots,p_n$. Furthermore, it can be equipped with canonical tangent vectors v_i at p_i: the expression $t_i=u_i^{-1}$ can be viewed as regular function in a neighborhood of p_i and gives rise to a formal parameter at it, so there is a unique v_i such that $\langle v_i,dt_i\rangle=1$. We will check in the proof of Theorem 4.1.2 below that $\pi(C_W,p_1,\ldots,p_n,v_1,\ldots,v_n)=W$.

Sketch of proof of Theorem 4.1.2. First, we will consider the important particular case $n=g\geq 1$. The case $n>g$ reduces to very similar considerations by considering the standard open covering of the Grassmannian by affine spaces.

The main part of the proof is the construction of a canonical presentation and a canonical basis of the ring $\mathcal{O}(C\setminus D)$, where $D=p_1+\ldots+p_g$, for a point $(C,p_1\ldots,p_g,v_1,\ldots,v_g)$ in $\widetilde{\mathcal{U}}^{ns}_{g,g}(k)$. By the Riemann-Roch theorem, the condition $H^1(\mathcal{O}(D)=0$ is equivalent to the condition $H^0(\mathcal{O}(D))=k$. Furthermore, for any $m\geq 1$ and $i=1,\ldots,g$, we still have $H^1(\mathcal{O}(D+mp_i))=0$, so $H^0(\mathcal{O}(D+mp_i))$ is $(m+1)$-dimensional. Let us choose at each point p_i a formal parameter t_i such that $\langle v_i,dt_i\rangle=1$. Then we can choose rational functions $x_i \in H^0(\mathcal{O}(D+p_i))$ (resp., $y_i \in H^0(\mathcal{O}(D+2p_i))$) with the Laurent expansion at p_i starting with $1/t_i^2$ (resp., $1/t_i^3$). The ambiguity in choosing x_i and y_i is the following: we can change x_i to $x_i + a_i$ and y_i to $y_i + b_i x_i + c_i$, for some constants a_i, b_i, c_i. We will fix this ambiguity later. Note that for each $m \geq 1$, the elements $x_1^m, \ldots, x_g^m$ (resp., $x_1^{m-1}y_1, \ldots, x_g^{m-1}y_g$) project to a basis of $H^0(\mathcal{O}(2mD))/H^0(\mathcal{O}((2m-1)D))$ (resp., $H^0(\mathcal{O}((2m+1)D))/H^0(\mathcal{O}(2mD))$). Hence, the elements

$$(x_i^m, x_i^m y_i), \text{ for } m \geq 0, \ i = 1, \ldots, g, \tag{4.1.4}$$

form a k-basis of the space $H^0(C \setminus D, \mathcal{O})$.

Now the functions y_i^2, $x_i x_j$, $x_i y_j$ and $y_i y_j$, where $i \neq j$, have some expressions as linear combinations of this basis. By taking into account what we know about the poles of (x_i) and (y_i), we obtain equations of the form

$$\begin{aligned}
&x_i x_j = \alpha_{ji} y_i + \alpha_{ij} y_j + \gamma_{ji} x_i + \gamma_{ij} x_j + \textstyle\sum_{k \neq i,j} c_{ij}^k x_k + a_{ij}, \\
&x_i y_j = d_{ij} x_j^2 + t_{ji} y_i + v_{ij} y_j + r_{ji} x_i + \delta_{ij} x_j + \textstyle\sum_{k \neq i,j} e_{ij}^k x_k + b_{ij}, \\
&y_i y_j = \beta_{ji} x_i^2 + \beta_{ij} x_j^2 + \varepsilon_{ji} y_i + \varepsilon_{ij} y_j + \psi_{ji} x_i + \psi_{ij} x_j + \textstyle\sum_{k \neq i,j} l_{ij}^k x_k + u_{ij}, \\
&y_i^2 = x_i^3 + q_i x_i y_i + r_i x_i^2 + u_i y_i + \textstyle\sum_{j \neq i} g_i^j y_j + \pi_i x_i + \textstyle\sum_{j \neq i} k_i^j x_j + s_i,
\end{aligned} \tag{4.1.5}$$

where $i \neq j$. Conversely, any algebra with generators (x_i), (y_i) and relations of this form is spanned by the elements (4.1.4). The condition that they are linearly independent is equivalent to a system of polynomial equations on the coefficients (these equations can be found explicitly by applying the theory of Gröbner bases to an appropriate order on monomials, see e.g., [9, Thm. 15.8]).

Furthermore, we can use the above equations to normalize our generators (x_i), (y_i). Namely, if 2 and 3 are invertible, then we can choose them uniquely so that $q_i = r_i = u_i = 0$. If only 2 is invertible and $g \geq 2$, then we can still make $q_i = u_i = 0$ and in addition we can make $\gamma_{ii_0} = 0$ for $i \neq i_0$ and $\gamma_{i_0 i_1} = 0$ for some fixed indices i_0 and i_1, $i_0 \neq i_1$.

It is not hard to check that the obtained affine scheme is equivalent to our stack $\widetilde{\mathcal{U}}_{g,g}^{ns}$. We use the natural filtration on the algebra given by the above equations to construct the corresponding projective curve, and we use x_i/y_i as a formal parameter at the ith point at infinity.

In order to treat the case $n > g$ we consider the covering of $\widetilde{U}_{g,n}^{ns}$ by the open substacks $\pi^{-1}(U_S)$, where $S \subset [1, n]$, $|S| = g$. Over $\pi^{-1}(U_S)$ we will similarly construct a canonical basis of $H^0(C \setminus D, \mathcal{O})$, where now $D = p_1 + \ldots + p_n$. Namely, first we consider generators x_i, y_i constructed as above for $i \in S$ (where we use only the points (p_i) with $i \in S$). Then for each $j \notin S$, we add a generator $x_{S,j} \in H^0(C, \mathcal{O}(p_j + \sum_{i \in S} p_i))$, such that $x_{S,j} = \frac{1}{t_j} + \ldots$, defined uniquely up to an additive constant. Then it is easy to see that the elements

$$(x_i^m, x_i^m y_i, x_{S,j}^{m+1}), \quad i \in S, j \notin S, m \geq 0$$

form a basis of $\mathcal{O}(C \setminus D)$. We normalize these elements as before, except in the case $g = 1$, $n \geq 2$: here we normalize x_i by the condition $x_i(p_{j_0}) = 0$ for some $j_0 \notin S$.

Furthermore, in addition to relations between $(x_i, y_i)_{i \in S}$ as above we will have relations in the algebra $\mathcal{O}(C \setminus D)$ describing the expressions of $x_{S,j} x_{S,j'}$, for $j \neq j'$, $x_i x_{S,j}$ and $y_i x_{S,j}$ in terms of the basis. We can normalize $x_{S,j}$ by requiring that $x_{S,j}(p_{i_0}) = 0$ for some fixed

$i_0 \in S$. Applying the theory of Gröbner bases we get an identification of $\pi^{-1}(U_S)$ with an affine scheme of finite type over U_S.

Note that the Laurent expansion of $x_{S,j}$ at p_i, where $i \in S$, should have form

$$x_{S,j} \equiv \frac{a_{S,ij}}{t_i} + \dots \tag{4.1.6}$$

It is easy to see that the functions $(a_{S,ij})_{i\in S, j\notin S}$ are precisely the pull-backs of the standard affine coordinates on U_S. One checks using the form of the relations in $\mathcal{O}(C \setminus D)$ that the remaining coordinates in the affine embedding of $\pi^{-1}(U_S)$ have positive weights with respect to $\mathbb{G}_m$. In the case of a curve C_W, for each $j \notin S$, the element

$$u_j + \sum_{i\in S} a_{S,ij} u_i \in W$$

can be viewed as an element of the algebra $A_W = \mathcal{O}(C_W \setminus D)$. Since u_i^{-1} are parameters at p_i, we deduce that $\pi(C_W)$ is the point of U_S with the coordinates $(a_{S,ij})$, i.e., $\pi(C_W) = W$. The fact that these are the only $\mathbb{G}_m$-invariant points in $\pi^{-1}(U_S)$ follows from the above observation about the $\mathbb{G}_m$-weights of the affine coordinates. $\square$

Remark 4.1.3. Let us consider the cuspidal curve C_1^{cusp} of arithmetic genus 1, with the affine part given by $y^2 = x^3$, over a field k of characteristic 2. Then we observe that the derivation ∂_y is well-defined on the algebra of functions on the affine part of C_1^{cusp} (due to $\partial_y(y^2) = 0$). Furthermore, in terms of the parameter $u = y/x$ on the normalization we have $\partial_y = \partial_u/u^2$. Thus, at the point $u = \infty$ this vector field is regular and has a zero of order 4. It follows that the curve $C_1^{\text{cusp}} \times \operatorname{Spec}(k[\epsilon]/\epsilon^2)$ over the dual numbers has an automorphism preserving the point at infinity and acting trivially on the tangent space at it. Thus, in characteristic 2 the stack $\widetilde{\mathcal{U}}_{1,1}^{ns}$ cannot be equivalent to a scheme.

4.2. **Setup for moduli of A_∞-structures.** Next, we want to consider a moduli problem for A_∞-structures on a family of graded associative algebras over the Grassmannian $G(n-g, n)$, which we will then relate to the moduli space of curves $\widetilde{\mathcal{U}}_{g,n}^{ns}$.

Let Q_n be the quiver with $n+1$ vertices marked as $\mathcal{O}, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n}$ and with the arrows

$$A_i : \mathcal{O} \to \mathcal{O}_{p_i}, \quad B_i : \mathcal{O}_{p_i} \to \mathcal{O}, \ i = 1, \dots, n.$$

We denote by $k[Q_n]$ the corresponding path algebra in which we write the paths from right to left. Let J_0 be the two-sided ideal in the path algebra $k[Q_n]$ of Q_n, generated by the elements

$$A_iB_iA_i, B_iA_iB_i, A_iB_j,$$

where $i \neq j$. For an $(n-g)$-dimensional subspace $W \subset k^n$ we define $J_W \subset k[Q_n]$ to be the ideal generated by J_0 together with the additional relations

$$\sum x_i B_i A_i = 0 \quad \text{for every } \sum x_i e_i \in W,$$

and consider the corresponding quotient algebra

$$E_W = k[Q_n]/J_W. \tag{4.2.1}$$

We equip E_W with the $\mathbb{Z}$-grading by $\deg(A_i) = 0$, $\deg(B_i) = 1$.

The similar definition makes sense when W is an R-point of the Grassmanian $G(n-g,n)$, where R is a commutative ring. Then E_W is an algebra over R, which is projective as an R-module.

Lemma 4.2.1. *Let $(C,p_\bullet,v_\bullet)$ be an R-point of the moduli scheme $\widetilde{\mathcal{U}}^{ns}_{g,n}$, where R is a commutative ring. Then there is a natural isomorphism of graded algebras*

$$\mathrm{Ext}^*(G,G)\simeq E_W,$$

where G is the sheaf (4.1.1) *and $W=\pi(C,p_\bullet,v_\bullet)$ is the corresponding R-point of the Grassmannian.*

Proof. For every i we have a natural identification $\mathrm{Hom}(\mathcal{O},\mathcal{O}_{p_i})=R\cdot A_i$, where $A_i=1\in H^0(\mathcal{O}_{p_i})$. On the other hand, the tangent vector v_i induces a trivialization of $\mathcal{O}(p_i)|_{p_i}$, and we define $B_i\in\mathrm{Ext}^1(\mathcal{O}_{p_i},\mathcal{O})$ as the class of the extension

$$0\to\mathcal{O}\to\mathcal{O}(p_i)\to\mathcal{O}(p_i)|_{p_i}\to 0.$$

This gives an identification $\mathrm{Ext}^1(\mathcal{O}_{p_i},\mathcal{O})=R\cdot B_i$. The composition B_iA_i is precisely the image of the ith basis vector under the coboundary homomorphism

$$R^n=\bigoplus_{i=1}^n\mathcal{O}(p_i)|_{p_i}\to H^1(\mathcal{O}).$$

This easily leads to the required identification. ☐

Remark 4.2.2. If we assume that $H^1(C,\mathcal{O}(\sum_{i\in S}p_i))=0$ for $S\subset[1,n]$, $|S|=g$, and use the corresponding functions $a_{S,ij}$ defined by (4.1.6), then the relations in the corresponding algebra $\mathrm{Ext}^*(G,G)$ take form

$$B_jA_j=\sum_{i\in S}a_{S,ij}B_iA_i, \tag{4.2.2}$$

for every $j\notin S$.

The algebras E_W are fibers of the natural sheaf of $\mathcal{O}$-algebras $\mathcal{E}_{g,n}$ over $G(n-g,n)$. Furthermore, the group $\mathbb{G}_m^n$ acts naturally on $G(n-g,n)$ (as diagonal matrices), and for $\lambda=(\lambda_1,\ldots,\lambda_n)=\mathbb{G}_m^n$ we have a natural isomorphism

$$E_W\to E_{\lambda\cdot W}:A_i\mapsto A_i, B_i\mapsto\lambda_iB_i.$$

This equips the sheaf $\mathcal{E}_{g,n}$ with a $\mathbb{G}_m^n$-equivariant structure.

We consider the moduli functor $\mathcal{M}_\infty$ for minimal A_∞-structures on the family of algebras $\mathcal{E}_{g,n}$, as in Sec.3.1.

4.3. **Construction of a morphism.** We are going to construct a natural morphism of functors

$$\widetilde{\mathcal{U}}^{ns}_{g,n}\to\mathcal{M}_\infty. \tag{4.3.1}$$

The idea is to associate with every curve $(C,p_1,\ldots,p_n,v_1,\ldots,v_n)$ the gauge equivalence class of the minimal A_∞-structure on $E_W\simeq\mathrm{Ext}^*(G,G)$, obtained by the homological perturbation.

Let us explain how this can be done in a family with an affine base $\mathrm{Spec}(R)$. Let $\pi : C \to \mathrm{Spec}(R)$ be a flat projective family of curves, $p_i : \mathrm{Spec}(R) \to C$ are disjoint sections, such that π is smooth near p_i, and $U = C \setminus D$ is affine, where $D = p_1 + \ldots + p_n$. We use the resolution $P = [\mathcal{O}_C(-D) \to \mathcal{O}_C]$ of the sheaf $\mathcal{O}_D = \bigoplus_i \mathcal{O}_{p_i}$, so we get a locally free sheaf of dg-algebras over C,

$$\mathcal{A}_G := \underline{\mathrm{End}}(\mathcal{O}_C \oplus P).$$

Now we need to get some dg-model for the cohomology of $\mathcal{A}_G$. One possibility is to pick a finite affine open covering $C = \cup_i U_i$ and to consider the corresponding Cech (total) complex (which is equipped with a structure of a dg-algebra in a standard way). However, the problem is that it is not clear why the obtained complex is homotopy equivalent to its cohomology, so we can not get the input data needed for the homological perturbation construction.

Instead we are going to consider a version of the Cech complex which uses the open subset $U = C \setminus D$ together with relative formal disks around p_i. Assuming that $\mathcal{O}_C(U)$ is a projective R-module, we will get in this way a complex which is homotopy equivalent to its cohomology. Note that this projectivity condition is satisfied for the universal family over an appropriate open covering of the moduli space $\widetilde{\mathcal{U}}^{ns}_{g,n}$ (over $\mathbb{Z}[1/2]$ for $n \geq 2$, or over $\mathbb{Z}[1/6]$ for $n = g = 1$).

Then for every quasicoherent sheaf $\mathcal{F}$ on C we can consider the two-term complex $K^{\bullet}_D(\mathcal{F})$ with

$$\begin{aligned} K^0_D(\mathcal{F}) &= \varprojlim_n H^0\big(C, \mathcal{F}/\mathcal{F}(-nD)\big) \oplus H^0(U, \mathcal{F}), \\ K^1_D(\mathcal{F}) &= \varinjlim_m \varprojlim_n H^0\big(C, \mathcal{F}(mD)/\mathcal{F}(-nD)\big) \end{aligned}$$

and the differential

$$d(s_0, s) = \kappa(s) - \iota(s_0),$$

where we use natural maps $\iota : H^0\big(C, \mathcal{F}/\mathcal{F}(-nD)\big) \to K^1(\mathcal{F})$ and $\kappa : H^0(U, \mathcal{F}) \to K^1_D(\mathcal{F})$.

The construction of $K^{\bullet}_D(\mathcal{F})$ immediately generalizes to the case when $\mathcal{F}$ is a bounded complex of vector bundles (by taking the total complex of the corresponding bicomplex). Furthermore, if $\mathcal{A}$ is a complex of quasicoherent sheaves equipped with a structure of an $\mathcal{O}$-dg-algebra then we can equip the complex $K^{\bullet}_D(\mathcal{A})$ with a structure of a dg-algebra by using the natural componentwise multiplication on $K^0_D(\mathcal{A})$ and using the multiplications

$$\begin{aligned} &K^0_D(\mathcal{A}) \otimes K^1_D(\mathcal{A}) \to K^1_D(\mathcal{A}) : (s_0, s) \cdot u = \iota(s_0) \cdot u, \\ &K^1_D(\mathcal{A}) \otimes K^0_D(\mathcal{A}) \to K^1_D(\mathcal{A}) : u \cdot (s_0; s) = u \cdot \kappa(s), \end{aligned} \tag{4.3.2}$$

where on the right-hand side we use the natural product on $K^1(\mathcal{A})$.

Applying this construction to $\mathcal{A} = \mathcal{A}_G$ we get a dg-model $K^{\bullet}_D(\mathcal{A}_G)$ for the Ext-algebra of G. We want to show that it is possible to get the input data for the homological perturbation construction on this dg-algebra. In [35] we constructed explicit cohomology representative and the homotopy needed for this. Here we will show how this can be deduced using some simple properties of this setup.

