GROUPS GENERATED BY INVOLUTIONS, GELFAND-TSETLIN PATTERNS AND COMBINATORICS OF YOUNG TABLEAUX

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ABSTRACT

We construct families of piecewise linear representations (cpl-representations) of the symmetric group S_n and the affine Weyl group $\widetilde{S}_n$ of type $A_{n-1}^{(1)}$ acting on the space of triangles X_n. We find a nontrivial family of local cpl-invariants for the action of the symmetric group S_n on the space X_n and construct one global invariant w.r.t. the action of the affine Weyl group $\widetilde{S}_n$ (so called cocharge). We find the continuous analogs for the Kostka-Foulkes polynomials and for the crystal graph. We give an algebraic version of some combinatorial transformations on the set of standard Young tableaux.

Introduction.

In this paper we define and study a new class of representations of the symmetric group S_n, namely, the continuous piecewise linear representations (cpl-representations) of S_n in the space of triangles X_n. By definition, a triangle $x \in X_n$ is a triangular array of real numbers $x = (x_{ij})$, $x_{ij} \in \mathbf{R}$, $1 \le i \le j \le n$. More precisely, following [GZ1],[GZ2],[BZ1], we consider the Gelfand-Tsetlin cone K_n consisting of all triangles $x \in X_n$ such that

$$\begin{aligned} &x_{ij} \ge x_{i+1,j+1}, \quad 1 \le i \le j \le n-1, \\ &x_{ij} \ge x_{i,j-1}, \quad 1 \le i \le j \le n, \\ &x_{ij} \ge 0, \quad 1 \le i \le j \le n. \end{aligned}$$

This is a nondegenerate convex polyhedral cone in the space $X_n \cong \mathbf{R}^{\frac{n(n+1)}{2}}$, having 2^n generators (see Remark 2.2). It is well known (e.g. [GZ1]) that the integral points set, $(K_n)_{\mathbf{Z}}$, of the cone K_n is in a one-to-one correspondence with the set $STY(n)$ of standard Young tableaux, having all entries not exceeding n. The *cpl*-action of the symmetric group S_n on the space X_n, given in our paper, is such that it conserves the Gelfand-Tsetlin cone K_n and that on the set $STY(n)$ it coincides with the action of the symmetric group on the set of standard Young tableaux given by A. Lascoux and M.-P. Schützenberger [LS2], [LS3] (see Theorem 2.3).

Our main observation is that a great many of combinatorial constructions on the set of standard Young tableaux, e.g. the Schützenberger involution [Sch1],[EG], [Ki1], the dual Schützenberger involution [Sch1], a promotion transformation [Sch1], [EG], the action of the symmetric group [LS2],[LS3], the crystal graph structure on the set $STY(\lambda, \leq n)$, [Ka1],[Ka2], the construction of cocharge [LS1],[Ki1], may be transfered to the Gelfand-Tsetlin cone K_n, and even to the whole space of triangles X_n. Our constructions are based on a consideration of "elementary transformations" $t_j,\ 1 \leq j \leq n-1$, and $T_\epsilon,\ \epsilon \in \mathbf{R}$.

Definition 0.1 Assume that $x \in X_n$, then $t_j(x) = \widetilde{x},\ \ T_\epsilon(x) = \widetilde{\widetilde{x}}$, where

$$\begin{aligned} &\widetilde{x}_{ik} = x_{ik}, \quad \text{if } k \neq j, \\ &\widetilde{x}_{ij} = min(x_{i,j+1},\ x_{i-1,j-1}) + max(x_{i,j-1}\ x_{i+1,j+1}) - x_{ij}, \end{aligned} \tag{0.1}$$

and we presuppose that $x_{0j} := +\infty,\ x_{j,j-1} := -\infty,\ \ 1 \leq j \leq n-1$;

$$\begin{aligned} &\widetilde{\widetilde{x}}_{ik} = x_{ik}, \quad \text{if } (i,k) \neq (1,1), \\ &\widetilde{\widetilde{x}}_{11} = x_{11} - \epsilon. \end{aligned}$$

