MATH 607: QUANTUM GROUPS. HOMEWORK 5

1. Recall that for any $\Psi \in Mat_{n^2}(k)$ the bialgebra $A(\Psi)$ is generated by $x_{ij}$, $i, j = 1, \ldots, n$ subject to the relations $\Psi X \bullet X = X \bullet X \Psi$, where $X \bullet X$ is an $n^2 \times n^2$-matrix with $(X \bullet X)_{(ij), (k\ell)} = x_{ik} x_{j\ell}$ for all $i, j, k, \ell = 1, \ldots, n$.
   (a) Find all matrices $\Psi \in Mat_4(k)$ such that $A(\Psi) = k_{p,q}[Mat_2]$.
   (b) How many of them satisfy the braid equation $\Psi_1 \Psi_2 \Psi_1 = \Psi_2 \Psi_1 \Psi_2$?

Let $B$ be a dual quasi-triangular bialgebra, i.e., $B$ has a self-pairing $\chi(\cdot, \cdot)$ such that
\[
\chi(ab,c) = \chi(b, c(1)) \chi(a, c(2)), \quad \chi(a, bc) = \chi(a(1), b) \chi(a(2), c),
\]
\[
a(1)b(1)\chi(a(2), b(2)) = \chi(a(1), b(1))b(2)a(2)
\]
for $a, b, c \in B$; and such that $\langle \cdot, \cdot \rangle$ is invertible as an element of the convolution algebra $Hom_k(B \otimes B, k)$.
Recall that these properties are equivalent to that the category of left $B$-comodules is naturally braided via $\Psi_{U,V} : U \otimes V \to V \otimes U$, where
\[
\Psi_{U,V}(u \otimes v) = \chi(u^{(-1)}, v^{(-1)}) \cdot v^{(0)} \otimes u^{(0)},
\]
where $\delta(u) = u^{(-1)} \otimes u^{(0)} \in B \otimes U$ and $\delta(v) = v^{(-1)} \otimes v^{(0)} \in B \otimes V$.

2. Let $B$ be a dual quasi-triangular bialgebra with the self-pairing $\langle \cdot, \cdot \rangle$.
   (a) Prove that each left $B$-comodule $V$ is also a right $B$-module via:
   \[
b(v) = \chi(v^{(-1)}, b) \cdot v^{(0)}.
\]
   (b) Prove that the for any $B$-comodule $V$ the opposite cross product $H = B \otimes T(V)$ (as in 3(a) with $H = B$, $\delta = \Delta$, but the order is reversed!) is a bialgebra with the co-multiplication $\Delta : H \to H \otimes H$ given by
   \[
   \Delta(v) = v \otimes 1 + \delta(v)
   \]
   for any $v \in V$ (the co-multiplication on $B$ is inherited) and the co-unit given by $\varepsilon(v) = 0$ for $v \in V$.
   (c) Show that if $B$ is a dual quasi-triangular Hopf algebra, then $H = B \otimes T(V)$ is also a Hopf algebra.

3. Let $H$ be a bialgebra over $k$ and let $A$ be a left $H$-module algebra, and $B$ be a left $H$-comodule algebra.
   (a) Define a "twisted" product (a.k.a. Takeuchi product) on $A \otimes B$ by
   \[
   (a' \otimes b)(a \otimes b') = a' \cdot (b^{(-1)}(a)) \otimes b^{(0)} \cdot b'
   \]
   for $a, a' \in A$, $b, b' \in B$, where we used the Sweedler notation $\delta(b) = b^{(-1)} \otimes b^{(0)}$ for the coaction. Prove that $A \otimes B$ is an associative algebra with this product.
   (b) For any $H$-module $U$ and any $H$-comodule $V$ one has a twisted product algebra structure on $T(U) \otimes T(V)$.

In the following problem we will deal with braided tensor categories $(C, \Psi)$. It is safe to assume that $(C, \Psi)$ is the category of left comodules over a dual quasi-triangular bialgebra $B$ as above.
4. If \((A, \Delta, \varepsilon)\) is a coalgebra in \((\mathcal{C}, \Psi)\), and \(R\) is an algebra in \((\mathcal{C}, \Psi)\), denote multiplication in \(R\) by \(m_R : R \otimes R \to R\), and define the convolution product on \(\text{Hom}_k(A, R)\) by:

\[ f \ast g = m_R \circ (f \otimes g) \circ \Delta . \]

(a) Show that the convolution product makes \(\text{Hom}_k(A, R)\) into an associative algebra in \((\mathcal{C}, \Psi)\) with unit element \(\varepsilon\) (where we consider \(\varepsilon\) as a map \(A \to R\) via \(k \hookrightarrow R\)).

(b) Assume from now on that \(A\) is a bialgebra in \((\mathcal{C}, \Psi)\). Prove that \(S \in \text{Hom}_k(A, A)\) is an antipode if and only if \(S\) is a two-sided inverse to the identity map \(id_A \in \text{Hom}_k(A, A)\) with respect to the convolution product. Deduce that an antipode is unique if it exists.

(c) Show that \(A \otimes \Psi A\) is a bialgebra with the usual product, and coproduct defined by

\[ \Delta(a \otimes a') = a_{(1)} \Psi(a_{(2)} \otimes a'_{(1)})a'_{(2)} \]

(using Sweedler notation with summation convention), and the counit \(\varepsilon(a \otimes a') = \varepsilon(a)\varepsilon(a')\).

(d) Let \(m^{op} : A \otimes A \to A\) be the opposite multiplication, \(m^{op}(a \otimes a') = m \circ \Psi(a \otimes a')\). If \(A\) has an antipode \(S\), prove that \(S \circ m^{op}\) is a 2-sided convolution inverse to \(m^{op}\) in \(\text{Hom}_k(A \otimes A, A)\).

(e) Show that in the situation of part (d), \(m \circ (S \otimes S)\) is also a 2-sided convolution inverse to \(m^{op}\). Deduce that an antipode \(S\) is necessarily an anti-homomorphism, i.e. the identity \(S(ab) = m \circ \Psi(S(a) \otimes S(b))\) holds.

(f) Show that an antipode is necessarily also an anti-homomorphism with respect to the coproduct, i.e., \(\Delta(S(a)) = \Psi(S(a_{(1)}) \otimes S(a_{(2)}))\) in Sweedler notation.

(g) Given a Hopf algebra \(H\) in \((\mathcal{C}, \Psi)\), show that the association

\[ (Ad a)(b) := a_{(1)}m\Psi(S(a_{(2)}) \otimes b) \]

defines a (braided) Hopf algebra action of \(H\) on itself, i.e., it satisfies:

\[ (Ad a)(bc) = ((Ad a_{(1)})(b'))((Ad a'_{(2)})(c)) \]

for all \(a, b, c \in H\), where \(b' \otimes a'_{(2)} = \Psi(a_{(2)} \otimes b)\). (The action \(Ad\) it is known as the adjoint action).