Solutions to selected homework problems. 3.

1.6.29. For $n = 1$ this is Theorem 1.13. Suppose $n > 1$ and the statement is true for $n-1$. Assume also we are given a and $b_1, \dots, b_n, c$ such that $a | b_1 \cdot b_2 \cdot \ldots \cdot b_n \cdot c$ and $(a, b_i) = 1$ for $i = 1, \dots, n$. Then we can apply the induction assumption to the numbers $b_1, \dots, b_{n-1}$ and $c' = b_n \cdot c$ to deduce that $a | b_n \cdot c$. Since $(a, b_n) = 1$, by Theorem 1.13, this implies that $a|c$.

1.6.44. For $n = 1$ this is true since $8|24 = 4!$. Suppose $n > 1$ and the statement is true for $n - 1$. Then we have $8^n = 8 \cdot 8^{n-1} | (4n)! \cdot 8$. But $(4(n+1))! = (4n+4)! = (4n)! \cdot (4n+1)(4n+2)(4n+3)(4n+4)$. It remains to observe that $8 | (4n+2)(4n+4)$, so $(4n)! \cdot 8 | (4n+4)!$.

1.6.60. Let $d = (a^n - 1, a^m + 1)$. Then $a^n \equiv 1 (\bmod d)$ and $a^m \equiv -1 (\bmod d)$. Raising the first congruence to the mth power and the second to the nth power we deduce that $a^{mn} \equiv 1 (\bmod d)$ and $a^{mn} \equiv (-1)^n (\bmod d)$. Hence $(-1)^n \equiv 1 (\bmod d)$. But n is odd, so we get $-1 \equiv 1 (\bmod d)$, i.e., $2 \equiv 0 (\bmod 2)$. This means that $d|2$, so $d = 1$ or 2, as required.

2.2.24. Let $d = (a, c)$. We can write $a = da'$, $c = dc'$, where $(a', c') = 1$. Note that $a^2 = d^2(a')^2$, $c^2 = d^2(c')^2$. Thus, the condition $a^2 | bc^2$ implies that $(a')^2 | b(c')^2$. Since $(a', c') = 1$, we also know that $((a')^2, (c')^2) = 1$ (problem 1.4.39). Hence, by Theorem 1.13, we deduce that $(a')^2 | b$. If $a' > 1$ this would imply that b is divisible by the square of some prime (pick a prime divisor of a'). Hence, we should have $a' = 1$, so $a = d$ divides c.