Lemma 4.3.1. *Let $\mathcal{A}$ be a sheaf of dg-algebras over C, which is bounded, i.e., concentrated in degrees $[-N, N]$ for some $N > 0$. Assume that every term $\mathcal{A}^i$ is a direct sum of line bundles of the form $\mathcal{O}_C(mD)$ and that $H^*(C, \mathcal{A})$ are projective R-modules. Assume also* *that $\mathcal{O}(U)$ is a projective R-module. Then $K^\bullet_D(\mathcal{A})$ is homotopy equivalent to a complex of R-modules with the trivial differential.*

Proof. We can think of $K^\bullet_D(\mathcal{A})$ as a total complex associated with the bicomplex

$$K^\bullet_D(\mathcal{A}^{-1}) \to K^\bullet_D(\mathcal{A}^0) \to K^\bullet_D(\mathcal{A}^1).$$

By Lemma 1.2.3(ii), it is enough to check that each $K^\bullet_D(\mathcal{A}^i)$ is homotopy equivalent to a complex of projective R-modules. It suffices to prove this for $K^\bullet_D(\mathcal{O}_C(mD))$. Choosing relative formal parameters at t_i, we can identify the latter complex with

$$\mathcal{O}(U) \oplus \mathcal{H}_{\geq -m} \xrightarrow{(\kappa,-\iota)} \mathcal{H}.$$

where $\mathcal{H} = \bigoplus_{i=1}^n R((t_i))$, $\mathcal{H}_{\geq -m} = \bigoplus_{i=1}^n t_i^{-m} R[[t_i]]$. Let us set $\mathcal{H}_{<-m} = \bigoplus_{i=1}^n t_i^{-m-1} R[t_i^{-1}]$, so that we have a decomposition

$$\mathcal{H} = \mathcal{H}_{\geq -m} \oplus \mathcal{H}_{<-m}.$$

Let $\kappa_{\geq -m} : \mathcal{O}(U) \to \mathcal{H}_{\geq -m}$ and $\kappa_{<-m} : \mathcal{O}(U) \to \mathcal{H}_{<-m}$ be the components of κ with respect to this decomposition. Then we have a natural projection

$$K^\bullet(\mathcal{O}_C(mD)) \xrightarrow{p} \left[\mathcal{O}(U) \xrightarrow{\kappa_{<-m}} \mathcal{H}_{<-m}\right].$$

We claim that it extends to a homotopy equivalence. Namely, we define the chain map

$$\left[\mathcal{O}(U) \xrightarrow{\kappa_{<-m}} \mathcal{H}_{<-m}\right] \xrightarrow{i} K^\bullet(\mathcal{O}_C(mD))$$

by $i(f) = (f, \kappa_{\geq -m}(f))$, $i(v) = v$, for $f \in \mathcal{O}(U)$, $v \in \mathcal{H}_{<-m}$. Then $p \circ i = \mathrm{id}$, while the homotopy between $i \circ p$ and id is given by $h(v) = v_{\geq -m}$ for $v \in \mathcal{H}$. This proves our claim.

It remains to note that $\mathcal{H}_{<-m}$ is a free R-module, while $\mathcal{O}(U)$ is projective by assumption. □

The terms $\mathcal{A}^i_G$ are direct sums of line bundles of the form $\mathcal{O}_C(mD)$, and the cohomology $H^*(K^\bullet_D(\mathcal{A}_G)) \simeq \mathrm{Ext}^*(G,G)$ are projective R-modules by Lemma 4.2.1. Hence, assuming that $\mathcal{O}(U)$ is a projective R-module, Lemma 4.3.1 is applicable in our case, and it gives a homotopy equivalence of $K^\bullet_D(\mathcal{A}_G)$ to its cohomology, which is needed to run the homological perturbation.

Thus, for every standard open affine cell $U_S \subset G(n-g,n)$, we can apply the above construction to the open affine subset

$$\widetilde{\mathcal{U}}^{ns}_{g,n}(U_S) := \pi^{-1}(U_S) \subset \widetilde{\mathcal{U}}^{ns}_{g,n}$$

and the sheaf of dg-algebras $\mathcal{A}_G|_{\pi^{-1}(U_S)}$. Thus, the homological perturbation gives a minimal A_∞-structure on $\pi^*\mathcal{E}_{g,n}|_{\pi^{-1}(U_S)}$. Note that the obtained A_∞-algebra is equivalent to the dg-algebra $K^\bullet_D(\mathcal{A}_G|_{\pi^{-1}(U_S)})$, so its gauge equivalence class does not depend on a choice of homotopies up to gauge equivalence.

In particular, over the intersections $\pi^{-1}(U_S \cap U_{S'})$ the restrictions of the minimal A_∞-structures from $\pi^{-1}(U_S)$ and $\pi^{-1}(U_{S'})$ are gauge equivalent. Thus, the map (4.3.1) is well defined.

Next, we recall that there is a $\mathbb{G}_m^n$-action on the moduli space $\widetilde{\mathcal{U}}^{ns}_{g,n}$, and that the open subsets $\pi^{-1}(U_S)$ are invariant under this action. Furthermore, the sheaf of dg-algebras

$\mathcal{A}_G$ is $\mathbb{G}_m^n$-equivariant, and the complexes $K_D^\bullet(\mathcal{A}_G|_{\pi^{-1}(U_S)})$ still carry the algebraic $\mathbb{G}_m^n$-action (compatible with the action on the base rings). The constructions involved in choosing a homotopy equivalence to the cohomology can all be made to be compatible with the $\mathbb{G}_m^n$-action. Thus, as a result we get minimal A_∞-algebras over $\pi^{-1}(U_S)$, which are $\mathbb{G}_m^n$-equivariant, and the gauge equivalences on the intersections are $\mathbb{G}_m^n$-invariant.

Lemma 4.3.2. *For each S, the canonical isomorphism $\mathrm{Ext}^*(G,G) \simeq \pi^*\mathcal{E}_W$ of algebras over $\widetilde{\mathcal{U}}_{g,n}^{ns}(U_S)$ (defined for the universal family as in Lemma 4.2.1) is compatible with the $\mathbb{G}_m^n$-action, where the action on the left comes from the $\mathbb{G}_m^n$-equivariant structure on G, while the action on the right is induced by the rescalings*

$$A_i \mapsto A_i, \ B_i \mapsto \lambda_i B_i, \tag{4.3.3}$$

for $(\lambda_1, \dots, \lambda_n) \in \mathbb{G}_m^n$.

Proof. This can be easily deduced from the fact that the isomorphism of $\mathrm{Ext}^*(G,G)$ with $\pi^*\mathcal{E}_W$ sends B_i to the generator of $\mathrm{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O})$, defined by the relative tangent vector v_i, and the $\mathbb{G}_m$-action rescales v_i by $\lambda_i^{-1} v_i$. □

Corollary 4.3.3. *The map* (4.3.1) *is compatible with the $\mathbb{G}_m^n$-actions, where the $\mathbb{G}_m^n$-action on $\mathcal{M}_\infty$ is induced by the rescalings* (4.3.3)*.*

We will especially care about the action of the diagonal subgroup $\mathbb{G}_m \subset \mathbb{G}_m^n$, which acts trivially on $G(n-g,n)$. Note that the action of $\mathbb{G}_m$ on $\mathcal{E}_W$ corresponds to the natural $\mathbb{Z}$-grading of $\mathcal{E}_W$ (i.e., it acts on degree m component with the weight m). Note that the induced action of $\mathbb{G}_m$ on A_∞-structures rescales m_n to $\lambda^{n-2} m_n$.

4.4. **Representability of the moduli of A_∞-structures.**

Next, we want to prove that $\mathcal{M}_\infty$ is represented by an affine scheme of finite type over $G(n-g,n)$. For this we want to apply the criterion of Theorem 3.3.1, which requires some information about the Hochschild cohomology of the algebras E_W. We will get this information geometrically by identifying $HH^*(E_W)$ with the Hochschild cohomology of the corresponding special curve C_W.

Lemma 4.4.1. *Let $C_W \in \widetilde{\mathcal{U}}_{g,n}^{ns}$ be the special curve corresponding to $W \in G(n-g,n)$ (see* (4.1.3)*). Then the minimal A_∞-structure on $\mathrm{Ext}^*(G,G) \simeq E_W$ coming from the homological perturbation is homotopically trivial. Hence, we have an equivalence*

$$\mathrm{Perf}(C_W) \simeq \mathrm{Perf}(E_W)$$

and therefore, an isomorphism

$$HH^*(C_W) \simeq HH^*(E_W),$$

where $W = \pi(C, p_1, \dots, p_n)$. The second grading on $HH^*(E_W)$ corresponds to the weights of the $\mathbb{G}_m$-action, coming from the natural $\mathbb{G}_m$-action on C_W.

Proof. Recall that the point C_W in the moduli space is $\mathbb{G}_m$-invariant. Hence, by Corollary 4.3.3, the minimal A_∞-structure on E_W gives a $\mathbb{G}_m$-invariant point of $\mathcal{M}_\infty$. But the $\mathbb{G}_m$-action simply rescales m_n to $\lambda^{n-2} m_n$. From this we can step by step deduce that all m_n with $n > 2$ can be made zero by a homotopy. Indeed, the class of $[m_3]$ in $HH^2(E_W)_{-1}$ is

$\mathbb{G}_m$-invariant with respect to the weight-1 action, hence, $[m_3]=0$. Thus, we can choose a gauge equivalent structure with $m_3=0$. Next, look at the class of $[m_4]$ in $HH^2(E_W)_{-2}$, etc.

The equivalence of the perfect derived categories follows since G is a generator of $\mathrm{Perf}(C_W)$. $\square$

Thus, to apply our criterion of representability of the functor of A_∞-structures, we need to calculate $HH^1(C_W)$, or at least, its part of negative weight. For this we need some geometric information about the curves C_W.

Lemma 4.4.2. *For each $W \in G(n-g,n)(k)$, where k is a field, the curve $C=C_W$ is a union of n irreducible components C_i, where $p_i \in C_i$. These components are joined in a single point q, which is the only singular point of C (with $p_i \in C_i \setminus \{q\}$). Each component C_i is either $\mathbb{P}^1$, or the cuspidal curve of arithmetic genus* 1.

Proof. Since the points p_i are smooth, it is enough to study the irreducible components of the affine curve $C_W \setminus D = \mathrm{Spec}(A_W)$. The component C_i corresponds to the image of the natural projections $A_W \to k[u_i]$. This image contains $k + u_i^2 k[u_i] = k[u_i^2, u_i^3]$, so it either equal to the latter subring, or is the entire $k[u_i]$. This implies that the corresponding irreducible component of $C_W \setminus D$ is either $\mathbb{A}^1$ or the cuspidal affine cubic, so that u_i is the affine coordinate on the normalization. $\square$

Recall that the Hochschild cohomology of a quasi-projective scheme can be calculated as

$$HH^*(X) = \mathrm{Ext}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X),$$

where $\Delta : X \to X \times X$ is the diagonal embedding. It is convenient to consider the sheafified version $\underline{\mathcal{H}}^*(X) := \underline{\mathrm{Ext}}^*(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$, which is a sheaf on X. There is a local-to-global spectral sequence

$$E_2^{pq} = H^p(X, \underline{HH}^*) \implies HH^{p+q}(X). \tag{4.4.1}$$

Note that we have $\underline{HH}^0 = \mathcal{O}$, and applying $\underline{\mathrm{Ext}}^*(\cdot, \Delta_*\mathcal{O}_X)$ to the exact sequence

$$0 \to \mathcal{J}_\Delta \to \mathcal{O}_{X\times X} \to \Delta_*\mathcal{O}_X \to 0,$$

where $\mathcal{J}_\Delta$ is the ideal sheaf of the diagonal, we immediately see that

$$\underline{HH}^1 \simeq \mathrm{Hom}_{X\times X}(\mathcal{J}_\Delta, \Delta_*\mathcal{O}_X) \simeq \mathrm{Hom}(\Omega_X, \mathcal{O}_X) \simeq \mathcal{T}_X,$$

where $\mathcal{T}_X$ is the tangent sheaf of X. In the case of a quasiprojective curve C spectral sequence (4.4.1) degenerates, and we have exact sequences

$$0 \to H^1(C, \underline{HH}^{i-1}) \to HH^i(C) \to H^0(C, \underline{HH}^i) \to 0 \tag{4.4.2}$$

for every i. In particular, for $i=1$, we get an exact sequence

$$0 \to H^1(C, \mathcal{O}) \to HH^1(C) \to H^0(C, \mathcal{T}) \to 0. \tag{4.4.3}$$

Thus, to study $HH^1(C)$, we need some information on the global derivations, as well as on $H^1(C, \mathcal{O})$.

Lemma 4.4.3. *Let C be a reduced projective curve over a field k with a $\mathbb{G}_m$-action, which is the union of irreducible components C_i, $i = 1, \ldots, n$, joined in a single point q. Assume that $C \setminus \{q\}$ is smooth and that each normalization map $\widetilde{C}_i \to C_i$ is a bijection, with $\widetilde{C}_i \simeq \mathbb{P}^1$. Assume also that the action of $\mathbb{G}_m$ on the Zariski tangent space at q has negative weights. Then*
(i) the action of $\mathbb{G}_m$ on $H^1(C, \mathcal{O}_C)$ has positive weights.
(ii) Assume in addition that $C = C_W$ for some subspace $W \subset k^n$, where $W = 0$ if $n = 1$. Assume also that either $W = k^n$ or $\mathrm{char}(k) \neq 2$. *If $n = g = 1$ then assume in addition* $\mathrm{char}(k) \neq 3$. *Then $H^1(C, \mathcal{T}) = 0$, and the action of $\mathbb{G}_m$ on $H^0(C, \mathcal{T})$ has weights* 0 *and* 1.
(iii) Keep the assumptions of (ii). Let $p_i \in C_i \setminus \{q\}$ be the unique $\mathbb{G}_m$-invariant point, and let $D = \sum_i p_i$. Then one has

$$H^0(C, \mathcal{T}(-D)) = H^0(C, \mathcal{T})^{\mathbb{G}_m}, \quad H^0(C, \mathcal{T}(-2D)) = 0.$$

Also, the natural map $H^0(C, \mathcal{T}(nD)) \to H^0(C, \mathcal{T}(nD)|_D)$ is surjective for $n \geq 0$.

Proof. (i) Let $V = C \setminus \{q\}$. We can choose a coordinate u_i on an affine part of $\widetilde{C}_i \simeq \mathbb{P}^1$ containing q such that $u_i(q) = 0$ and u_i has some weight $w_i > 0$ with respect to the $\mathbb{G}_m$-action. Let U be an affine neighborhood of q obtained by deleting on each C_i the point where u_i has a pole. We can calculate $H^1(C, \mathcal{O}_C)$ as the quotient of $\mathcal{O}(U \setminus \{q\})$ by $\mathcal{O}(V) + \mathcal{O}(U)$. Note that $U \setminus \{q\}$ is the disjoint union of n copies of $\mathbb{A}^1 \setminus \{0\}$, with the coordinates (u_i). Since every u_i^n with $n \leq 0$ extends to a regular function on V, we see that $H^1(C, \mathcal{O}_C)$ is spanned by positive powers of u_i's, so $\mathbb{G}_m$ has only positive weights on it.
(ii) As before, we use the coordinates u_i on affine parts of the normalizations $\widetilde{C}_i$. The space $H^0(C, \mathcal{T}_C)$ embeds into the space of vector fields on $V \simeq \sqcup_{i=1}^n (C_i \setminus \{q\})$, which are spanned by $u_i^m \partial_{u_i}$ with $m \leq 2$ (this comes from the condition of regularity at ∞).

We claim that if a vector field $v = (P_i(u_i, u_i^{-1})\partial_{u_i})$ on $U \setminus \{q\}$ extends to a derivation of $\mathcal{O}(U)$ then $P_i \in u_i k[u_i]$ for every i. Indeed, assume first that $n \geq 2$ and $\mathrm{char}(k) \neq 2$. Then applying v to the function on U corresponding to $u_i^2 \in A_W$ we get that $v(u_i^2)/2 = P_i(u_i)u_i \in A_W$, which implies that $P_i \in k[u_i]$. Furthermore, let $P_i = a_i \bmod u_i k[u_i]$. Then $P_i(u_i)u_i \equiv a_i u_i \bmod u_i^2 k[u_i^2]$, so the condition $v^2(u_i) \in A_W$ implies that $a_i v(u_i) = a_i P_i(u_i) \in A_W$, which is possible only if $a_i = 0$. This proves the claim in this case. In the case $W = k^n$ and k is arbitrary, we have $u_i \in A_W$, so the condition $v(u_i) \in A_W$ immediately gives $P_i \in u_i k[u_i]$. Finally, if $n = g = 1$ then applying v to u_1^2 we get $P_1(u_1)u_1 \in k[u_1^2, u_1^3]$, so $P_1 = a u_1^{-1} \bmod u_1 k[u_1]$. Now $v(u_1^3) = 3a u_1 + \ldots$, so using that $\mathrm{char}(k) \neq 3$ we deduce $a = 0$, and the claim follows.

Thus, if v extends to a global section of $\mathcal{T}_C$ then each P_i is a linear combination of u_i and u_i^2, which implies that the weights of $\mathbb{G}_m$ on $H^0(C, \mathcal{T}_C)$ are 0 and 1. Similarly, we see that if $P_i \in u_i^2 k[u_i]$ for every i then v extends to a derivation of $\mathcal{O}(U)$. Thus, $H^0(U, \mathcal{T})$ and $H^0(V, \mathcal{T})$ span $H^0(U \setminus \{q\}, \mathcal{T})$, which gives the vanishing of $H^1(C, \mathcal{T}_C)$.
(iii) A vector field on $U \setminus \{q\}$ has zero (resp., double zero) along D iff each $P_i \in u_i k[u_i^{-1}]$ (resp., $P_i \in k[u_i^{-1}]$). Together with calculations of (ii) this immediately implies our assertions about $H^0(C, \mathcal{T}(-D))$ and $H^0(C, \mathcal{T}(-2D))$. Next, similarly to (ii) we can represent sections of $H^0(C, \mathcal{T}_C(nD))$ as vector fields $v = (P_i(u_i)\partial_{u_i})$ on $U \setminus \{q\}$ with $\deg(P_i) \leq n+2$,

and the last assertion follows from the fact that v extends to a regular derivation of $\mathcal{O}(U)$ whenever $P_i \in u_i^2 k[u_i]$. □

Corollary 4.4.4. *Let k be a field of characteristic $\neq 2$ (resp., $\neq 2, 3$ if $n = 1$). Then for any subspace $W \subset k^n$, where $W = 0$ if $n = 1$, one has*

$$HH^0(E_W)_{<0} = HH^1(E_W)_{<0} = 0.$$

The same result holds for $W = k^n$, $n \geq 2$, with no restrictions on the characteristic.

