

Elliptic Curves

Exercises set I

Throughout, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ is a lattice, $T_\Lambda = \mathbb{C}/\Lambda$, and $\wp = \wp_\Lambda$.

Exercise 1. For $\tau \in \mathbb{C}$ with $\Im \tau > 0$, the *normalized torus* is

$$T_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

Write the isomorphisms between the normalized tori T_τ , $T_{\tau+1}$, and $T_{-1/\tau}$, and compute their pullbacks on the differential dz .

Exercise 2 (*). Using the lifting property of the covering $\mathbb{C} \rightarrow T_{\Lambda'}$, prove that every holomorphic map $T_\Lambda \rightarrow T_{\Lambda'}$ has the form $[z] \mapsto [az + b]$, where $a\Lambda \subseteq \Lambda'$. Show that it is a biholomorphism exactly when $a \neq 0$ and $a\Lambda = \Lambda'$.

Exercise 3. Let f be nonconstant with total pole order d at r distinct points of the torus. Prove that f' has total pole order $d + r$, and hence that many zeros, counted with multiplicity.

Exercise 4. Use theorems proved in class to find all possible classes a for which $2[a] - 2[0]$ could be principal. For $\Lambda = \mathbb{Z} + i\mathbb{Z}$, show that $[0] + [1/2] - [i/2] - [1/3]$ is not principal.

Exercise 5. For $n \geq 1$, let $L(n[0])$ be the space of elliptic functions with no pole outside 0, and a pole of order at most n there. Prove that $\dim_{\mathbb{C}} L(n[0]) \leq n$, using Laurent coefficients and the sum of residues.

Exercise 6. Using Exercise 5, prove that for every $n \geq 2$,

$$1, \wp, \wp', \dots, \wp^{(n-2)}$$

is a basis of $L(n[0])$. This proves the one-pole genus-one dimension formula without using Riemann–Roch.

Exercise 7. If $a\Lambda = \Lambda$, prove $\wp(az) = a^{-2}\wp(z)$. Work this out for the square lattice $\mathbb{Z} + i\mathbb{Z}$ and the hexagonal lattice $\mathbb{Z} + \mathbb{Z}\rho$, where $\rho = e^{2\pi i/3}$, and determine which Laurent coefficients are forced to vanish.

Exercise 8 (*). Let $\Lambda \subset \Lambda'$ have finite index, and let $A \subset \Lambda'$ be representatives for Λ'/Λ , including 0. Prove

$$\wp_{\Lambda'}(z) = \sum_{a \in A} \wp_{\Lambda}(z - a) - \sum_{a \in A \setminus \{0\}} \wp_{\Lambda}(a).$$

Explain why the expression is independent of the chosen representatives.

For Exercises 9–11, use the notation

$$G_{2k} = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k} \quad (k \geq 2), \quad g_2 = 60G_4, \quad g_3 = 140G_6.$$

Exercise 9. Let e_1, e_2, e_3 be the roots of $4X^3 - g_2X - g_3$. Prove that

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}.$$

Exercise 10. Using $\wp'' = 6\wp^2 - g_2/2$, compute the coefficient of z^6 in $\wp$. Deduce $G_8 = \frac{3}{7}G_4^2$.

Exercise 11. Recall that the nonzero half-periods are

$$h_1 = \frac{\omega_1}{2}, \quad h_2 = \frac{\omega_2}{2}, \quad h_3 = \frac{\omega_1 + \omega_2}{2} \pmod{\Lambda},$$

and set $e_i = \wp(h_i)$. Prove

$$\operatorname{div}(\wp - e_i) = 2[h_i] - 2[0], \quad \operatorname{div}(\wp') = [h_1] + [h_2] + [h_3] - 3[0].$$

Check the weighted-sum condition explicitly for these divisors.