We denote by $G_n =< t_1, \cdots, t_{n-1} >$ the group generated by $t_i,\ \ 1 \leq i \leq n-1$. The transformations $t_i,\ \ 1 \leq i \leq n-1$, satisfy the following relations (see Corollary 1.1):

$$\begin{aligned} &(i)\ t_i^2 = 1, \quad t_it_j = t_{ji}, \quad \text{if } |j-i| \geq 2, \\ &(ii)\ (t_1t_2)^6 = 1, \\ &(ii)\ (t_1q_i)^4 = 1, \quad \text{if } 3 \leq i \leq n-1, \end{aligned} \tag{0.2}$$

where $q_i := t_1 \underbrace{t_2t_1} \underbrace{t_3t_2t_1} \cdots \underbrace{t_it_{i-1}\cdots t_1}$.

The restriction of the involutions t_i to the set of standard Young tableaux $STY(\lambda,\beta)$ of a given shape λ and content β admits a simple combinatorial interpretation. It is these restrictions that are ordinary used in order to prove that Schur functions are the symmetric, e.g. [BK],[SW],[Sa] and Section 2A. We assume that the relations (0.2) are the defining relations for the group G_n. By any way, the group G_n seems to be very interesting. It is easy to see that the order of the group G_3 is equal to 12. But if $n \geq 4$ then G_n is infinite and for any N there exist an epimorphism of the group G_4 on the symmetric group S_N (see comments after Corollary 1.3). The group G_n admits an extension $\widetilde{G}_n$ by means of $\mathbf{R}^1$:

$$\widetilde{G}_n \ :=< t_1, \cdots, t_{n-1}, T_\epsilon, \epsilon \in \mathbf{R} > .$$

We have the following relations between the generators in the group $\widetilde{G}_n$:

$$
\begin{aligned}
&(i)\quad T_\epsilon \cdot T_\delta = T_{\epsilon+\delta},\\
&(ii)\quad (t_1 T_\epsilon)^2 = 1,\quad t_j T_\epsilon t_j = T_\epsilon,\quad 3 \le j \le n-1,\quad \epsilon \in \mathbf{R},\\
&(iii)\quad t_i t_j = t_j t_i,\quad \text{if } |j-i| \ge 2,\\
&(iv)\quad (T_\epsilon t_2 t_1 T_\delta t_2 t_1)^3 = 1,\quad \text{for any } \epsilon, \delta \in \mathbf{R},\\
&(\mathrm{v})\quad T_\epsilon t_2 T_{-\epsilon+\delta} t_2 T_{-\delta} t_2 = t_2 T_\delta t_2 T_{-\delta+\epsilon} t_2 T_{-\epsilon},\\
&(\mathrm{v}i)\quad (t_1 T_\epsilon q_j t_1 T_\delta q_j)^2 = 1,\quad 3 \le j \le n-1,\\
&(\mathrm{v}ii)\quad T_\epsilon q_j T_\delta q_j = q_j T_\delta q_j T_\epsilon,\quad 3 \le j \le n-1,\ \epsilon, \delta \in \mathbf{R},
\end{aligned}
\tag{0.3}
$$

where transformation q_j is defined in (0.2).

The proofs of the relations (i)-(v) are based on direct computations. The main difficulties arise in the proof of the (vi). In order to understand better the relations (0.2) and (0.3), let us consider the following elements in the group $\widetilde{G}_n$:

$$
s_i := q_i t_1 q_i^{-1},\quad 1 \le i \le n-1, \tag{0.4}
$$

$$
s_i^\epsilon = q_i T_{-\epsilon} q_i^{-1},\quad 1 \le i \le n-1, \tag{0.5}
$$

The main result of Section 1 is Theorem 1.1, which is equivalent to the relations (i)-(vi), and asserts that

1^0. The involutions s_i, $1 \le i \le n-1$, satisfy the relations of the symmetric group S_n, i.e.