Proof. By Lemma 4.4.1, we have $HH^1(E_W) \simeq HH^1(C_W)$, where C_W is the corresponding special curve, and the second grading is induced by the $\mathbb{G}_m$-action on C_W. Now $HH^0(C_W) = H^0(C_W, \mathcal{O})$ lives in degree 0. For HH^1 we use the exact sequence (4.4.3). Now the assertion follows from Lemma 4.4.3(i)(ii). □

Proposition 4.4.5. *Let us work over $\mathbb{Z}[1/2]$ if $n \geq 2$, or over $\mathbb{Z}[1/6]$ if $n = 1$, or over $\mathbb{Z}$ if $g = 0$. Assume that either $n \geq 2$ or $g = 1$. Then the functor $\mathcal{M}_\infty$ of A_∞-structures (up to a gauge equivalence) on the family (E_W) is represented by an affine scheme of finite type over $G(n - g, n)$.*

Proof. Due to Corollary 4.4.4, the criterion of Theorem 3.3.1(ii) implies that $\mathcal{M}_\infty$ (resp., $\mathcal{M}_n$) is represented by an affine scheme (resp., of finite type) over $G(n - g, n)$. Next, we note that by Lemma 4.4.1, $HH^i(E_W)$ is finite-dimensional for every i. Hence, by Theorem 3.3.1(iii), we derive that $\mathcal{M}_\infty \simeq \mathcal{M}_n$ for sufficiently large n, so it is of finite type over $G(n - g, n)$. □

4.5. Comparison of the moduli spaces via deformation theory.

Theorem 4.5.1. *Assume that either*

- *$n \geq g \geq 1$, $n \geq 2$ and we work over $\mathbb{Z}[1/2]$;*
- *$n = g = 1$ and we work over $\mathbb{Z}[1/6]$;*
- *$g = 0$, $n \geq 2$ and we work over $\mathbb{Z}$.*

Then the map (4.3.1) is an isomorphism of schemes, compatible with the $\mathbb{G}_m^n$-action.

Remark 4.5.2. It is plausible that that the (4.3.1) gives an equivalence of stacks over $\mathbb{Z}$. However, our method does not show it, since it is based on first proving representability of each side. One could imaging constructing the map in the inverse direction that would recover the affine part of the curve C as a certain moduli space of 1-dimensional A_∞-modules over $\mathrm{Ext}^*(G, G)$: a point $p \in C \setminus D$ would correspond to the A_∞-module $\mathrm{Ext}^*(G, \mathcal{O}_p)$.

The crucial part of the proof of Theorem 4.5.1 is the comparison between the deformation theories of the curves C_W (see (4.1.3)) and that of the trivial A_∞-structures on the algebras E_W.

Let us fix a field k and consider the category $\mathrm{Art}(k)$ of local Artinian S-algebras with fixed identifications of the residue field with k. Here S is our base ring, which is either $\mathbb{Z}$ or $\mathbb{Z}[1/2]$ or $\mathbb{Z}[1/6]$. Morphisms in this category are local homomorphisms inducing the identity on the residue field.

We use the following terminology from [24]. A *deformation functor* is a covariant functor $F : \mathrm{Art}(k) \to \mathrm{Sets}$ such that $F(k)$ is a set with one element, and for any fibered product diagram in $\mathrm{Art}(k)$,

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

with $B \to A$ surjective (resp., $A = k$), the induced map

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is surjective (resp., an isomorphism).

Given a curve $(C, p_1, \ldots, p_n, v_1, \ldots, v_n)$ with smooth distinct marked points and the nonzero tangent vectors at them, we have the corresponding deformation functor

$$\mathrm{Def}(C, p_\bullet, v_\bullet) : \mathrm{Art}(k) \to \mathrm{Sets}$$

associating with R the set of isomorphism classes of flat proper families of curves $\pi_R : C_R \to \mathrm{Spec}(R)$ with sections $p_1^R, \ldots, p_n^R$, and trivializations of the relative tangent bundle along them, such that the induced data over $\mathrm{Spec}(k) \subset \mathrm{Spec}(R)$ is $(C, p_\bullet, v_\bullet)$.

On the other hand, for any finite-dimensional minimal A_∞-algebra E we have the deformation functor

$$\mathrm{Def}(E) : \mathrm{Art}(k) \to \mathrm{Sets}$$

of extended gauge equivalence classes of minimal A_∞-algebras E_R over R, reducing to E over k. Let also, for a fixed $n-g$-dimensional subspace $W \subset k^n$,

$$\widetilde{\mathrm{Def}}(E_W) : \mathrm{Art}(k) \to \mathrm{Sets}$$

be the functor associating with R the set of pairs $(W_R, m_\bullet)$, where W_R is an R-point of $G(n-g, n)$, reducing to W over k, and $m_\bullet$ is a minimal A_∞-structure on E_{W_R}, reducing to the trivial A_∞-structure on E_W, viewed up to a gauge equivalence reducing to the identity modulo the maximal ideal. Note that we have a natural forgetful morphism

$$\widetilde{\mathrm{Def}}(E_W) \to \mathrm{Def}(E_W).$$

Lemma 4.5.3. *(i) Let $\mathcal{M}_\infty$ be the functor of A_∞-moduli associated with the family $\mathcal{E}_W$ over the Grassmannian. For every $W \in G(n-g, n)(k)$, we have a natural identification of $\widetilde{\mathrm{Def}}(E_W)(R)$ with the fiber of $\mathcal{M}_\infty(R) \to \mathcal{M}_\infty(k)$ over the equivalence class of the trivial A_∞-structure on E_W.*
(ii) The functor $\widetilde{\mathrm{Def}}(E_W)$ is prorepresentable.

Proof. (i) First, we have to check that if a minimal R-linear A_∞-structure m on E_{W_R} reduces to an A_∞-structure on E_W that is gauge equivalent to the trivial one, then there exists a gauge transformation f over R such that $f * m$ reduces to the trivial A_∞-structure under $R \to k$. This immediately follows from the fact that we can lift any gauge transformation defined over k to a gauge transformation defined over R.

It remains to show that if we have minimal R-linear A_∞-structures m and m' on E_{W_R}, reducing to the trivial one on E_W, and a gauge equivalence f such that $f * m = m'$ then

there exists a gauge equivalence f' reducing to the identity on E_W and such that we still have $f' * m = m'$. Let $\overline{f}$, $\overline{m}$, etc., denote the reduction with respect to $R \to k$. Thus, $\overline{m} = \overline{m'}$ is the trivial A_∞-structure on E_W. Since $HH^1(E_W)_{<0} = 0$ (see Corollary 4.4.4), by Lemma 3.3.5, there exists a homotopy $\overline{h} = (\overline{h}_n)$ over k from the identity to $\overline{f}$. We can lift $\overline{h}$ to a homotopy h over R from the identity transformation of m' to some gauge transformation f_1 with $f_1 * m' = m'$ and $\overline{f_1} = \overline{f}$. Then setting $f' = f_1^{-1} \circ f$ gives the gauge transformation with the required properties.
(ii) Since $\mathcal{M}_\infty$ is representable by a scheme, part (i) implies that the functor $\widetilde{\mathrm{Def}}(E_W)$ is prorepresentable by the completion of the algebra of functions on $\mathcal{M}_\infty$ at the k-point corresponding to the trivial A_∞-structure on E_W. □

For each special curve $C_W = (C_W, p_\bullet, v_\bullet) \in \widetilde{\mathcal{U}}^{ns}_{g,n}$ corresponding to a subspace $W \in G(n-g, n)(k)$, where k is a field, the morphism (4.3.1) induces a morphism of deformation functors

$$\mathrm{Def}(C_W) \to \widetilde{\mathrm{Def}}(E_W). \tag{4.5.1}$$

We stress that here $\mathrm{Def}(C_W)$ denotes the functor of deformations of not just a curve but a curve with marked points and tangent vectors at them. The key step in the proof of Theorem 4.5.1 is that under some assumptions on the characteristic of k, the morphism (4.5.1) is an isomorphism.

Proposition 4.5.4. *Assume that either $n \geq 2$ and $g = 0$, or $n \geq 2$ and the characteristic of k is $\neq 2$, or $n = g = 1$ and the characteristic of k is $\neq 2, 3$. Then the morphism* (4.5.1) *is an isomorphism.*

Recall that the *tangent space* to a functor $F : \mathrm{Art}(k) \to \mathrm{Sets}$ is $t_F := F(k[\epsilon]/(\epsilon^2))$. A morphism of deformation functors $F \to G$ is called *smooth* if it satisfies the following lifting property: for every surjection $B \to A$ in $\mathrm{Art}(k)$, the induced map

$$F(B) \to G(B) \times_{G(A)} F(A)$$

is surjective. A morphism $F \to G$ is called *étale* if it is smooth and induces an isomorphism $t_F \xrightarrow{\sim} t_G$.

The main idea of the proof of Proposition 4.5.4 is that it is enough to check that the morphism (4.5.1) is étale. Indeed, this is a consequence of the following general result.

Lemma 4.5.5. *(cf.* [24, Cor. 2.11]*) Let $\phi : F \to G$ be an étale morphism of deformation functors, where G is prorepresentable. Then ϕ is an isomorphism.*

Proof. First, since ϕ is smooth, applying the lifting property to the surjection $A \to k$, we see that $F(A) \to G(A)$ is surjective for every A. Secondly, we claim that the fact that the induced map $t_F \to t_G$ is injective, together with prorepresentability of G, imply that $F(A) \to G(A)$ is injective. Indeed, it is enough to prove that if we have a small extension

$$0 \to M \to B \to A \to 0$$

and $F(A) \to G(A)$ is injective then $F(B) \to G(B)$ is also injective. To this end we observe that there is an isomorphism of rings

$$B \times_A B \simeq B \times_k (k \oplus M) : (b, b') \mapsto (b, (\overline{b}, b' - b)),$$

where $k \oplus M$ is the trivial small extension of k, and we denote by $\bar{b}$ the reduction of $b \in B$ modulo the maximal ideal. Therefore, we get a canonical morphism

$$\eta_F : F(B) \times F(k \oplus M) \simeq F(B \times_k (k \oplus M)) \simeq F(B \times_A B) \to F(B) \times_{F(A)} F(B).$$

Using this map (which is surjective by the definition of a deformation functor), we can construct a transitive action of $F(k \oplus M) \simeq t_F \otimes_k M$ on every fiber of the map $F(B) \to F(A)$. Namely, given $\xi \in F(B)$ and $x \in F(k \oplus M)$ we define $x * \xi \in F(B)$, in the fiber containing ξ, so that

$$\eta_F(\xi, x) = (\xi, x * \xi).$$

These actions for F and G are compatible but for G we also know that η_G is an isomorphism since G is prorepresentable. This easily implies the required injectivity of $F(B) \to G(B)$. □

Thus, we will need to study the tangent spaces to our deformation functors and also the smoothness of the map between them. For studying smoothness the following notion is extremely useful. A *complete obstruction theory* with values in a k-vector space V for such a functor is the data, for every small extension e,

$$0 \to M \to B \to A \to 0 \tag{4.5.2}$$

in Art_k (this means that M is an ideal in B annihilated by the maximal ideal of B), of a map $v_e : F(A) \to V \otimes_k M$ such that and element $\xi \in F(A)$ lifts to $F(B)$ if and only if $v_e(\xi) = 0$. In addition, we require the obstruction map to be compatible with morphisms of small extensions.

We will use the following standard smootheness criterion for a morphism $\phi : F \to G$ of deformation functors: If ϕ extends to a compatible morphism of obstruction theories $V_F \to V_G$, which is injective, while the induced map of tangent spaces $t_F \to t_G$ is surjective, then ϕ is smooth (see [24, Prop. 2.17]). The proof is an easy exercise using the action of $t_F \otimes_k M$ on the fibers $F(B) \to F(A)$ for a small extension (4.5.2), as in the proof of Lemma 4.5.5.

We will also need the following standard result (it is proved in [24, Prop. 2.18] using universal obstruction theories).

Lemma 4.5.6. *Let $F \to G \to H$ be morphisms of deformation functors, such that $F \to H$ is smooth and the induced map on tangent spaces $t_F \to t_G$ is surjective. Then $F \to G$ is smooth.*

Proof. Given a small extension (4.5.2), and an element $(\xi_F, \eta_G) \in F(A) \times_{G(A)} G(B)$ we want to lift it to an element $\eta_F \in F(B)$. Let $\eta_H \in H(B)$ be the image of η_G. By smoothness of $F \to H$ we can lift the element $(\xi_F, \eta_H) \in F(A) \times_{H(A)} H(B)$ to an element $\widetilde{\eta}_F \in F(B)$. The problem is that the image of $\widetilde{\eta}_F$ in $G(B)$ differs from η_G. However, it lies in the same fiber of the map to $G(A)$, so the image of $\widetilde{\eta}_F$ in $G(B)$ differs from η_G by an action of an element in $t_G \otimes_k M$. Thus, using the surjectivity of the map $t_F \to t_G$, we can correct $\widetilde{\eta}_F$ by an action of an element in $t_G \otimes_k M$, so that the resulting element η_F projects to η_G (without changing its image in $F(A)$). □

We have the following standard obstruction theory for deformations of A_∞-structures. For an A_∞-algebra E, for every integer n, the Hochschild cochains of internal degree $\leq n$

form a subcomplex $CH^\bullet(E)_{\le n}$ in $CH^\bullet(E)$. We denote by $HH^*(E)_{\le n}$ its cohomology (note that the map $HH^*(E)_{\le n} \to HH^*(E)$ is not necessarily injective).

Lemma 4.5.7. *(i) Let E_k be a minimal A_∞-algebra over k. Let us consider the functor on* Art_k *associating with $R \in \mathrm{Art}_k$ the set of deformations of E_k to a minimal A_∞-algebra structure on a R-algebra E_R (where E_R is flat over R) up to extended gauge transformations (see Definition 1.1.4). Note that here we allow to deform m_2 as well. Then the tangent space to this deformation functor is naturally identified with $HH^2(E_k)_{\le 0}$. Furthermore, there is a complete obstruction theory for this functor with values in $HH^3(E_k)_{\le 0}$. Similar statements hold for deformations of a small minimal A_∞-category.*
(ii) The tangent space to the functor $\widetilde{\mathrm{Def}}(E_W)$ can be identified with

$$HH^2(E_W)_{<0} \oplus T_W G(n-g,n).$$

There is a complete obstruction theory for this functor with values in $HH^3(E_W)_{<0}$.

Proof. (i) Any A_∞-structure of $E \otimes k[\epsilon]/(\epsilon^2)$ extending $m = (m_n)$ has form $m + \epsilon c$, where $c = (c_2, c_3, \ldots) \in CH^2(E)_{\le 0}$ satisfies $[m, c] = 0$. It is easy to see that the extended gauge transformations amount to changing c by a Hochschild coboundary.

Given a small extension

$$0 \to M \to B \to A \to 0$$

in Art_k, and a minimal A_∞-algebra E_A, deforming E_k, we can lift each m_i to some Hochschild cochain $\widetilde{m}_i \in CH^2(E_B)_{2-i}$ (such a lifting exists since E_B is free as a B-module, as any flat module over an Artinian local ring is free, see [50, 051E]). Let D be the coderivation of $\mathrm{Bar}(E_B)$ associated with $\widetilde{m}$. The A_∞-equations hold modulo M, hence

$$D^2 = D_\phi$$

for some $\phi \in M \otimes_B CH^3(E_B)_{\le 0} = M \otimes_k CH^3(E_k)_{\le 0}$. We have

$$[D, D^2] = [D, D_\phi] = 0,$$

so ϕ is a Hochschild cocycle. If we choose different liftings of m_i then D would change to $D + D'$ where D' takes values in $M \otimes_k \mathrm{Bar}(E_k)$. Then

$$(D + D')^2 = D_\phi + [D, D'] + (D')^2 = D_\phi + [D, D']$$

since $M^2 = 0$, so ϕ would change by a Hochschild coboundary. Thus, the class of ϕ in $M \otimes HH^3(E_k)_{\le 0}$ is well defined. Conversely, if this class is zero then we can correct our choice of $\widetilde{m}$ to make $\phi = 0$, so that $\widetilde{m}$ defines an A_∞-structure. Thus, ϕ is a complete obstruction for the functor of A_∞-structures. Since any extended gauge transformation over A lifts to the one over B, the same obstruction works for the extended gauge equivalence classes of A_∞-structures.
(ii) The tangent space classifies pairs $(f, m_\bullet)$, where $f : \mathrm{Spec}(k[t]/(t^2)) \to G(n-g,n)$ is a morphism sending the closed point to W, and $m_\bullet$ is a minimal A_∞-structure on $f^*\mathcal{E}$, extending the given m_2, reducing to the trivial one modulo (t), up to a gauge equivalence. Then f corresponds to a tangent vector in $T_W G(n-g,n)$, while the class of $(m_3, m_4, \ldots)$ is an element in $HH^3(E_W)_{<0}$. The obstruction theory is obtained from the usual obstruction theory for A_∞-structures in part (i), using the fact that $G(n-g,n)$ is smooth. □

Remark 4.5.8. In general the spaces $HH^i(E_W)_{<0}$ are given by the products of the components $HH^i(E_W)_j$ for $j < 0$. However, in our case the spaces $HH^i(E_W)$ are finite-dimensional, by Lemma 4.4.1, so for each i there is only finitely many j with $HH^i(E_W)_j \neq 0$.

For a scheme S over a field k we denote by L_S the cotangent complex of S over k. Recall that in the case when S is smooth, this is just the sheaf of Kähler differentials Ω_S. In general, it should be viewed as a nonadditive derived functor of Ω. For example, in the affine case $S = \mathrm{Spec}(A)$ it can be computed by taking a simplicial resolution by free commutative algebras $P_\bullet \to A$ and setting $L_A = \Omega_{P_\bullet} \otimes_{P_\bullet} A$ (see [50, 08P5]).

It is well known that the deformation theory of S is governed by $\mathrm{Ext}^1(L_S, \mathcal{O}_S)$, which is the tangent space to deformations, and $\mathrm{Ext}^2(L, \mathcal{O}_S)$ which is where the obstructions take values. Thus, we need to understand these spaces for our curves $C = C_W$, or rather the corresponding map to the same spaces for deformations of A_∞-structures. One of tricks will be to reduce to considering the affine curves $U = C \setminus D$. The point is that in this case there are no higher products, so we just consider the map from deformations of $\mathcal{O}(U)$ as a commutative algebra to its deformations as an associative algebra. One has the following result about the induced map on the tangent spaces and obstuction theories.

Lemma 4.5.9. *Let C be a projective connected reduced curve over a field k, $D = p_1 + \ldots + p_n \subset C$ a finite subset of smooth points, such that $U = C \setminus D$ is affine. Then the natural map*

$$\mathrm{Ext}^i(L_U, \mathcal{O}_U) \to HH^{i+1}(U) \tag{4.5.3}$$

is an isomorphism for $i = 1$, and is an injection for $i = 2$.

Proof. By [41, Thm. 8.1], there is a spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p(\bigwedge\nolimits^q L_U, \mathcal{O}_U) \implies HH^{p+q}(U)$$

where $\bigwedge^\bullet(?)$ denotes the exterior power functor on bounded above complexes. Since $L_U \in D^{\le 0}$, it follows that $\bigwedge^i L_U \in D^{\le 0}$, so $E_2^{pq} \neq 0$ only for $p \geq 0$ (and $q \geq 0$). Since U is affine, we also have $E_2^{p0} = 0$ for $p > 0$. We claim also that $E_2^{0q} = 0$ for $q > 1$. Indeed, we have

$$\mathrm{Hom}(\bigwedge\nolimits^q L_U, \mathcal{O}_U) = \mathrm{Hom}\big(\underline{H}^0(\bigwedge\nolimits^q L_U), \mathcal{O}_U\big).$$

Note that the coherent sheaf $\underline{H}^0(\bigwedge^q L_C)$ on C is supported on the singular locus of C (since $q > 1$), which is contained in U. Therefore, we have

$$\mathrm{Hom}\big(\underline{H}^0(\bigwedge\nolimits^q L_U), \mathcal{O}_U\big) \simeq \mathrm{Hom}\big(\underline{H}^0(\bigwedge\nolimits^q L_C), \mathcal{O}_C\big).$$

But $\mathcal{O}_C$ cannot have subsheaves with finite support since all global functions on C are constant, so the above space is zero, and our claim follows.

Thus, the spectral sequence implies that the map (4.5.3) is an isomorphism for $i = 1$, while for $i = 2$ it fits into an exact sequence

$$0 \to \mathrm{Ext}^2(L_U, \mathcal{O}_U) \to HH^3(U) \to \mathrm{Ext}^1(\bigwedge\nolimits^2 L_U, \mathcal{O}_U) \to 0.$$

□

The following lemma shows that we do not loose any information on the tangent spaces and obstruction spaces by passing from the projective curves $C = C_W$ to the affine curves $U = C \setminus D$.

Lemma 4.5.10. *Assume that either* $W = k^n$ *and* $n \geq 2$*, or* k *has characteristic* $\neq 2$ *(resp.,* $\neq 2, 3$ *if* $n = 1$*). Let* $C = C_W$ *be a special curve over* k*, where* $W = 0$ *if* $n = 1$*, and let* $D = p_1 + \ldots + p_n$*,* $U = C \setminus D$*. Then the natural morphism*

$$\mathrm{Ext}^1(L_C, \mathcal{O}_C(-2D)) \to \mathrm{Ext}^1(L_C, \mathcal{O}_C(-D)) \tag{4.5.4}$$

is surjective, while the natural morphism

$$\mathrm{Ext}^1(L_C, \mathcal{O}_C(-D)) \to \mathrm{Ext}^1(L_U, \mathcal{O}_U)$$

is an isomorphism. The natural morphism

$$\mathrm{Ext}^2(L_C, \mathcal{O}_C(-2D)) \to \mathrm{Ext}^2(L_U, \mathcal{O}_U)$$

is an isomorphism.

Proof. (i) First, we observe that since L_C is a locally free sheaf on the smooth part of C, it follows that

$$\mathrm{Ext}^{>0}(L_C, \mathcal{O}_D) = 0.$$

Thus, applying the functor $\mathrm{Ext}^\bullet(L_C, \cdot)$ to the exact sequences

$$0 \to \mathcal{O}_C(nD) \to \mathcal{O}_C\big((n+1)D\big) \to \mathcal{O}_D((n+1)D) \to 0$$

we get that the natural maps

$$\mathrm{Ext}^i\big(L_C, \mathcal{O}_C(nD)\big) \to \mathrm{Ext}^i\big(L_C, \mathcal{O}_C((n+1)D)\big)$$

are isomorphisms for $i = 2$ and are surjective for $i = 1$. Furthermore, since $L_C \in D^{\leq 0}(C)$ and $\underline{H}^0(L_C) = \Omega_C$, the map

$$\mathrm{Hom}(L_C, \mathcal{O}_C(nD)) \to \mathrm{Hom}(L_C, \mathcal{O}_D(nD)) \tag{4.5.5}$$

can be identified with the map

$$H^0(C, \mathcal{T}(nD)) \to H^0(D, \mathcal{T}(nD)|_D)$$

which is surjective for $n \geq 0$ by Lemma 4.4.3(iii). This implies that the map $\mathrm{Ext}^1\big(L_C, \mathcal{O}_C(nD)\big) \to \mathrm{Ext}^1(L_C, \mathcal{O}_C((n+1)D))$ is an isomorphism for $n \geq -1$.

It remains to prove that the natural map

$$\varinjlim_n \mathrm{Ext}^i\big(L_C, \mathcal{O}_C(nD)\big) \to \mathrm{Ext}^i_C(L_C, \mathcal{O}_U) \simeq \mathrm{Ext}^i_U(L_U, \mathcal{O}_U)$$

is an isomorphism for any i. This would be immediate if L_C were a perfect complex but we do not know this.[2] We will show that it is enough to use the fact L_C is perfect in some open neighborhood U' of D. Since C is smooth near D, we can just take as U' the smooth locus in C. Now C is covered by U and U'. Now the long exact sequences associated with Cech resolutions

$$0 \to \mathcal{O}_C(nD) \to \mathcal{O}_U \oplus \mathcal{O}_{U'}(nD) \to \mathcal{O}_{U \cap U'} \to 0$$

---

[2] Here we correct a mistake in the proof of [35, Lem. 4.4.5](ii).

show that it is enough to see that a similar assertion is true with $\mathcal{O}_C(nD)$ replaced by $\mathcal{O}_{U'}(nD)$. But in this case it reduces to the fact that the map

$$\varinjlim_n \mathrm{Ext}^i\big(L_C, \mathcal{O}_{U'}(nD)\big) \to \mathrm{Ext}^i(L_C, \mathcal{O}_{U\cap U'})$$

is an isomorphism for any i, which is true since $L_C|_{U'}$ is perfect. □

We also need similar assertions for Hochschild cohomology.

Lemma 4.5.11. *(i) Let C be a reduced projective curve over a field k, $U \subset C$ a complement to a finite number of smooth points. Then the natural morphism*

$$HH^i(C) \to HH^i(U)$$

is an isomorphism for $i \geq 3$.
(ii) Now let $C = C_W$ and $U = C \setminus D$. Then under the assumptions of Lemma 4.5.10, the map

$$HH^2(C) \to HH^2(U)$$

is an isomorphism

Proof. (i) Note that for $i \geq 2$, the sheaf $\underline{\mathcal{H}}^i$ has finite support (since it is zero on the smooth part of C), so it does not have higher cohomology. Thus, for $i \geq 3$, the horizontal arrows in the commutative diagram

$$\begin{tikzcd} HH^i(C) \arrow[r] \arrow[d] & H^0(C, \underline{\mathcal{H}}^i) \arrow[d] \\ HH^i(U) \arrow[r] & H^0(U, \underline{\mathcal{H}}^i) \end{tikzcd}$$

are isomorphisms. Since $\underline{\mathcal{H}}^i$ has finite support in U, the right vertical arrow is an isomorphism, hence, the left vertical arrow is also an isomorphism.
(ii) We have $H^1(U, \underline{\mathcal{H}}^1) = 0$ since U is affine. On the other hand, $H^1(C, \underline{\mathcal{H}}^1) = H^1(C, \mathcal{T}) = 0$ for $C = C_W$, by Lemma 4.4.3(ii). Thus, the argument of part (i) still applies for $i = 2$ and $C = C_W$. □

Proof of Proposition 4.5.4 . Let U_W be the affine curve $C_W \setminus D$, where $D = p_1 + \ldots + p_n$. Let also $\mathcal{C}$ be the (non-full) subcategory in the A_∞-enhancement of the derived category of $\mathrm{Qcoh}(C_W)$ with the objects $(G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_n}, \mathcal{O}_{U_W})$, all endomorphisms (including Ext^*) of G, and all morphisms from G to $\mathcal{O}_{U_W}$ and all endomorphisms of $\mathcal{O}_{U_W}$ (but we do not include morphisms from $\mathcal{O}_{U_W}$ to G). We have the following commutative diagram of functors

(4.5.6)
$$\begin{tikzcd} \mathrm{Def}(U_W) \arrow[d] & \mathrm{Def}(C_W) \arrow[l] \arrow[r] \arrow[d] & \widetilde{\mathrm{Def}}(E_W) \arrow[d] \\ \mathrm{Def}(\mathcal{O}(U_W)) & \mathrm{Def}(\mathcal{C}) \arrow[l] \arrow[r] & \mathrm{Def}(E_W) \end{tikzcd}$$

where in the lower we consider deformations of A_∞-algebras (A_∞-category in the case of $\mathcal{C}$). Note that since $\mathcal{O}(U_W)$ lives in degree 0, this means that $\mathrm{Def}(\mathcal{O}(U_W))$ is the functor of deformations of $\mathcal{O}(U_W)$ as an associative algebra. On the other hand, $\mathrm{Def}(U_W)$ can be thought of as deformations of $\mathcal{O}(U_W)$ as a commutative algebra.

It is easy to see that each functor in this diagram is a deformation functor, as it describes deformations of some algebraic structure (to check the first condition in the definition one can use the appropriate obstruction theory).

As was mentioned earlier, the key statement need to prove is that the morphism (4.5.1) is étale. To this end we will establish the following properties of the functors in our diagram 4.5.6:

$$\begin{tikzcd}
\mathrm{Def}(U_W) \arrow[d, "\text{ét (Step 2)}"'] & \mathrm{Def}(C_W) \arrow[l, "\text{sm (Step 2)}"'] \arrow[r, "\text{isom on } t\text{(Step 4)}"] \arrow[d, "\text{sur on } t\text{(Step 3)}"'] & \widetilde{\mathrm{Def}}(E_W) \arrow[d] \\
\mathrm{Def}(\mathcal{O}(U_W)) & \mathrm{Def}(\mathcal{C}) \arrow[l] \arrow[r, "\text{ét (Step 1)}"] & \mathrm{Def}(E_W)
\end{tikzcd}$$

where “sm” (resp., “ét”) means “smooth” (resp., “étale”); “isom on t” (resp., “sur on t”) means isomorphism (resp., surjection) on tangent spaces. Then the fact that (4.5.1) is étale will follow formally from Lemma (4.5.6) (see Step 5 below).

Step 1. The map $\mathrm{Def}(\mathcal{C}) \to \mathrm{Def}(E_W)$ is étale. To prove that this morphism is étale it is enough to check that it induces an isomorphism on tangent spaces and an embedding on obstruction spaces. By Lemma 4.5.7, these spaces are given by $HH^2_{\leq 0}$ and $HH^3_{\leq 0}$ (applied to $\mathcal{C}$ and E_W), respectively. Note that our morphism corresponds to the embedding of the full subcategory on the object G into $\mathcal{C}$. Since $\mathcal{O}(U_W)$ is the algebra of endomorphisms of the A_∞-module $\mathrm{Hom}(G, \mathcal{O}_{U_W})$, it follows that this embedding induces an isomorphism

$$HH^*(\mathcal{C}) \xrightarrow{\sim} HH^*(E_W)$$

(see [15], [23, Thm. 4.1.1]).

Hence, it is enough to check that the vertical arrows in the commutative diagram

$$\begin{tikzcd}
HH^i(\mathcal{C})_{\leq 0} \arrow[r] \arrow[d] & HH^i(E_W)_{\leq 0} \arrow[d] \\
HH^i(\mathcal{C}) \arrow[r] & HH^i(E_W)
\end{tikzcd}$$

are isomorphisms for $i = 2$ and that the map $HH^3(\mathcal{C})_{\leq 0} \to HH^3(\mathcal{C})$ is an embedding.

We have an exact sequence

$$\begin{aligned}(4.5.7)\qquad & HH^1(\mathcal{C}) \to H^1\big(CH(\mathcal{C})_{\geq 1}\big) \to HH^2(\mathcal{C})_{\leq 0} \to HH^2(\mathcal{C}) \to H^2\big(CH(\mathcal{C})_{\geq 1}\big) \to \\ & HH^3(\mathcal{C})_{\leq 0} \to HH^3(\mathcal{C}),\end{aligned}$$

where $CH(?)_{\geq i} := CH(?)/CH(?)_{\leq i-1}$. Since E_W has trivial higher products, we have a canonical decomposition $HH^i(E_W) = \prod_j HH^i(E_W)_j$, so the similar exact sequence for E_W has trivial connecting homomorphisms. Thus, it is enough to check that

$$H^2\big(CH(\mathcal{C})_{\geq 1}\big) = H^2\big(CH(E_W)_{\geq 1}\big) = 0 \tag{4.5.8}$$

and that the map $HH^1(\mathcal{C}) \to H^1\big(CH(\mathcal{C})_{\geq 1}\big)$ is surjective.

Since $\mathcal{C}$ has morphisms only of degree 0 and 1, the only possible cochains in $CH^{s+t}(\mathcal{C})_t$ with $s + t = 2$ and $t \geq 1$ correspond to $(s, t) = (1, 1)$. But such cochains should have form $\mathrm{Hom}^0(X, Y) \to \mathrm{Hom}^1(X, Y)$, so they all vanish (since we exclude the identities in the Hochschild complex). The same argument works for E_W, so we get the vanishings (4.5.8).

Similar considerations with cochains show that

$$H^1\big(CH(\mathcal{C})_{\geq 1}\big) = CH^1(\mathcal{C})_1 = \mathrm{Ext}^1(\mathcal{O},\mathcal{O}) \oplus \bigoplus_{i=1}^{g} \mathrm{Ext}^1(\mathcal{O}_{p_i},\mathcal{O}_{p_i}) \oplus \bigoplus_{i=1}^{g} \mathrm{Ext}^1(\mathcal{O}_{p_i},\mathcal{O}),$$

and the same formula holds for E_W. Thus, in the commutative square

$$\begin{tikzcd} HH^1(\mathcal{C}) \arrow[r] \arrow[d] & H^1\big(CH(\mathcal{C})_{\geq 1}\big) \arrow[d] \\ HH^1(E_W) \arrow[r] & H^1\big(CH(E_W)_{\geq 1}\big) \end{tikzcd}$$

both vertical arrows are isomorphisms. Since the bottom horizontal arrow is surjective, the top horizontal arrow is surjective too, as required.

Step 2. The map $\mathrm{Def}(C_W) \to \mathrm{Def}(U_W)$ is smooth, while the map $\mathrm{Def}(U_W) \to \mathrm{Def}(\mathcal{O}(U_W))$ is étale.

First, we observe that the maps on tangent spaces induced by these maps are

$$\mathrm{Ext}^1(L_{C_W},\mathcal{O}(-2D)) \to \mathrm{Ext}^1(L_{U_W},\mathcal{O}_{U_W}) \to HH^2(U_W),$$

the first of which is surjective by Lemma 4.5.10, while the second is an isomorphism by Lemma 4.5.9. Similarly the maps of obstruction spaces are

$$\mathrm{Ext}^2(L_{C_W},\mathcal{O}(-2D)) \to \mathrm{Ext}^2(L_{U_W},\mathcal{O}_{U_W}) \to HH^3(U_W),$$

of which the first is an isomorphism by Lemma 4.5.10, while the second is injective by Lemma 4.5.9. Hence, the maps $\mathrm{Def}(C_W) \to \mathrm{Def}(U_W)$ and $\mathrm{Def}(U_W) \to \mathrm{Def}(\mathcal{O}(U_W))$ are smooth and the second is étale.

Step 3. The map $\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{C})$ induces a surjection on tangent spaces.

Indeed, Step 2, together with the commutativity of diagram (4.5.6), implies that $\mathrm{Def}(\mathcal{C}) \to \mathrm{Def}(\mathcal{O}(U_W))$ induces a surjection on tangent spaces. But

$$HH^2(U_W) \simeq HH^2(C_W) \simeq HH^2(E_W)$$

by Lemmas 4.5.11 and 4.4.1, so the dimensions of tangent spaces are the same. Hence, $\mathrm{Def}(\mathcal{C}) \to \mathrm{Def}(\mathcal{O}(U_W))$ induces an isomorphism on tangent spaces.

It follows that the maps induced on tangent spaces by $\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{C})$ and by $\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{O}(U_W))$ are isomorphic, so the required surjectivity follows from Step 2.

Step 4. The map $\mathrm{Def}(C_W) \to \widetilde{\mathrm{Def}}(E_W)$ induces an isomorphism on tangent spaces.