$$
\begin{aligned}
&a)\quad s_i^2 = 1,\quad 1 \le i \le n-1,\\
&b)\quad (s_i s_{i+1})^3 = 1,\quad 1 \le i \le n-2,\\
&c)\quad s_i s_j = s_j s_i \quad \text{if } |i-j| \ge 2.
\end{aligned}
\tag{0.6}
$$

2^0. The transformations $s_i^{(\epsilon)}$, $1 \le i \le n-1$, $\epsilon \in \mathbf{R}$, satisfy the colored braid relations, i.e. for any $\epsilon, \delta \in \mathbf{R}$ we have

$$
\begin{aligned}
&a)\quad s_i^{(\epsilon)} \cdot s_i^{(\delta)} = s_i^{(\epsilon+\delta)},\quad 1 \le i \le n-1,\\
&b)\quad s_i^{(\epsilon)} \cdot s_{i+1}^{(\epsilon+\delta)} \cdot s_i^{(\delta)} = s_{i+1}^{(\delta)} \cdot s_i^{(\epsilon+\delta)} \cdot s_{i+1}^{(\epsilon)},\quad 1 \le i \le n-2,\\
&c)\quad s_i^{(\epsilon)} \cdot s_j^{(\delta)} = s_j^{(\delta)} \cdot s_i^{(\epsilon)},\quad \text{if } |i-j| \ge 2.
\end{aligned}
\tag{0.7}
$$

The relation (0.3),(vi) is equivalent to the statement that the transformations $s_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute if $|i-j| \ge 2$. In oder to prove the last statement, we use another expression for $s_i^{(\epsilon)}$, $1 \le i \le n-1$, as a product of the Lusztig involutions [Lu2]. We recall the corresponding definitions, because we use not exactly the same involutions that contained in [Lu2], but their analogs for the space of triangles X_n.

Definition 0.2. For each triple of integers (ijk), $1 \le i < j < k \le n$, let us define a transformations $R_{ijk}: X_n \to X_n$ in the following manner

$$R_{ijk}(x) = \widetilde{x}, \quad x \in X_n,$$

where

$$\begin{aligned} &\widetilde{x}_{i,j} = x_{i,j} + x_{i,k} - x_{i,k-1} - min(x_{j,k} - x_{j,k-1},\ x_{i,j} - x_{i,j-1}), \\ &\widetilde{x}_{j,j} = x_{j,j} - x_{i,k} + x_{i,k-1} + min(x_{j,k} - x_{j,k-1},\ x_{i,j} - x_{i,j-1}), \\ &\widetilde{x}_{\alpha\beta} = x_{\alpha\beta}, \quad \text{if } (\alpha, \beta) \ne (i,j) \ \text{ or } (j,j). \end{aligned} \tag{0.8}$$

Let us denote by L_n the group generated by all R_{ijk} with $1 \le i < j < k \le n$. We have the following relations between the generators in the group L_n:

$$\begin{aligned} &(i) \quad (R_{ijk})^2 = 1, \\ &(ii) \quad R_{ijk} \cdot R_{i'j'k'} = R_{i'j'k'} \cdot R_{ijk}, \quad \text{if } |(ijk) \cap (i'j'k')| \ne 2, \\ &(iii) \ (R_{ijk} R_{ijl} R_{ikl} R_{jkl})^2 = 1, \\ &(\text{iv}) \quad (R_{ijl} R_{ikl})^3 = 1, \quad \text{if } 1 \le i < j < k < l \le n. \end{aligned} \tag{0.9}$$

We assume that the relations (0.9) are the defining ones for the group L_n.