Note that by Steps 1 and 3, we know that the map $\mathrm{Def}(C_W) \to \mathrm{Def}(E_W)$ induces a surjection on tangent spaces. Hence, the same is true for $\widetilde{\mathrm{Def}}(E_W) \to \mathrm{Def}(E_W)$. We claim that there is a commutative diagram with exact rows

(4.5.9)
$$\begin{tikzcd} k^n \arrow[r, "\alpha"] \arrow[d, "="'] & \mathrm{Ext}^1(L_{C_W},\mathcal{O}(-2D)) \arrow[r, "\beta"] \arrow[d, "\gamma"'] & HH^2(\mathcal{C}) \arrow[r] \arrow[d, "\gamma'"'] & 0 \\ k^n \arrow[r, "\alpha'"] & HH^2(E_W)_{<0} \oplus T_W G(n-g,n) \arrow[r, "\beta'"] & HH^2(E_W)_{\leq 0} \arrow[r] & 0 \end{tikzcd}$$

where the arrow α (resp., α') is induced by the $\mathbb{G}_m^n$-action on the functor $\mathrm{Def}(C_W)$ (resp., $\widetilde{\mathrm{Def}}(E_W)$), while the right commutative square is induced by the right commutative square in (4.5.6) (flipped about the diagonal). Note that we already know that γ' is an isomorphism and β' is surjective. To see the exactness of the top row we observe that by Steps 1 and 2, the map β can be identified with the morphism

$$\mathrm{Ext}^1(L_{C_W}, \mathcal{O}(-2D)) \to \mathrm{Ext}^1(L_{C_W}, \mathcal{O}(-D)) \xrightarrow{\sim} \mathrm{Ext}^1(L_{U_W}, \mathcal{O}_{U_W}),$$

where the second arrow is an isomorphism by Lemma 4.5.10. Hence, its kernel is the image of the coboundary map $H^0(C_W, \mathcal{T}(-D)|_D) \to \mathrm{Ext}^1(L_{C_W}, \mathcal{O}(-2D))$, which can be identified with α. The exactness of the bottom row in (4.5.9) would follow from the exactness in the middle of the sequence

$$k^n \to T_W G(n-g,n) \to HH^2(E_W)_0 \to 0,$$

where the second arrow is the tangent map to the map $W \to E_W$, and the first arrow corresponds to the $\mathbb{G}_m^n$-action on $G(n-g,n)$. But this follows from the observation that a $k[t]/(t^2)$-point of $G(n-g,n)$, $\mathcal{W}$, can be recovered from the isomorphism class of the corresponding algebra $E_{\mathcal{W}}$ up to a $\mathbb{G}_m^n$-action.

Note that diagram (4.5.9), together with the fact that γ' is an isomorphism, immediately implies that γ is surjective. It remains to prove that the restriction of γ to $\mathrm{im}(\alpha)$ is injective. To this end we use the fact that each point $C_W \in \widetilde{\mathcal{U}}_{g,n}^{ns}$ lies in the section $\sigma(G(n-g,n))$ of the projection to $G(n-g,n)$, and that the $\mathbb{G}_m^n$-orbit of C_W still lies in $\sigma(G(n-g,n))$. Hence, the tangent space to this orbit maps injectively to $T_W G(n-g,n)$, which implies our assertion.

Step 5. The composition

$$\mathrm{Def}(C_W) \to \mathrm{Def}(U_W) \to \mathrm{Def}(\mathcal{O}(U_W)),$$

is smooth since both arrows are smooth by Step 2. Hence, applying Lemma 4.5.6 to the composition

$$\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{C}) \to \mathrm{Def}(\mathcal{O}(U_W))$$

and using Step 3, we deduce that the morphism $\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{C})$ is smooth.

Next, we deduce that the composition

$$\mathrm{Def}(C_W) \to \mathrm{Def}(\mathcal{C}) \to \mathrm{Def}(E_W)$$

is smooth since the second arrow is smooth by Step 1. Hence, applying Lemma 4.5.6 to the composition

$$\mathrm{Def}(C_W) \to \widetilde{\mathrm{Def}}(E_W) \to \mathrm{Def}(E_W)$$

and using Step 4, we deduce that $\mathrm{Def}(C_W) \to \widetilde{\mathrm{Def}}(E_W)$ is smooth. Since it induces an isomorphism on tangent spaces, we get that it is étale. But the functor $\widetilde{\mathrm{Def}}(E_W)$ is prorepresentable (see Lemma 4.5.3), hence, by Lemma 4.5.5, the morphism $\mathrm{Def}(C_W) \to \widetilde{\mathrm{Def}}(E_W)$ is an isomorphism. □

Proof of Theorem 4.5.1. We know that both schemes are affine of finite type over $G(n-g,n)$ (by Theorem 4.1.2 and Proposition 4.4.5), and that the morphism (4.3.1) is compatible with $\mathbb{G}_m$-action. Furthermore, the $\mathbb{G}_m$-invariant loci of each scheme provide a section of the projection to $G(n-g,n)$, and the weights of $\mathbb{G}_m$ are non-negative on the spaces of functions (locally over $G(n-g,n)$). Thus, locally over $G(n-g,n)$ our morphism corresponds to a homomorphism $f : A \to B$ of non-negatively graded algebras such that $f_0 : A_0 \to B_0$ is an isomorphism. Furthermore, by Proposition 4.5.4, for every point of $\mathrm{Spec}(A_0) \simeq \mathrm{Spec}(B_0)$, the map f induces an isomorphism of deformation functors. Hence, applying Lemma 4.5.12 below we deduce that f is an isomorphism. □

Lemma 4.5.12. *Let $f : A \to B$ be a morphism of degree zero of non-negatively graded algebras such that the induced map $A_0 \to B_0$ is an isomorphism. Assume that A_0 is Noetherian, A and B are finitely generated as algebras over $A_0 \simeq B_0$, and for every maximal ideal $\mathfrak{m} \subset A_0$ the map f induces an isomorphism $\hat{A} \to \hat{B}$ of the completions with respect to the maximal ideals $\mathfrak{m} + A_{>0}$ and $\mathfrak{m} + B_{>0}$, respectively. Then f is an isomorphism.*

Proof. It is enough to prove that f induces an isomorphism $A/A^N_{>0} \to B/B^N_{>0}$ for each $N > 0$. Note that $A/A^N_{>0}$ (resp., $B/B^N_{>0}$) is a finitely generated module over A_0 (resp., B_0). Note that for any maximal ideal $\mathfrak{m} \subset A_0 \simeq B_0$, the $(\mathfrak{m} + A_{>0})$-adic topology on $A/A^N_{>0}$ is equivalent to the $\mathfrak{m}$-adic topology, and similarly on $B/B^N_{>0}$. Thus, we have a morphism

$$A/A^N_{>0} \to B/B^N_{>0}$$

of finitely generated A_0-modules, inducing an isomorphism of $\mathfrak{m}$-adic completions of localizations at every maximal ideal $\mathfrak{m} \subset A_0$. Since A_0 is Noetherian, such a morphism is an isomorphism. □

4.6. HH^3 **as an invariant of a curve singularity.** The isomorphism of Theorem 4.5.1 suggests to look at the stratification of the moduli spaces $\widetilde{\mathcal{U}}^{ns}_{g,n}$ given by the ranks of the Hochschild cohomology groups of the corresponding A_∞-algebras. The natural question is whether these strata have some geometric interpretation. Here we show that HH^3 gives some interesting information about the singularities of the curve.

Note that when a curve $(C, p_\bullet, v_\bullet) \in \widetilde{\mathcal{U}}^{ns}_{g,n}$ corresponds to an A_∞-algebra structure m on E_W then there is an equivalence of categories

$$\mathrm{Perf}(C) \simeq \mathrm{Perf}(E_W, m),$$

where on the right we have the category of perfect A_∞-modules over the A_∞-algebra (E_W, m). Indeed, this follows from the fact that $G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_n}$ is a generator of $\mathrm{Perf}(C)$ (see Sec. 4.1). This implies that the Hochschild cohomology of these two categories are the same:

$$HH^*(C) \simeq HH^*(E_W, m).$$

Proposition 4.6.1. *(i) For a reduced connected projective curve C over an algebraically closed field k one has*

$$HH^3(C) \simeq \bigoplus_{q \in \mathrm{Sing}\, C} HH^3(\mathcal{O}_{C,q}).$$

Furthermore, for every singular point q, one has $HH^3(\mathcal{O}_{C,q}) \neq 0$. In particular, C is smooth if and only if $HH^3(C) = 0$.

(ii) For a reduced plane curve singularity $q \in C$ one has

$$\dim HH^3(\mathcal{O}_{C,q}) = \tau(q),$$

where $\tau(q)$ is the Tyurina number of the singularity.

Lemma 4.6.2. *(i) Let C be a reduced curve over a field k. Then for any closed point $q \in C$ one has $\mathrm{Ext}^1(\mathcal{O}_q, \mathcal{O}_C) \neq 0$.*
(ii) If C is a Gorenstein curve then for any torsion coherent sheaf T on C one has $\dim \mathrm{Ext}^1(T, \mathcal{O}_C) = \ell(T)$.

Proof. (i) This follows immediately from the fact that C is Cohen-Macaulay and from the cohomological characterization of depth (see [25, 15.D]). Here is a simple proof. First of all, we can replace $\mathcal{O}_C$ by the local ring $A = \mathcal{O}_{C,q}$, and $\mathcal{O}_q$ by its residue field A/M. The equality $\mathrm{Ext}^1(A/M, A) = 0$ would imply that $\mathrm{Ext}^1(T, A) = 0$ for any finitely generated A-module annihilated by some power of M. Now we can take a non-zero-divisor $x \in M$ and set $T = A/(x)$. Then we get $0 = \mathrm{Ext}^1(A/(x), A) \simeq A/(x)$, which is a contradiction.
(ii) This immediately follows from Serre duality,

$$\mathrm{Ext}^1(T, \mathcal{O}_C) \simeq H^0(C, T)^*.$$

□

Proof of Proposition 4.6.1. (i) Since for $i \geq 2$ the sheaves $\underline{HH}^i$ are supported on the singularities of C, the exact sequence (4.4.2) gives an isomorphism

$$HH^3(C) \simeq H^0(C, \underline{HH}^i) = \bigoplus_{q \in \mathrm{Sing}\, C} HH^3(\mathcal{O}_{C,q}).$$

It remains to prove that if U is a singular affine curve then $HH^3(U) \neq 0$. As in the proof of Lemma 4.5.9, we have a surjection

$$HH^3(U) \to \mathrm{Ext}^1(\bigwedge^2 L_U, \mathcal{O}_U), \tag{4.6.1}$$

so it is enough to prove that $\mathrm{Ext}^1(\bigwedge^2 L_U, \mathcal{O}_U) \neq 0$.

Note that $\bigwedge^2 L_U$ has coherent cohomology supported on the singular locus of U. The exact triangle

$$\tau_{\leq -1}(\bigwedge^2 L_U) \to \bigwedge^2 L_U \to \underline{H}^0(\bigwedge^2 L_U) \to \dots$$

leads to an exact sequence

$$0 = \mathrm{Hom}(\tau_{\leq -1}(\bigwedge^2 L_U), \mathcal{O}_U) \to \mathrm{Ext}^1(\underline{H}^0(\bigwedge^2 L_U), \mathcal{O}_U) \to \mathrm{Ext}^1(\bigwedge^2 L_U, \mathcal{O}_U) \to \dots \tag{4.6.2}$$

so it is enough to check that $\mathrm{Ext}^1(\underline{H}^0(\bigwedge^2 L_U), \mathcal{O}_U) \neq 0$.

Note that for any nonzero sheaf $\mathcal{F}$ supported at one point $q \in U$ (possibly singular), one has $\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_U) \neq 0$. Indeed, any surjection $\mathcal{F} \to \mathcal{O}_q$ induces an embedding $\mathrm{Ext}^1(\mathcal{O}_q, \mathcal{O}_U) \hookrightarrow \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_U)$ (since there are no morphisms from torsion sheaves to $\mathcal{O}_U$), so this follows from Lemma 4.6.2(i).

It remains to prove that $\underline{H}^0(\bigwedge^2 L_U)$ is nonzero near a singular point q. To this end we observe that L_U can be represented by a complex

$$\ldots \to P_1 \xrightarrow{d_1} P_0 \to 0$$

of vector bundles. Furthermore, since $\operatorname{coker}(d_1) \simeq \Omega_U$, and q is singular, we deduce that $\dim \operatorname{coker}(d_1(q)) \geq 2$. Now $\bigwedge^2 L_U$ is represented by the complex

$$\ldots \to P_1 \otimes P_0 \xrightarrow{d} \bigwedge^2 P_0,$$

where $d(p_1 \otimes p_0) = d_1(p_1) \wedge p_0$. Now let us consider the natural surjective map

$$\pi : \bigwedge^2 P_0|_q \to \bigwedge^2 \operatorname{coker}(d_1(q)) \neq 0.$$

We have $\pi \circ d(q) = 0$, so $d(q)$ is not surjective. Hence, $\underline{H}^0(\bigwedge^2 L_U)|_q \neq 0$, as required.
(ii) We follow the same steps as in (i). Note first that in this case $\operatorname{Ext}^2(L_U, \mathcal{O}_U) = 0$, as for any locally complete intersection, so the map (4.6.1) is an isomorphism (see the proof of Lemma 4.5.9). Next, we observe that in the exact sequence (4.6.2) the term

$$\operatorname{Ext}^1(\tau_{\leq -1}(\bigwedge^2 L_U), \mathcal{O}_U) \simeq \operatorname{Hom}(\underline{H}^{-1} \bigwedge^2 L_U, \mathcal{O}_U)$$

vanishes since $\underline{H}^{-1}\bigwedge^2 L_U$ is a torsion sheaf. Hence, we have

$$\operatorname{Ext}^1(\bigwedge^2 L_U, \mathcal{O}_U) \simeq \operatorname{Ext}^1(\underline{H}^0(\bigwedge^2 L_U), \mathcal{O}_U).$$

Finally, using that $\mathcal{O}_U$ is given by $f = 0$ in a smooth surface S, we get that

$$L_U \simeq [\mathcal{O}_U \xrightarrow{df} \Omega^1_S|_U], \text{ and hence,}$$

$$\bigwedge^2 L_U \simeq [\mathcal{O}_U \xrightarrow{df} \Omega^1_S|_U \xrightarrow{\wedge df} \Omega^2_S|_U].$$

It follows that $\underline{H}^0(\bigwedge^2 L_U$ is isomorphic to the quotient of $\mathcal{O}_U$ by the ideal generated by the partial derivatives of f. Hence, $\ell(\underline{H}^0(\bigwedge^2 L_U)$ is exactly the Tyurina number (for U containing only one singular point q). Now the result follows from Lemma 4.6.2(ii). ☐

Note that if we replace $HH^3(C)$ with the obstruction space to commutative deformations, $\operatorname{Ext}^2(L_C, \mathcal{O}_C)$, then it will not be true that the vanishing of this space is equivalent to smoothness. For example, $\operatorname{Ext}^2(L_C, \mathcal{O}_C) = 0$ for any curve which is a locally complete intersection.

It is interesting to compute $hh^3(q) := HH^3(\mathcal{O}_{C,q})$ for some reduced curve singularities.

Proposition 4.6.3. *(i) For the coordinate cross in 3-space, one has $hh^3(q) =$???.*
(ii) For the elliptic n-fold singularity, one has $hh^3(q) = n + 1$???

4.7. **More on m_3.** Let $\mathcal{M}_n(\mathcal{E})$ denote the moduli space of A'_n-structures for the family $\mathcal{E} = (E_W)$ over $G(n-g,n)$. We assume that the assumptions of Theorem 4.5.1 are satisfied, so all $\mathcal{M}_n(\mathcal{E})$ and $\mathcal{M}_\infty(\mathcal{E})$ are affine of finite type over $G(n-g,n)$. Note that $\mathcal{M}_3(\mathcal{E})$ is the total space of a coherent sheaf over $G(n-g,n)$ with the fibers $HH^2(E_W)_{-1}$. More precisely, this coherent sheaf is the kernel of a morphism of vector bundles over $G(n-g,n)$,

$$CH^2(\mathcal{E}/G(n-g,n))_{-1}/\operatorname{im}(\delta^1) \xrightarrow{\delta^2} CH^3(\mathcal{E}/G(n-g,n)).$$

We have a natural projection $\mathcal{M}_\infty(\mathcal{E}) \to \mathcal{M}_3(\mathcal{E})$. It is interesting to study it from the point of view of the moduli space of curves $\widetilde{\mathcal{U}}^{ns}_{g,n}$. For each subset $S \subset [1,n]$, $|S| = g$, let $\mathcal{M}_\infty(\mathcal{E}, S) \subset \mathcal{M}_\infty(\mathcal{E})$ denote the preimage of the open cell $U_S \subset G(n-g,n)$. Recall that we have a natural affine embedding of $\mathcal{M}_\infty(\mathcal{E}, S)$, obtained by considering a section of the gauge transformation group on the variety of all A_∞-structures, or equivalently, certain normal forms of A_∞-structures. Among these affine coordinates, those coming from m_3 are exactly the coordinates that have weight 1 with respect to the $\mathbb{G}_m$-action (recall that m_n has weight $n-2$). Thus, we have to look at the affine coordinates of weight 1 on $\widetilde{\mathcal{U}}^{ns}_{g,n}(S) := \pi^{-1}(U_S) \subset \widetilde{\mathcal{U}}^{ns}_{g,n}$.

In the case $n = g$, the only coordinates of weight 1 on $\widetilde{\mathcal{U}}^{ns}_{g,g}$ are the functions α_{ij} defined by the Laurent expansions

$$x_i \equiv \frac{\alpha_{ij}}{t_j} + \dots$$

at p_j, for $i \neq j$ (this is equivalent to the definition of α_{ij} as a coefficient of y_j in the expansion of $x_i x_j$ in the canonical basis; see (4.1.5)).

In the case $n > g$, we have in addition the functions $x_{S,j}(p_{j'})$, for $j, j' \notin S$, $j \neq j'$, as well as the coefficients of y_i in the expansion of $y_i x_{S,j}$ for $i \in S$, $j \notin S$ (which can be normalized to be zero for a fixed $i_0 \in S$).

We conjecture that a generic smooth curve can be recovered from these coordinates for sufficiently large g and n. Here is a more precise statement. Let $\widetilde{\mathcal{M}}_{g,n}$ denote the $\mathbb{G}_m^n$-torsor over $\mathcal{M}_{g,n}$ corresponding to choices of nonzero tangent vectors at the marked points.

Conjecture. *Let us work over an algebraically closed field of characteristic* 0. *The projection from $\widetilde{\mathcal{M}}_{g,n}$ to $\mathcal{M}_3(\mathcal{E})$ is birational onto its image whenever it is possible by dimension consideration.*

In the case $n = g$ the dimension of $\mathcal{M}_3(E) = HH^2(E)_{-1}$ is $g^2 - g$, where $\widetilde{\mathcal{M}}_{g,g}$ has dimension $5g-3$, so the dimension of $\mathcal{M}_3(E)$ is bigger when $g \geq 6$. It was proved in [8, Thm. 3.2.1] that indeed the conjecture holds in this case (using some computer calculations to establish the case $g = 6$). We will discuss this proof (with some modifications) in Sec. **??**.