To go further, let us define the transformations $T_\epsilon^{(i)}$ acting on the space X_n:

$$T_\epsilon^{(i)} = \widetilde{x}, \quad x \in X_n, \quad \text{where}$$
$$\widetilde{x}_{ii} = x_{ii} - \epsilon, \ \widetilde{x}_{\alpha\beta} = x_{\alpha\beta}, \quad \text{if } (\alpha, \beta) \ne (i,j).$$

The statement that $s_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute if $|i-j| \ge 2$, follows from the following expression for $s_i^{(\epsilon)}$, $1 \le i \le n-1$, (Theorem 1.2):

$$s_i^{(\epsilon)} = R_{i-1,i} R_{i-2,i} \cdots R_{1,i} \, T_\epsilon^{(i)} \, R_{1,i} \cdots R_{i-2,i} R_{i-1,i}, \tag{0.10}$$

where $R_{jk} := R_{jkk+1}$, $1 \le j < k \le n-1$. The proof of identity (0.10) is based on induction and on the recurence formula for $s_i^{(\epsilon)}$ (see (1.29)).

In order to obtain the corresponding results for the involutions s_i (see (1.28a)), we use

Crucial Lemma. Assume $x \in X_n$, $\beta(x) = (\beta_1, \cdots, \beta_n)$, then

$$s_i(x) = s_i^{(\beta_i - \beta_{i+1})}(x), \tag{0.11}$$

where $\beta(x)$ is a weight of the triangle $x \in X_n$, i.e. $\beta_i(x) := |x^{(i)}| - |x^{(i-1)}|$, $1 \le i \le n$, $x^{(0)} := \phi$.

The relations (0.3) between generators of the group $\widetilde{G}_n$ allow us to construct many interesting subgroups in $\widetilde{G}_n$. For example

1^0. Let $\epsilon \in \mathbf{R}$ be fixed, then the elements $s_1 s_1^{(\epsilon)}$, $s_2 s_2^{(\epsilon)}$, $\cdots$, $s_{n-1} s_{n-1}^{(\epsilon)}$ are the standard generators of the symmetric group S_n.

2^0. Let $\epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1}$ be the real numbers, $\epsilon_0 \neq 0$, and $\widetilde{s}_i := \widetilde{s}_i^{(\epsilon_i)} = s_i s_i^{(\epsilon_i)}$ for $i = 1, \cdots, n-1$. Let us put

$$\widetilde{s}_0 := \widetilde{s}_0^{(\epsilon_0)} = \widetilde{s}_{n-1}\widetilde{s}_{n-2}\cdots\widetilde{s}_1 s_1^{(\epsilon_0)}\widetilde{s}_2\cdots\widetilde{s}_{n-1}.$$

Then $\widetilde{s}_0, \widetilde{s}_1, \cdots, \widetilde{s}_{n-1}$ are the standard generators of the affine Weyl group of the type $A_{n-1}^{(1)}$. It is important that the cocharge $\overline{c}_n(x)$ of a triangle $x \in X_n$ (Section 3) is an invariant w.r.t. the action of any involutions $\widetilde{s}_i^{(\epsilon_i)}$, $0 \leq i \leq n-1$.

Let us summarize the content of the Section 1 of our paper. We construct a family of the *cpl*-representations of the symmetric group, find a family of the *cpl*-representations of the affine Weyl group of the type $A_{n-1}^{(1)}$ (see item after Corollary 1.3) and construct a *cpl*-representation of the colored braid relations. We developed a geometric techniques for proving some (non trivial!) identities between piecewise linear functions (see Theorem 1.3). Our next step is to give a combinatorial interpretation of the transformations under consideration and to study the continuous piecewise linear invariants w.r.t. the action of the symmetric group generated by $s_1 \cdots, s_{n-1}$, on the space of triangles X_n. In Section 3 we prove (see Theorem 3.2) that the following *cpl*-functions on the space of triangles X_n are s_j-invariants:

$$\begin{aligned} \psi_{1j}(x) &= min(x_{1,j+1} - x_{1,j},\ x_{j,j} - x_{j+1,j+1}), \\ \psi_{ij}(x) &= min(x_{i-1,j} - x_{i-1,j-1},\ x_{i,j+1} - x_{i,j}), \end{aligned} \tag{0.12}$$