Another case when the conjecture is known is $g = 1$ and $n > 1$. In fact, in this case the dimension of $\mathcal{M}_3(\mathcal{E})$ is bigger than that of $\widetilde{\mathcal{M}}_{1,n}$ starting with $n \geq 5$. We will discuss this case in Sec. 4.9 below.

It is instructive to calculate explicitly some of the functions of weight 1 on our moduli spaces in terms of the products m_3 on the algebras E_W. We consider two examples: the functions α_{ij}, where $i, j \in S$, and the functions $x_{S,j}(p_k) - x_{S,j}(p_l)$, where $j, k, l \notin S$ ($x_{S,j}$ is defined uniquely up to adding a constant, so these differences do not depend on any choices).

Recall that A_i, B_i denote generators of E_W. Let us also set $\psi_i = A_i B_i \in \mathrm{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$.

Proposition 4.7.1. *(i) The functions α_{ij} on $\widetilde{\mathcal{U}}^{ns}_{g,n}(S)$, where $i, j \in S$, $i \neq j$, are determined by the condition*

$$m_3(B_i, \psi_i, A_i) = -\sum_{j\in S} \alpha_{ij} B_j A_j \tag{4.7.1}$$

(here α_{ii} maybe nonzero but it does not have an invariant interpretation).
(ii) The functions $x_{S,j}(p_k)$ on $\widetilde{\mathcal{U}}^{ns}_{g,n}(S)$, for $j, k \notin S$, $j \neq k$, satisfy

$$x_{S,j}(p_k) - x_{S,j}(p_l) = m_3(A_k, B_j, A_j) - m_3(A_l, B_j, A_j) - \sum_{i\in S} a_{S,ij}(m_3(A_k, B_i, A_i) - m_3(A_l, B_i, A_i)),$$

where $a_{S,ij}$ are defined by (4.1.6).

Proof. (i) Set $D_S = \sum_{i\in S} p_i$. We can realize α_{ij} as the composition

$$\mathcal{O} \xrightarrow{x_i} \mathcal{O}(D_S + p_i) \longrightarrow \mathcal{O}_{p_j}.$$

Now one can check that $\mathcal{O}(D_S + p_i)$ can be represented by the twisted object

$$[\mathcal{O}(D_S + p_i)] := (\mathcal{O}_{p_i} \oplus t\mathcal{O}_{D_S} \oplus t\mathcal{O}, t\psi_i + \sum_{i\in S} B_i).$$

Morphisms of degree 0 in $\mathrm{Hom}(\mathcal{O}, [\mathcal{O}(D_S + p_i)])$ have form $xA_i + \sum_{j\in S} y_j(tA_j) + z(t\,\mathrm{id})$ and the differential has form

$$\delta(xA_i + \sum_{j\in S} y_j(tA_j) + z(t\,\mathrm{id})) = x \cdot m_3(B_i, \psi_i, A_i)t + \sum_{j\in S} y_j B_j A_j t.$$

Thus, we can represent x_i by the closed morphism $A_i - \sum_{j\in S} \widetilde{\alpha}_{ij}(tA_j)$, where $\widetilde{\alpha}_{ij}$ are determined from

$$m_3(B_i, \psi_i, A_i) = \sum_{j\in S} \widetilde{\alpha}_{ij} B_j A_j.$$

On the other hand, the morphism $\mathcal{O}(D_S + p_i) \to \mathcal{O}_{p_j}$ is induced by the natural projection $e_j : t\mathcal{O}_{D_S} \to \mathcal{O}_{p_j}$. Hence, the composition with x_i is given by $-\widetilde{\alpha}_{ij} A_j$, which gives our assertion.
(ii) As in part (i), we realize $\mathcal{O}(D_S + p_j)$ by the twisted object

$$[\mathcal{O}(D_S + p_j)] = (\mathcal{O}_{p_j} \oplus \mathcal{O}_{D_S} \oplus \mathcal{O}, B_j + \sum_{i\in S} B_i).$$

Up to an additive constant, the element $x_{S,j}$ can be represented by some closed element of the form $A_j + \sum_i x_i A_i$. Calculating the differential we get

$$\delta(A_j + \sum_i x_i A_i) = B_j A_j + \sum_i x_i B_i A_i.$$

Thus, in order for our element to be closed, we should take $x_i = -a_{S,ij}$ (due to the relation (4.2.2)). The morphism $[\mathcal{O}(D_S + p_j)] \to \mathcal{O}_{p_k}$, for $k \notin S$, $k \neq j$, corresponds to the projection $A_k : \mathcal{O} \to \mathcal{O}_{p_k}$. Thus, the composition

$$\mathcal{O} \xrightarrow{x_{S,j}} \mathcal{O}(D_S + p_j) \longrightarrow \mathcal{O}_{p_l}$$

is given by

$$m_3(A_k, B_j, A_j) - \sum_{i \in S} a_{S,ij} m_3(A_k, B_i, A_i),$$

which immediately leads to our formula. ☐

Remark 4.7.2. Another way to see the relation (4.7.1) between the functions α_{ij} and m_3 is to study the triple Massey products

$$\mathcal{O} \to \mathcal{O}_{p_i} \xrightarrow{[1]} \mathcal{O}_{p_i} \to \mathcal{O}$$

(see [8, Sec. 2.4]).

4.8. **Relation to canonical embedding.** Here we work over an algebraically closed field k of characteristic zero.

Let C be a smooth projective (connected) curve with distinct marked points $p_1, \ldots, p_g$ such that $H^1(C, \mathcal{O}(D)) = 0$, where $D = p_1 + \ldots + p_g$. By Serre duality, the latter condition is equivalent to $H^0(\omega_C(-D)) = 0$. This implies that for every $i = 1, \ldots, n$, the space $H^0(\omega_C(-D + p_i))$ is 1-dimensional, and its generator ω_i satsifies $\omega_i|_{p_i} \neq 0$. Thus, a choice of nonzero tangent vectors $(v_1, \ldots, v_g)$ at the marked points is equivalent to a choice of nonzero 1-forms $\omega_1, \ldots, \omega_g$ such that $\omega_i(p_j) = 0$ for $i \neq j$ (the connection with the choice of v_i is given by $\langle \omega_i(p_i), v_i \rangle = 1$).

Furthermore, by considering the restriction to the points p_i, we immediately see that $(\omega_1, \ldots, \omega_g)$ are linearly independent, so they form a basis of $H^0(C, \omega_C)$. Since we are in characteristic zero, we can define a formal parameter t_i at each point p_i uniquely by requiring that $\omega_i = dt_i$ on the formal disk around p_i. It turns out that the same formal parameters can be characterized in a different way.

Proposition 4.8.1. *(i) The formal parameters t_i are characterized by the property that for each $m \geq 2$ there exists a function $f_i[m] \in H^0(C, \mathcal{O}(D + (m-1)p_i))$, such that the polar part of $f_i[m]$ at p_i is $\frac{1}{t_i^m}$.*
(ii) For $i \neq j$, let $p_{ij}[n]$ denote the coefficient of $\frac{1}{t_j}$ in the Laurent series of $f_i[n]$ at p_j. Then the expansion of ω_i near p_j (where $i \neq j$) has form

$$\omega_i = -\sum_{n \geq 2} p_{ji}[n] t_j^{n-1} dt_j. \tag{4.8.1}$$

Proof. (i) First, we can check the existence of formal parameters $\widetilde{t}_i$ for which the required functions $f_i[m]$ exist. We start with arbitrary formal parameters t_i and then gradually improve them. First, we look at the polar parts of $f_i[2]$: to kill the coefficient of $\frac{1}{t_i}$ we use an appropriate change of the form $t_i \mapsto t_i + ct_i^2$. Then we similarly use $f_i[3]$ to correct t_i by the cubic term, etc. (see [35, Lem. 2.1.1] for details).

The rational differential $f_i[n]\omega_i$ can have a pole only at p_i, so by the residue theorem, we get $\mathrm{Res}_{p_i}(f_i[n]\omega_i) = 0$ for every $n \geq 2$. Thus, if we write $\omega_i = \phi_i(t_i)\widetilde{dt_i}$ at the formal neighborhood of p_i then we deduce that $\phi_i = 1$.
(ii) This follows immediately from the residue theorem applied to the rational differentials $f_j[n]\omega_i$ for $i \neq j$, since $\mathrm{Res}_{p_i}(f_j[n]\omega_i) = p_{ji}[n]$ while $\mathrm{Res}_{p_j}(f_j[n]\omega_i)$ is equal to the coefficient of $t_j^{n-1}dt_j$ in the expansion of ω_i at p_j. ☐

Now we can combine the classical fact due to Petri that for $g \geq 4$, a generic curve can be defined by explicit quadratic and cubic equations in its canonical embedding (whereas for $g \geq 5$ one only needs quadratic equations) with the expansions (4.8.1), to see that a generic curve $(C, p_\bullet, v_\bullet)$ of genus $g \geq 4$ (resp., $g \geq 5$) is determined by the values of the functions $(p_{ij}[m])$ with $m \leq 4$ (resp., $m \leq 3$) on it.

Lemma 4.8.2. *Assume that $g \geq 4$ and $H^0(C, \omega_C^{\otimes 2}(-3D)) = 0$. If $g = 4$ then assume in addition that $H^0(C, \omega_C^{\otimes 3}(-4D)$. Then the quadratic (resp., quadratic and cubic, if $g = 4$) relations in the canonical algebra $\bigoplus_{n\geq 0} H^0(C, \omega_C^{\otimes n})$ are determined by the values of $(p_{ij}[m])$ with $m \leq 3$ (resp., $m \leq 4$).*

Proof. By assumption, the map $H^0(C, \omega_C^{\otimes 2}) \to \bigoplus_{i=1}^g \omega_C^{\otimes 2}|_{3p_i}$ is injective. Thus, the quadratic relations coincide with the kernel of the composed map

$$H^0(C, \omega_C)^{\otimes 2} \to \bigoplus_{i=1}^{g} \omega_C^{\otimes 2}|_{3p_i}.$$

Thus, it is enough to know expansions of each ω_i at each point p_j modulo $(t_j^3)dt_j$, and the assertion follows from Proposition 4.8.1(ii). For cubic relations in the case $g = 4$, the argument is similar using the map

$$H^0(C, \omega_C)^{\otimes 3} \to \bigoplus_{i=1}^{4} \omega_C^{\otimes 3}|_{4p_i}.$$

☐

Lemma 4.8.3. *One has for $i \neq j$,*

$$p_{ij}[4] = 2\alpha_{ij}\overline{\gamma}_{ij} - \sum_{k\neq i,j} \alpha_{ik}^2\alpha_{kj},$$

with $\alpha_{ij} = p_{ij}[2]$ and $\overline{\gamma}_{ij} := \gamma_{ij} - \gamma_{ii}$, where γ_{ij} are determined from the expansions

$$x_i = f_i[2] = \frac{\alpha_{ij}}{t_j} + \gamma_{ij} + \ldots, \quad x_i = \frac{1}{t_i^2} + \gamma_{ii} + \ldots$$

Proof. Apply the residue theorem to the rational differential $x_i^2\omega_j$. ☐

Proposition 4.8.4. *For $g \geq 4$, the generic curve $(C, p_\bullet, v_\bullet)$ in $\widetilde{\mathcal{M}}_{g,g}$ is determined by the corresponding values of $\alpha_{ij} = p_{ij}[2]$, $\beta_{ij} = p_{ij}[3]$ and $\overline{\gamma}_{ij}$. Hence, the corresponding A_∞-structure is determined by m_3 and m_4.*

Proof. By the classical theorem of Petri, the image of the canonical embedding of a generic curve C of genus $g \geq 5$ can be recovered from the quadratic relations between the differentials (ω_i). In the case $g = 4$ the same is true if we consider quadratic and cubic relations. Furthermore, knowing (ω_i) we can also recover the points p_i (as common zeros of $(\omega_j)_{j \neq i}$) and the tangent vectors v_i. It remains to apply Lemmas 4.8.2 and 4.8.3. To translate this into a statement about A_∞-structures we just have to observe that m_3 and m_4 correspond to functions of weight ≤ 2 with respect to $\mathbb{G}_m$, and both β_{ij} and $\overline{\gamma}_{ij}$ have weight 2 (recall that α_{ij} has weight 1). □

To deduce that a generic curve of genus $g \geq 6$ is determined by the values of (α_{ij}) alone (see [8, Thm. 3.2.1]), one has to use in addition the following equations:

$$\alpha_{ik}(\overline{\gamma}_{jk} - \overline{\gamma}_{ji}) + \alpha_{jk}(\overline{\gamma}_{ik} - \overline{\gamma}_{ij}) - \alpha_{ji}\beta_{ik} - \alpha_{ij}\beta_{jk} = \sum_{l \neq i,j,k} \alpha_{il}\alpha_{jl}\alpha_{lk}$$

for every distinct i, j, k (obtained by applying the residue theorem to the rational differentials $x_i x_j \omega_k$). We view these elements as linear equations on (β_{ij}) and $\overline{\gamma}_{ij}$. The claim is that the solution is unique. Thus, we have to prove that for a generic curve the corresponding matrix is nondegenerate. We prove this in [8] by reducing the problem to the case $g = 6$, in which case we present an explicit nodal curve (rational with 6 nodes) for which the matrix is nondegenerate.

4.9. **Case of genus** 1. In the case of genus 1 the moduli space $\widetilde{\mathcal{U}}^{ns}_{1,n}$ has a $\mathbb{G}_m^n$-map to the projective space $\mathbb{P}^{n-1}$. Let us consider the preimage V_n of the open subset $x_1 \ldots x_n \neq 0$ in $\mathbb{P}^{n-1}$. Since the latter open subset is an open orbit of $\mathbb{G}_m^n$, isomorphic to $\mathbb{G}_m^n/\mathbb{G}_m$, we can expect that V_n is obtained from a smaller moduli space $\overline{V}_n$ with just $\mathbb{G}_m$-action, as

$$V_n = \overline{V}_n \times_{\mathbb{G}_m} \mathbb{G}_m^n.$$

This is indeed the case, with $\overline{V}_n$ parametrizing $(C, p_1, \ldots, p_n, \omega)$, where C is a reduced projective connected curve of arithmetic genus 1 with n smooth marked points, ω is a nonzero section of the dualizing sheaf ω_C, such that $\mathcal{O}(p_1 + \ldots + p_n)$ is ample and $H^1(C, \mathcal{O}(p_i)) = 0$ for every $i = 1, \ldots, n$. The latter condition is equivalent to nonvanishing of $\omega(p_i)$ for every i (see [20, Lem. 1.1.1]).

Note that $\overline{V}_n$ has a unique $\mathbb{G}_m$-invariant point, which is an elliptic n-fold curve C_n (with one smooth marked point). This curve is defined as follows: C_1 is a cuspidal cubic in $\mathbb{P}^2$; C_2 is a union of two $\mathbb{P}^1$, tangent at the point of intersection (forming a tacnode); C_m is the union of m generic lines through a point in $\mathbb{P}^{m-1}$.

For $n \geq 3$, the affine part of the curve C_n can be described by the following equations in the $\mathbb{A}^{n-1}$ with coordinates $(x_2, \ldots, x_n)$:

$$\begin{aligned} x_i x_j &= x_{i'} x_{j'} \quad \text{for } i \neq j, i' \neq j', \\ x_2 x_3^2 &= x_2^2 x_3 \end{aligned} \tag{4.9.1}$$

(to get the projective curve C_n one adds n points at infinity, which become the marked points).

It turns out (see [20, Sec. 1.1]) that for $n \geq 3$, for each curve $(C, p_1, \ldots, p_n, \omega)$ in $\overline{V}_n$, one can describe an embedding of the affine curve $C \setminus D$, where $D = p_1 + \ldots + p_n$, into $\mathbb{A}^{n-1}$ with the defining equations deforming (4.9.1). Namely, for $i = 2, \ldots, n$, we can find

$x_i \in \mathcal{O}_C(p_1+p_i)$ such that $\operatorname{Res}_{p_1}(x_i\omega)=1$. These functions are unique up to additive constants, and we can normalize them by requiring that

$$x_3(p_2)=0,\ x_i(p_3)=0 \ \text{ for } i\neq 3.$$

Then the functions

$$1, x_2^m x_3, x_2^m, \dots, x_n^m, \quad \text{for } m\geq 1 \tag{4.9.2}$$

form a basis of $\mathcal{O}(C\setminus D)$, and we have the following defining equations for $C\setminus D$:

$$\begin{aligned} x_2x_i &= x_2x_3+c_ix_i+\overline{c}_ix_2-c,\ 4\leq i\\ x_3x_i &= x_2x_3+(a+c_i+\overline{c}_i)(x_i-\overline{c}_i)+b,\ 4\leq i\\ x_ix_j &= x_2x_3+c_{ij}x_j+c_{ji}x_i-c,\ 4\leq i<j,\\ x_2x_3^2 &= x_2^2x_3+ax_2x_3+bx_2+cx_3+d, \end{aligned} \tag{4.9.3}$$

for some constants $(a,b,c,d,c_i,\overline{c}_i,c_{ij})$, where $i\neq j$, $i,j\geq 4$. For example, for $n=3$, we only have one equation (the last one), and a,b,c,d are independent variables on the moduli space $\overline{V}_3\simeq\mathbb{A}^4$.

Using the fact that (4.9.2) is a basis of $\mathcal{O}(C\backslash D)$, we get some equations on $(a,b,c,d,c_i,\overline{c}_i,c_{ij})$, which can be found explicitly (see [20, Prop. 1.1.5]). In particular, we get that for $n\geq 5$, all of them can be expressed universally in terms of $(a,c_i,\overline{c}_i,c_{ij})$, where $4\leq i<j$. Now we observe that the latter functions on $\overline{V}_n$ have weight 1 with respect to the natural $\mathbb{G}_m$-action. Thus, for $n\geq 5$, we get an affine embeding of $\overline{V}_n$ into some affine space with the weight 1 action of $\mathbb{G}_m$.

Recalling the isomorphism of $\widetilde{\mathcal{U}}^{ns}_{1,n}$ with the moduli space of A_∞-structures and the fact that m_n has weight $n-2$ with respect to $\mathbb{G}_m$, we derive that over the open subset corresponding to $V_n\subset\widetilde{\mathcal{U}}^{ns}_{1,n}$, every A_∞-structure is determined by m_3. In particular, we get a stronger version of the Conjecture from Sec. 4.7 in this case: for $g\geq 5$, the projection $\widetilde{\mathcal{M}}_{1,n}\to\mathcal{M}_3(\mathcal{E})$ is a locally closed embedding.

One can also say precisely which curves of arithmetic genus 1 can appear in the moduli space $\overline{V}_m$.

Proposition 4.9.1. *(see* [20, Thm. 1.5.7]*) For every $(C,p_1,\dots,p_n,\omega)$ in $\overline{V}_n$, the curve C is Gorenstein with $\omega_C\simeq\mathcal{O}_C$. More precisely, C is either smooth, or a standard nodal m-gon, where $1\leq m\leq n$ (for $m=1$ this means that it is an irreducible nodal curve), or an elliptic m-fold curve C_m, for $1\leq m\leq n$.*

Proof(sketch). The first key observation is that each curve C_n is Gorenstein. This is a simple local computation done in [48, Prop. 2.5] (one has to check that $\dim\omega_C/\mathfrak{m}\omega_C=1$ near the singular point, where ω_C is the dualizing sheaf, and for this one can use an explicit description of ω_C in terms of the normalization). Next, we use the $\mathbb{G}_m$-action on $\overline{V}_n$ (corresponding to rescaling of ω). It is easy to see that the affine coordinates on $\overline{V}_n$ have positive $\mathbb{G}_m$-weights, so the closure of every $\mathbb{G}_m$-orbit contains the point where all these coordinates are zero. This gives a flat family degenerating every curve in $\overline{V}_n$ into C_n. Since the Gorenstien property is open, we deduce that every curve in $\overline{V}_n$ is Gorenstein.

If C is irreducible then it is either smooth, or standard 1-gon, or isomorphic to C_1, so assume C is reducible.

Let $C = C_0 \cup C'$, where C_0 is irreducible, C' is connected, and $\xi = C_0 \cap C'$ is a finite subscheme of C. The exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C'} \to \mathcal{O}_\xi \to 0$$

shows that

$$1 = h^1(\mathcal{O}_C) = h^1(\mathcal{O}_{C_0}) \oplus h^1(\mathcal{O}_{C'}) + \ell(\xi) - 1.$$

Thus, either both both C_0 and C' are of arithmetic genus 0 and $\ell(\xi) = 2$, or one of the subcurves has genus 1 and the other 0 and $\ell(\xi) = 1$. We claim that the latter case is impossible. Indeed, then we could find an irreducible subcurve $C_1 \subset C$, which is isomorphic to $\mathbb{P}^1$ and is joined to the union of the remaining components at one point transversally. But then for a marked point $p_i \in C_1$ (which exists by the definition of $\overline{V}_n$) we would have $h^0(C, \mathcal{O}(p_i)) \geq 2$, hence $h^1(C, \mathcal{O}(p_i)) \neq 0$, which is a contradiction.

It follows that all irreducible components of C are isomorphic to $\mathbb{P}^1$. Furthermore, if there exists an irreducible component $C_i \subset C$, such that $C_i \cap \overline{C \setminus C_i}$ is one point q (with the scheme structure of length 2), then $C' = \overline{C \setminus C_i}$ is a connected curve of arithmetic genus 0. Now the above observation implies that every irreducible component of C' contains q, and it is easy to deduce that C is isomorphic to C_n.

In the case when for every component C_i the intersection $C_i \cap \overline{C \setminus C_i}$ consists of two points, one easily checks that C contains a standard nodal m-gon as a subcurve C'. Using the above observation again, one can see that in fact $C = C'$. ☐

Remark 4.9.2. One can give a different proof of Proposition 4.9.1 using classification of Gorenstein singularities of genus 1 (see [48, Lem. 3.3]).

The associative algebras E_W corresponding to the points of V_n are all isomorphic. Namely, they are isomorphic to the algebra $E_1([1, n])$ of the quiver with relations $Q([1, n])$ given in Fig. 1 below, where the relations are $B_1A_1 = \ldots = B_nA_n$, $A_jB_i = 0$ for $i \neq j$. We denote the quiver in this way because we want to consider the subquivers $Q[S]$ (isomorphic to $Q([1, |S|])$) corresponding to subsets $S \subset [1, n]$.

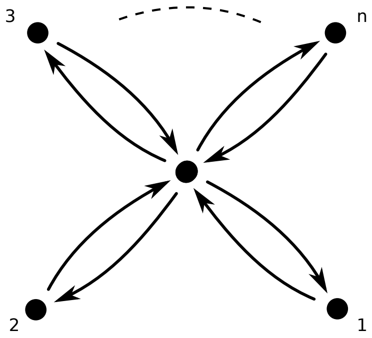

FIGURE 1. The quiver $Q([1, n])$

By Theorem 4.5.1, a curve $(C, p_1, \ldots, p_n, \omega)$ in $\overline{V}_n(k)$, where $n \geq 2$ and $\mathrm{char}(k) \neq 2$ (or $n = 1$ and $\mathrm{char}(k) \neq 2, 3$), is determined by the corresponding A_∞-structure m on the algebra $E_1([1, n])$.

In this setup we can give a characterization of nodal curves in terms of the corresponding A_∞-structures. The key idea is that given a subset $S \subset [1, n]$ and an A_∞-structure on $E_1([1, n])$, we can consider the induced A_∞-structure on the subquiver Q_S (i.e., on the algebra $E_1(S)$). Geometrically this operation corresponds to passing to a curve $(C, (p_i)_{i \in S}, \omega)$. However, it may happen that the divisor $D_S := \sum_{i \in S} p_i$ is not ample on C, so in fact, one has to replace the curve C by a certain contraction $\overline{C}$. More precisely, $\overline{C}$ is obtained as the Proj of the Rees algebra of the filtered algebra $H^0(C \setminus D_S, \mathcal{O})$. Looking at the basis of the latter algebra one can deduce that $(\overline{C}, (p_i)_{i \in S}, \omega)$ is a well defined point of the moduli space $\overline{V}_m$, where $m = |S|$.

In fact, it will be enough to consider the restrictions to one-element subsets $S = \{i\}$, in which case we look at subquivers $Q(i)$ with just two vertices.

For an A_∞-algebra structure on $E_1([1, n])$ we denote by $P_0, P_1, \ldots, P_n$ the natural perfect A_∞-modules associated with the vertices $\mathcal{O}, \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_n}$.

Proposition 4.9.3. *(i) A curve $(C, p_1, \ldots, p_n, \omega)$ in $\overline{V}_n(k)$ has only nodal singularities if and only if for every $i = 1, \ldots, n$, the restriction of the corresponding A_∞-structure on $E_1([1, n])$ to $Q(i)$ is homotopically nontrivial.*
(ii) A curve $(C, p_1, \ldots, p_n, \omega)$ is nodal with n irreducible components (i.e., isomorphic to the standard n-gon G_n, with one marked point on each component) if and only if for every i the condition in (i) is satisfied, and in addition, the subcategories $\langle P_0, P_i \rangle$ (in the derived category of perfect A_∞-modules) are distinct for different indices i.

Proof. (i) Assume first that C is either smooth or nodal. Then the restriction of the corresponding A_∞-structure to $Q(i)$ corresponds to replacing C with the contraction of all components not containing p_i, which will give an irreducible nodal curve with one point. Since the homotopically trivial A_∞-structure on $Q(i)$ corresponds to the cuspidal curve, we deduce the required nontriviality.

On the other hand, if C has a non-nodal singularity, then by Proposition 4.9.1, it is isomorphic to the elliptic m-fold curve C_m for some m. Let us choose one marked point on each component of C. Without loss of generality we can assume that these points are $p_1, \ldots, p_m$. Then replacing $(C, p_1, \ldots, p_n)$ with $(C, p_1, \ldots, p_m)$ we get a $\mathbb{G}_m$-invariant point of the moduli space that corresponds to the trivial A_∞-structure on $Q([1, m])$. Now restricting further to $Q(1)$ gives the trivial A_∞-structure.
(ii) By Proposition 4.9.1, the condition in (i) is equivalent to C being the standard m-gon for some m. Suppose $m < n$. Then we can find $i < j$ such that p_i and p_j belong to the same irreducible component C' of C. It is easy to see that this implies that the subcategories $\langle \mathcal{O}_C, \mathcal{O}_{p_i} \rangle$ and $\langle \mathcal{O}_C, \mathcal{O}_{p_j} \rangle$ are the same. Namely, we have a contraction $\pi : C \to \overline{C}$ that contracts every irreducible component of C other than C' to a point, and both these subcategories are $\pi^* \mathrm{Perf}(\overline{C})$. Under the equivalence ??? these subcategories correspond to $\langle P_0, P_i \rangle$ and $\langle P_0, P_j \rangle$. Conversely, if C is the standard n-gon, so that all marked points lie on different components then we claim that $\langle \mathcal{O}_C, \mathcal{O}_{p_i} \rangle \neq \langle \mathcal{O}_C, \mathcal{O}_{p_j} \rangle$ for $i \neq j$. Indeed, let $\iota_i : C_i \hookrightarrow C$ be the embedding of the irreducible component containing

p_i. Then the functor $L\iota_i^*$ sends $\mathcal{O}_{p_j}$ to zero, $\mathcal{O}_{p_i}$ to $\mathcal{O}_{p_i}$, and $\mathcal{O}_C$ to $\mathcal{O}_C$. Thus, the images of our subcategories under this functor are distinct. ☐

Remark 4.9.4. The characterization of Proposition 4.9.3(ii) can be used to prove the equivalence of the perfect derived category of the standard n-gon with the (compact) exact Fukaya category of the n-punctured torus, see [21, Thm. B(i)]. Namely, we consider in *loc. cit.* a natural set of exact Lagrangians $(L_0, \dots, L_n)$ such that morphisms between them have the same dimensions as for the objects $(\mathcal{O}_C, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n})$ on the standard n-gon (where p_i lies on the ith irreducible component), and the composition maps

$$\mathrm{Hom}^1(L_i, L_0) \otimes \mathrm{Hom}^0(L_0, L_i) \to \mathrm{Hom}^1(L_0, L_0), \ \ \mathrm{Hom}^0(L_0, L_i) \otimes \mathrm{Hom}^1(L_i, L_0) \to \mathrm{Hom}^1(L_i, L_i),$$

for $i \geq 1$, are nondegenerate. This implies that the corresponding associative algebra is isomorphic to $E([1, n])$. The more nontrivial part of the proof is to check that the objects $(L_0, \dots, L_n)$ generate the Fukaya category. The argument is based on the fact that the Dehn twists around the curves L_i generate the pure mapping class group (see [21, Lem. 3.1.1]).

5. PAIRS OF 1-SPHERICAL OBJECTS, A_∞-STRUCTURES, AND YANG-BAXTER EQUATIONS

Now we will transition to studying another moduli space of A_∞-structures which is related to solutions of the associative Yang-Baxter equation (AYBE) (see (2.1.2)). Namely, our motivation is to find a generalization of the triple Massey product calculation over an elliptic curve considered in Sec. 2.1, which gives rise to all solutions AYBE in a reasonable class. First, we will address this problem formally, i.e., we will show that this Massey product can be defined for any pair of 1-spherical objects in a cyclic A_∞-category. Then we will consider the question of realizing all such pairs of 1-spherical objects geometrically.

5.1. 1-spherical objects.

One of the key properties used in the Massey product calculation in Sec. 2.1 was the form of the Serre duality for the elliptic curve. The more general setup involves 1-spherical objects, as defined in [46].

Definition 5.1.1. An object X of an A_∞-category $\mathcal{C}$ over a field k is called *n-spherical* if $\mathrm{Hom}^i(X, X) = 0$ for $i \neq 0, n$, $\mathrm{Hom}^0(X, X) = \mathrm{Hom}^n(X, X) = k$, and for any object Y of $\mathcal{C}$ the pairing between the morphism spaces in the cohomology category $H^*\mathcal{C}$,

$$\mathrm{Hom}^{n-i}(Y, X) \otimes \mathrm{Hom}^i(X, Y) \to \mathrm{Hom}^1(X, X),$$

induced by m_2, is perfect. In particular, these morphism spaces are finite-dimensional.

For example, a vector bundle V over an elliptic curve C is 1-spherical in $D^b(C)$ provided it is endosimple, i.e., $\mathrm{Hom}(V, V) = k$.

In the case when $\mathcal{C}$ is a triangulated A_∞-category, one can associate with an n-spherical object a twist A_∞-endofunctor of $\mathcal{C}$, T_E defined by

$$T_E(X) = \mathrm{Cone}(\hom(E, X) \otimes E \xrightarrow{ev} X)$$

which is an autoequivalence (see [46], [43, I.5]).

We are interested in pairs of 1-spherical objects (E, F) such that $\mathrm{Hom}(E, F)$ is concentrated in degree 0. We refer to such (E, F) as a 1-*spherical pair.*

Note that in this case we have two perfect pairings,

$$\begin{aligned} &\mathrm{Hom}^1(F,E)\otimes\mathrm{Hom}^0(E,F)\to\mathrm{Hom}^1(E,E)\simeq k \ \text{ and}\\ &\mathrm{Hom}^0(E,F)\otimes\mathrm{Hom}^1(F,E)\to\mathrm{Hom}^1(F,F)\simeq k. \end{aligned} \tag{5.1.1}$$

We set $V=\mathrm{Hom}(E,F)$ and use the second pairing to identify $\mathrm{Hom}^1(F,E)$ with the dual space $V^\vee$. Then the first pairing gets identified with the map

$$V^\vee\otimes V\to k: v^*\otimes v\mapsto\langle v^*,gv\rangle$$

for an element $g\in\mathrm{GL}(V)$. Note that the definition of this element depends on the choice of trivializations of 1-dimensional spaces $\mathrm{Hom}^1(E,E)$ and $\mathrm{Hom}^1(F,F)$, however, the corresponding element in $\mathrm{PGL}(V)$ is an invariant of an isomorphism class of the pair (E,F). We say that (E,F) is a *symmetric* 1*-spherical pair* if g is a scalar multiple of identity.

In the case when $\mathcal{C}$ is the derived category of coherent sheaves on an elliptic curve, as well as in some other geometric examples, we have an additional structure: the A_∞-structure on $\mathcal{C}$ is cyclic with respect to the symmetric pairings between $\mathrm{Hom}^i(a,b)$ and $\mathrm{Hom}^{1-i}(b,a)$ provided by the Serre duality. Thus, every 1-spherical pair of objects in a cyclic A_∞-category is symmetric. However, we will also consider not necessarily symmetric pairs.

Assume that (E,F) is a 1-spherical pair in a minimal A_∞-category, with fixed trivializations $\mathrm{Hom}^1(E,E)\simeq\mathrm{Hom}^1(F,F)\simeq k$. Then we get a minimal A_∞-algebra $\mathrm{End}(E\oplus F)$. The underlying associative graded algebra $\mathcal{S}(V,g)$ depends only on the space V and the element $g\in GL(V)$. Namely, $\mathcal{S}(V,g)$ is the algebra of the following graded quiver with relations. It has two vertices E and F. The space of arrows from E to F (resp., from F to E) is V in degree 0 (resp., $V^\vee$), with defining relations

$$v^*\circ v=\langle v^*,g(v)\rangle\xi_F,\quad v\circ v^*=\langle v^*,v\rangle\xi_E,$$

where ξ_E (resp., ξ_F) is an arrow of degree 1 from E (resp., F) to itself.

Conversely, given a minimal A_∞-algebra on $\mathcal{S}(V,g)$ we get a 1-spherical pair in the corresponding A_∞-category with two objects. By a standard argument, it will still be a 1-spherical pair in the triangulated A_∞-envelope of this category. Thus, we have a correspondence between 1-spherical pairs and A_∞-structures on $\mathcal{S}(V,g)$. We can now study the moduli spaces of such A_∞-structures using the tools we developed.

5.2. Solutions of the Associative Yang-Baxter Equation (AYBE) and 1-spherical pairs.

Now we want to generalize the construction of an associative r-matrix (i.e., a solution of the AYBE) from the triple Massey products on elliptic curves associated with a family of stable vector bundles over an elliptic curve (see Sec. 2.1) to a certain Massey product associated with any 1-spherical pair (E,F) in a cyclic A_∞-category $\mathcal{C}$. The point is that the construction only used some formal categorical properties. However, in the general setup we do not have an analog of a pair of nonisomorphic stable vector bundles f (resp., pair of distinct points). Instead, we consider families of formal deformations of E and F and invert some formal expressions to make the Massey product well defined.

Let $\mathcal{C}$ be an A_∞-category over a field k, such the spaces $\hom^i(?,?)$ are finite-dimensional. Suppose we have a k-algebra R, complete with respect to I-adic topology, for some ideal $I\subset R$. Then we can consider twisted objects over the R-linear A_∞-category $\mathcal{C}\otimes R$ of the following kind: (X, δ_X), where X is an object of $\mathcal{C}$ and $\delta_X \in I \otimes \hom^1(X,X)$ is an element satisfying the Maurer-Cartan equation

$$\sum_{i \geq 2} (-1)^{\binom{i}{2}} m_i(\delta_X, \ldots, \delta_X) = 0.$$

Note that the sum converges in $R \otimes \hom^2(X,X)$, since the term with m_i belongs to $I^i \otimes \hom^2(X,X)$. As in Sec. 1.5.2, such twisted objects form a (non-minimal) A_∞-category. We denote by m_n^t the A_∞-products for morphisms between twisted objects.

Now, given a 1-spherical object E in a minimal A_∞-category $\mathcal{C}$ over k, an I-adically complete k-algebra R, and an element $x \in I$, we can consider a twisted object $(E, x \cdot \xi_E)$ over $\mathcal{C} \otimes R$, where ξ_E is a generator of the 1-dimensional space $\operatorname{Ext}^1(E,E)$. Note that the Maurer-Cartan equation is satisfied trivially since $\operatorname{Ext}^2(E,E) = 0$. For example, we can take $R = k[[x]]$ and the twisted object $(E, x \cdot \xi_E)$ over $\mathcal{C} \otimes k[[x]]$. We can think of the latter object as an incarnation of the universal formal deformation of E.

The formal analog of considering pairs of non-isomorphic objects in this context is the following. We can take $R = k[[x_1, x_2]][(x_1 - x_2)^{-1}]$ and a pair of twisted objects $E_1 = (E, x_1 \cdot \xi_E)$ and $E_2 = (E, x_2 \cdot \xi_E)$. Note that the morphism space $\hom(E_1, E_2)$ is the complex

$$\operatorname{Hom}^0(E,E) \otimes R \xrightarrow{m_1^t} \operatorname{Hom}^1(E,E) \otimes R,$$

where $m_1^t(\mathrm{id}_E) = (x_2 - x_1)\xi_E$. Since $(x_1 - x_2)$ is invertible in R, the differential m_1^t is an isomorphism, so the complex is exact.

Now we can construct the analog of the triple Massey product from Sec. 2.1, associated with a 1-spherical pair (E, F) in a minimal A_∞-category $\mathcal{C}$ over k. For this we take $R = k[[x_1, x_2, y_1, y_2]][\Delta_2^{-1}]$, where $\Delta_2 = (x_1 - x_2)(y_1 - y_2)$, and consider the twisted objects

$$E_1 = (E, x_1 \cdot \xi_E), \quad E_2 = (E, x_2 \cdot \xi_E), \quad F_1 = (F, y_1 \cdot \xi_F), \quad F_2 = (F, y_2 \cdot \xi_F)$$

Note that we have canonical identifications

$$\operatorname{Hom}^0(E_i, F_j) = \hom(E_i, F_j) = \operatorname{Hom}^0(E,F) \otimes R, \ \operatorname{Hom}^1(F_j, E_i) = \hom(F_j, E_i) = \operatorname{Hom}^1(F,E) \otimes R.$$

Thus, given an element $\theta \in V := \operatorname{Hom}(E,F)$, we can view it as a closed morphism $\theta[ij] \in \operatorname{Hom}^0(E_i, F_j)$ for $i, j \in \{1, 2\}$. Similarly, an element $\eta \in V^\vee \simeq \operatorname{Hom}^1(F,E)$ gives rise to $\eta[ij] \in \operatorname{Hom}^1(F_i, E_j)$. Now we want to consider the triple Massey product corresponding to the composable arrows

$$E_1 \xrightarrow{\theta[11]} F_1 \xrightarrow{\eta[12]} E_2 \xrightarrow{\theta[22]} F_2.$$

Note that the compositions $m_2^t(\eta[12], \theta[11])$ and $m_2(\theta[22], \eta[12])$ are not zero on the nose, however, they are coboundaries:

$$m_2^t(\eta[12], \theta[11]) = \langle \eta, g\theta \rangle \xi_E = m_1^t\Big(\frac{\langle \eta, g\theta \rangle}{x_2 - x_1} \cdot \mathrm{id}_E\Big),$$

$$m_2^t(\theta[22], \eta[12]) = \langle \eta, \theta \rangle \xi_F = m_1^t\Big(\frac{\langle \eta, \theta \rangle}{y_2 - y_1} \cdot \mathrm{id}_F\Big).$$

Assuming that our A_∞-category $\mathcal{C}$ is strictly unital, the products m_2^t containing identity elements can be replaced by the original products m_2. Thus, the Massey product is given by

$$MP(\theta[22], \eta[12], \theta[11]) = m_3^t(\theta[22], \eta[12], \theta[11]) \pm \frac{\langle \eta, g\theta \rangle}{x_2 - x_1} \cdot \theta[22] \pm \frac{\langle \eta, \theta \rangle}{y_2 - y_1} \cdot \theta[11].$$

We can think of this Massey product as an element of

$$\big(\mathrm{Hom}^0(E_1, F_2) \otimes_R \mathrm{Hom}^0(E_2, F_2)^\vee\big) \otimes_R \big(\mathrm{Hom}^1(F_1, E_2)^\vee \otimes_R \mathrm{Hom}^0(E_1, F_1)^\vee\big) \simeq \mathrm{End}(V) \otimes \mathrm{End}(V) \otimes R.$$

Choosing a basis (θ_α) in V, and letting (η_α) to be the dual basis in $V^\vee$, we can write this element as

$$r_{y_1 y_2}^{x_1 x_2} = \sum_{\alpha, \alpha', \beta, \beta'} \langle \eta_{\beta'}, \mathrm{MP}(\theta_{\alpha'}[22], \eta_\beta[12], \theta_\alpha[11]) \rangle \cdot e_{\beta' \alpha'} \otimes e_{\beta \alpha}, \tag{5.2.1}$$

where we use the identification of $\mathrm{Hom}(E_1, F_2)$ with $V \otimes R$.

Similarly, we can consider the triple Massey product corresponding to the composable arrows

$$F_2 \xrightarrow{\eta[21]} E_1 \xrightarrow{\theta[11]} F_1 \xrightarrow{\eta[12]} E_2.$$

We have

$$m_2^t(\eta[21], \theta[11]) = m_1^t\big(\frac{\langle \eta, g\theta \rangle \cdot \mathrm{id}_F}{y_1 - y_2}\big), \quad m_2^t(\theta[11], \eta[12]) = m_1^t\big(\frac{\langle \eta, \theta \rangle \cdot \mathrm{id}_E}{x_2 - x_1}\big),$$

so we get

$$MP(\eta[21], \theta[11], \eta[12]) = m_3^t(\eta[21], \theta[11], \eta[12]) \pm \frac{\langle \eta, \theta \rangle}{x_2 - x_1} \cdot \eta[21] \pm \frac{\langle \eta, g\theta \rangle}{y_1 - y_2} \cdot \eta[12].$$

Let us record this Massey product by the element of $\mathrm{End}(V) \otimes \mathrm{End}(V) \otimes R$,

$$\widetilde{r}_{y_1 y_2}^{x_1 x_2} = \sum_{\alpha, \alpha', \beta, \beta'} \langle \mathrm{MP}(\eta_\beta[12], \theta_\alpha[11], \eta_{\beta'}[21]), \theta_{\alpha'} \rangle \cdot e_{\beta' \alpha'} \otimes e_{\beta \alpha}, \tag{5.2.2}$$

Next, we can consider twisted objects $E_i = (E, x_i \cdot \xi_E)$, $F_i = (F, y_i \cdot \xi_F)$, for $i = 1, 2, 3$, over a bigger ring $R_3 := k[[x_1, x_2, x_3, y_1, y_2, y_3]][\Delta_3^{-1}]$, where $\Delta_3 = \prod_{i<j}(x_i - x_j)(y_i - y_j)$. As in Sec. 2.1, we want to apply the A_∞-identity to the composable arrows

$$E_1 \to F_1 \to E_2 \to F_2 \to E_3 \to F_3$$

to deduce some identity fo $r_{y_1 y_2}^{x_1 x_2}$ and $\widetilde{r}_{y_1 y_2}^{x_1 x_2}$. As a first step we can replace the A_∞-structure on the subcategory of our twisted objects with the minimal one using homological perturbation (see Remark 1.2.2). By the functoriality of the Massey products (see Proposition 1.4.1), we see that the obtained products of the form $m_3(\theta[jj], \eta[ij], \theta[ii])$ and $m_3([\eta[ji], \theta[ii], \eta[ij]])$, for $i \neq j$, coincide with the corresponding Massey products in the original category. Thus, we get the identity of the form

$$MP(\theta[33], \eta[23], MP(\theta[22], \eta[12], \theta[11])) \pm MP(\theta[33], MP(\eta[23], \theta[22], \eta[12]), \theta[11]) \pm$$
$$MP(MP(\theta[33], \eta[23], \theta[22]), \eta[12], \theta[11]) = 0.$$

Now assume that our A_∞-category is cyclic. Then the element g can be taken to be id, and $\widetilde{r}^{x_1x_2}_{y_1y_2} = \pm r^{x_1x_2}_{y_1y_2}$. Thus, the above identity for the Massey products leads to the following equation in $\mathrm{End}(V) \otimes \mathrm{End}(V) \otimes R_3$:

$$(r^{x_2x_1}_{y_2y_3})^{12}(r^{x_1x_3}_{y_1y_3})^{13} - (r^{x_1x_3}_{y_1y_2})^{23}(r^{x_2x_3}_{y_2y_3})^{12} + (r^{x_2x_3}_{y_1y_3})^{13}(r^{x_1x_2}_{y_1y_2})^{23} = 0, \tag{5.2.3}$$

which we call the *formal set-theoretic AYBE*. In addition, because of the cyclicity of the A_∞-structure, this tensor satisfies the following *skew-symmetry condition*:

$$(r^{x_1x_2}_{y_1y_2})^{21} = -r^{x_2x_1}_{y_2y_1}. \tag{5.2.4}$$

Note that the polar part of $r^{x_1x_2}_{y_1y_2}$ is $\frac{\mathrm{id}\otimes\mathrm{id}}{x_2-x_1} + \frac{\mathrm{P}}{y_1-y_2}$, where $\mathrm{P} = \sum e_{ij} \otimes e_{ji}$ is the permutation matrix.

One can check that cyclic A_∞-equivalence between 1-spherical pairs (E, F) lead to the following equivalence relation between solutions of the set-theoretic AYBE. For every $\varphi^x_y \in \mathrm{id} + (x, y) \in \mathrm{Mat}_n(k) \otimes k[[x, y]]$, we have a transformation

$$r^{x_1x_2}_{y_1y_2} \mapsto (\varphi^{x_2}_{y_1} \otimes \varphi^{x_1}_{y_2}) r^{x_1x_2}_{y_1y_2} (\varphi^{x_1}_{y_1} \otimes \varphi^{x_2}_{y_2})^{-1}. \tag{5.2.5}$$

Theorem 5.2.1. ([22, Thm. A]) *The above construction gives a bijection between cyclic A_∞-structures on $\mathcal{S}(k^n, \mathrm{id})$, up to a cyclic A_∞-equivalence, and equivalence classes of solutions of the set-theoretic AYBE in $k[[x, y]][\Delta_2^{-1}]$, with the polar part $\frac{\mathrm{id}\otimes\mathrm{id}}{x_2-x_1} + \frac{\mathrm{P}}{y_1-y_2}$, satisfying the skew-symmetry condition.*

The main point is that the pair of formal series (with coefficients in $\mathrm{End}(V)^{\otimes 2}$),

$$\langle m_3^t(\theta[22], \eta[12], \theta[11]), \eta\rangle, \langle m_3^t(\eta[21], \theta[11], \eta[12]), \theta\rangle \in \mathrm{End}(V)^{\otimes 2}[[x_1, x_2, y_1, y_2]],$$

can be viewed as generating functions for all possibly nonzero higher products on the A_∞-category with the objects E and F. Indeed, by considering the degrees we see that all nontrivial higher products have one of the two forms,

$$m_{a+b+c+d+3}((\xi_F)^a, \theta, (\xi_E)^b, \eta, (\xi_F)^c, \theta, (\xi_E)^d) \text{ and } m_{a+b+c+d+3}((\xi_E)^a, \eta, (\xi_F)^b, \theta, (\xi_E)^c, \eta, (\xi_F)^d),$$

which are directly related to the coefficients of the above two formal series (recall that we considered the first of these products in Theorem 2.2.1 in the case when E and F are line bundles over an elliptic curve). Note that due to cyclic symmetry, the second type of products is determined by the first.

5.3. **Classification of nondegenerate trigonometric solutions of the AYBE.**

Let us assume that in an associative r-matrix the variables x_i, y_j are complex numbers and that $r^{x_1x_2}_{y_1y_2}$ depends only on their differences:

$$r^{x_1x_2}_{y_1y_2} = r(x_1 - x_2, y_1 - y_2),$$

where $r(u, v)$ is a meromorphic function in a neighborhood of $(0, 0)$ in $\mathbb{C}^2$ with values in $\mathrm{Mat}_n(\mathbb{C})^{\otimes 2}$, where $n \geq 2$. For example, this assumption is satisfied for the solutions associated with bundles over an elliptic curve. Then the equation (5.2.3) and the skew-symmetry condition take form

$$r^{12}(-u', v) r^{13}(u + u', v + v') - r^{23}(u + u', v') r^{12}(u, v) + r^{13}(u, v + v') r^{23}(u', v') = 0, \tag{5.3.1}$$

$$r^{21}(-u, -v) = -r(u, v). \tag{5.3.2}$$

It is easy to check that if $r(u,v)$ satisfies these equations then so does

$$\hat{r}(u,v) := r(v,u)^t \cdot \mathrm{P},$$

where $(a\otimes b)^t = a^t \otimes b^t$, and a^t denotes the transpose of a.

Let us say that $r(u,v)$ is *strongly nondegenerate* if the tensors $r(u,v)$ and $\hat{r}(u,v)$ have the maximal rank n^2 for generic (u,v). One can check (see [22, Prop. 1.4.4]) that if $r(u,v)$ is strongly nondegenerate then after rescaling the variables we will get that the polar part of $r(u,v)$ at $u=0$ (resp., $v=0$) has form $\frac{1\otimes 1}{u}$ (resp., $\frac{\mathrm{P}}{v}$). Now we can look at the Laurent series of $r(u,v)$ in u near $u=0$:

$$r(u,v) = \frac{1\otimes 1}{u} + r_0(v) + \dots \tag{5.3.3}$$

Let us denote by $\mathrm{pr} : \mathrm{Mat}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ the projection along $\mathbb{C}\cdot 1$. Then one can check that $\overline{r}(v) := (\mathrm{pr}\otimes\mathrm{pr})(r_0(v))$ is a nondegenerate solution of the *classical Yang-Baxter equation (CYBE)* with values in $\mathfrak{sl}_n(\mathbb{C})\otimes\mathfrak{sl}_n(\mathbb{C})$,

$$[\overline{r}^{12}(v), \overline{r}^{13}(v+v')] + [\overline{r}^{12}(v), \overline{r}^{23}(v')] + [\overline{r}^{13}(v+v'), \overline{r}^{23}(v')] = 0.$$

Here "nondegenerate" simply means that the tensor $r(v)$ is nondegenerate for generic v

Recall that Belavin and Drinfeld in the seminal paper [**?**] classified nondegenerate solutions of the classical Yang-Baxter equation for all simple complex Lie algebras, up to some natural equivalence. They showed that they can be either elliptic or trigonometric or rational, and further classified trigonometric solutions in terms of some combinatorial data, involving so called Belavin-Drinfeld triples.

Similarly, one can pose the problem of classifying nondegenerate solutions $r(u,v)$ of the AYBE (and of its formal set-theoretic version). It was shown in [34] that if $\overline{r}(v)$ is either elliptic or trigonometric then $r(u,v)$ is determined by $\overline{r}(v)$, up to some natural transformations. Furthermore, all elliptic solutions of the CYBE extend to those of the AYBE and are obtained using our Massey product construction with bundles over elliptic curves.

As for trigonometric solutions, it was proved in [34] that nondegenerate solutions of the AYBE, with the Laurent expansion at $u=0$ of the form (5.3.3) and such that $\overline{r}(v)$ is a trigonometric solution of the CYBE, admit a classification in terms of the following combinatorial data.

Definition 5.3.1. An *associative Belavin-Drinfeld structure (BD-structure)* (S, C_1, C_2, A) consists of a finite set S, a pair of transitive cyclic permutations $C_1, C_2 : S \to S$ and a proper subset $A \subset S$ such that for all $a\in A$, one has :

$$C_1(C_2(a)) = C_2(C_1(a)).$$

The trigonometric solution of the AYBE corresponding to an associative BD-structure (S, C_1, C_2, A) is given by

(5.3.4)

$$\begin{aligned} r(u,v) &= \frac{1}{\exp(u)-1}\sum_i e_{ii}\otimes e_{ii} + \frac{1}{1-\exp(-v)}\sum_i e_{ii}\otimes e_{ii} \\ &+ \frac{1}{\exp(u)-1}\sum_{0<k<n,i} \exp(\frac{ku}{n}) e_{C_1^k(i),C_1^k(i)}\otimes e_{ii} + \frac{1}{\exp(v)-1}\sum_{0<m<n,i}\exp(\frac{mv}{n}) e_{i,C_2^m(i)}\otimes e_{C_2^m(i),i} \\ &+ \sum_{0<k,0<m;a\in A(k,m)} \Big\{ \exp(-\frac{ku+mv}{n}) e_{C_2^m(a),a}\otimes e_{C_1^k(a),C_1^kC_2^m(a)} - \exp(\frac{ku+mv}{n}) e_{C_1^k(a),C_1^kC_2^m(a)}\otimes e_{C_2^m(a),a}\Big\}, \end{aligned}$$

where we denote by $A(k,m) \subset A$ the set of all $a \in A$ such that $C_1^i C_2^j(a) \in A$ for all $0 \le i < k, 0 \le j < m$.

In [34] we also computed all the solutions of the AYBE coming from vector bundles over the nodal degenerations of elliptic curves, i.e., cycles of projective lines (aka standard m-gons). These solutions are trigonometric and correspond to some of the combinatorial data above. However, it turned out that not all trigonometric solutions of the AYBE appear in this way. Namely, it was also shown in [34] that the trigonometric solution of the AYBE, corresponding to the data (S, C_1, C_2, A), arises from a simple vector bundle on a cycle of projective lines if and only if the corresponding cyclic permutations C_1 and C_2 commute (equivalently, $C_2 = C_1^k$ for some k).

In [22] we observed that all trigonometric solutions of the AYBE corresponding to the associative BD-structures can be constructed by looking at the Massey products between appropriate objects in compact Fukaya categories of open Riemann surfaces. Namely, starting from an associative BD-structure (S, C_1, C_2, A), we construct a square-tiled surface Σ with a local symplectomorphism

$$\pi : \Sigma \to \mathbb{T}$$

to the square torus $\mathbb{T}$. In the case $A = \emptyset$, Σ is just the $n-$fold covering space of the punctured torus $\mathbb{T}_0$ associated to the permutations C_1, C_2. In the case of general A we fill in the holes in this n-fold covering, corresponding to elements of A. Lifts of standard Lagrangian curves in $\mathbb{T}$ to Σ give a pair of exact Lagrangians L_1 and L_2 in Σ. Now, we consider triple products between $(L_1^{x_1}, L_2^{y_1}, L_1^{x_2}, L_2^{y_2})$, where (L_1^x) and (L_2^y) are certain 1-parameter deformations of L_1 and L_2, and show that the corresponding solution of the AYBE is exactly the trigonometric solution associated with (S, C_1, C_2, A).

In order to find a purely algebro-geometric construction of trigonometric solutions, we will undertake a more systematic study of the corresponding moduli space of A_∞-structures.

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