where $2 \leq i \leq j \leq n-1$,

$$\begin{aligned} \overline{\psi}_{1j}(x) &= (min(x_{j-1,j} + x_{j,j} - x_{j-1,j-1} - x_{j,j+1}, \\ &\qquad x_{1,j+1} + x_{j+1,j+1} - x_{1,j} - x_{j,j}))_+, \\ \overline{\psi}_{2j}(x) &= (min(x_{1,j} + x_{j,j} - x_{1,j+1} - x_{j+1,j+1}, \\ &\qquad x_{1,j-1} + x_{2,j+1} - x_{1,j} - x_{2,j}))_+, \\ \overline{\psi}_{ij}(x) &= (min(x_{i-2,j} + x_{i-1,j} - x_{i-2,j-1} - x_{i-1,j+1}, \\ &\qquad x_{i-1,j-1} + x_{i,j+1} - x_{i-1,j} - x_{i,j}))_+, \end{aligned} \tag{0.13}$$

where $3 \leq i \leq j \leq n-1$, and for any $a \in \mathbf{R}$, $(a)_+ := max(a, 0)$. We recall that a *cpl*-function φ defined on the space X_n is said to be an s_j-invariant if $\varphi(s_j(x)) = \varphi(x)$ for all $x \in X_n$.

It seems a very interesting task to find a fundamental system I_j of *cpl*-invariants for s_j, that is a system such that any *cpl*-invariant under the action of s_j is a $min - max$ linear combination of that from I_j. It is not clear whether or not this set is finite, because we have the trivial examples of the following kind

$$x_{ik}, \quad \text{if } k \neq j, \quad \text{or} \quad min(\varphi(x),\ \varphi(s_j(x))),$$

where $\varphi(x)$ is any *cpl*-function on X_n. However, the number of fundamental *cpl*-s_j-invariants having a complexity bounded by some integer k, is finite. Here we define the complexity of a *cpl*-function as the least number of min and max needed for its representation. [*Remark.* In a previous definition of complexity we considered min and max as functions of two variables, e.g. $min\ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. Thus, the complexity of the function $min(x_1, x_2, \cdots, x_m)$ is equal to $m-1$]. By any way, we may construct (see the proof of Theorem 3.2) other nontrivial examples of *cpl*-s_j-invariants and it is not clear whether or not they are independent (in *cpl*-sense) from the invariants (0.12) and (0.13).

The next interesting problem is to describe the fundamental system of continuous piecewise polynomial functions (*cpp*-functions) on the space X_n, which are invariant w.r.t. the action of the subgroup $S_n \subset G_n$. Of course, we have such trivial example as: $x_{in},\ \ 1 \le i \le n$, or $min\{\ \varphi(\sigma(x)) \mid \sigma \in S_n\}$, or $\sum_{\sigma \in S_n} \varphi(\sigma(x))$, where $\varphi(x)$ is any *cpp*-function on X_n. But the question is the following one: does there exist nontrivial *cpl*-invariants under the action of the symmetric group S_n on the space X_n? In Section 3 we give an affirmative answer to this question (see Theorem 3.1). In fact we define a stable (ibid), *cpl*-invariant w.r.t. the action of a family of the affine Weyl groups of type $A_{n-1}^{(1)}$, namely, $\widetilde{S}_n := < \widetilde{s}_0, \widetilde{s}_1, \cdots, \widetilde{s}_{n-1} >$, see Corollary 1.3, which takes nonnegative values on the Gelfand-Tsetlin cone K_n. Note that the complexity of the invariant under consideration is equal to $(n-1)! - 1$, if $n \ge 3$. An origin of such invariant is very clear. We have a distinguished S_n-invariant function on the set $STY(n)$, namely, the cocharge $\overline{c}_n(T)$ of a tableau $T \in STY(n)$ as defined by A. Lascoux and M.-P. Schützenberger [LS1]. Note that it is possible (at least for $n \le 5$ and hypothetically for all n) to reformulate the definition of the cocharge of a tableau $T \in STY(n)$ in terms of the corresponding Gelfand-Tsetlin pattern $x(T)$ and obtain some function $\overline{c}_n$ on the set $(K_n)_{\mathbf{Z}}$. We consider an extension of the function $\overline{c}_n$ to the whole space X_n in a natural manner, changing its domain of definition from $(K_n)_{\mathbf{Z}}$ to X_n. This gives us the desired *cpl*-$\widetilde{S}_n$-invariant, $\overline{c}_n(x),\ \ x \in X_n$, which we still call cocharge. Details are contained in Section 3. We define another *cpl*-S_n-invariant $\psi(x)$ (which does't $\widetilde{S}_n$-invariant!) by setting $\psi(x) = \overline{c}_n(q_{n-1}(x)),\ x \in X_n$, where the involution q_{n-1} is defined in (0.2). As an example we give an expression for cocharge $\overline{c}_4$:

$$\overline{c}_4(x) = min(x_{13} - x_{12},\ x_{22} - x_{33}) + x_{14} - x_{13} + x_{33} - x_{44} +$$
$$+ min(x_{23} - x_{34},\ x_{24} - x_{23},\ x_{13} - x_{12},\ x_{22} - x_{33},\ \beta_4(x) - x_{34},\ x_{24} - \beta_4(x)).$$

It is known (e.g. [H]) that the volume of the convex polytope $K^{\lambda}(\beta)$ consisting of all Gelfand-Tsetlin patterns with highest weight λ and weight β (see Section 1) may be considered as a continuous analog of the weight multiplicity $K_{\lambda,\beta} := dimV_{\lambda}(\beta)$. We define a continuous analog of the Kostka-Foulkes polynomial (q-analog of the weight multiplicity) [LS1],[Ma],[Ki1],by means of following integral

$$K_{\lambda,\beta}(q) = \int_{K^{\lambda}(\beta)} exp(h\overline{c}_n(x))dx, \quad q = exp(h). \tag{0.14}$$

We expect to study the properties of the integral (0.14) and the toric variety corresponding to the Gelfand-Tsetlin cone K_n in a separate publication. In Remark

3.2 we give a generalization of the Lascoux-Schützenberger algorithm for computing the charge of a dominant weight standard Young tableau to the case of a standard Young tableau with an arbitrary weight.

In Section 2 we study the restrictions of the transformations considered in Section 1 to the integral points set $(K_n)_{\mathbf{Z}}$ of the Gelfand-Tsetlin cone K_n. We show that the involution $q_i \ : \ STY(n) \to STY(n), \ 1 \le i \le n-1$, (see (0.2)) coincides with the partial Schützenberger involution $\mathbf{S}_i \ : \ STY(n) \to STY(n)$. Here for a given tableau $T \in STY(n)$, the involution $\mathbf{S}_i$ acts non trivially only on the part $T_{\le i+1}$ of the tableau T filling by the numbers $1, \cdots, i+1$, and on this part coincides with the ordinary Schützenberger's involution [Sch1] or Section 2C. Let us note that the group generated by the involutions $q_i, \ \ 1 \le i \le n-1$, coincides with the group $G_n =< t_1, \cdots, t_{n-1} >$, and thus we may describe the relations between the partial Schützenberger involutions (see Remark 1.3). Further we have the following relation between the partial Schützenberger involution q_i and the action of the symmetric group S_n on the set $STY(n)$ as defined by A. Lascoux and M.-P. Schützenberger [Sch2],[Sch3]:

$$s_i = q_i q_1 q_i, \quad 1 \le i \le n-1, \tag{0.15}$$

where $s_i := (i, \ i+1) \in S_n$ is a simple transposition.

The Schützenberger involution $\mathbf{S}$ posesses many interesting properties in connection with the Robinson-Schensted correspondence [Sch2], with a rigged configurations [Ki1] and so on. In addition we explain in Remark 2.4 that the involution $\mathbf{S}$ allows to give a simple pure combinatorial proof of the following symmetry property of the Littlewood-Richardson numbers (see Proposition 2.8)

$$c_{\lambda\mu}^{\nu} = c_{\lambda\nu*}^{\mu*}. \tag{0.16}$$

All other symmetries of the LR-numbers follow from (0.16) and the symmetries of the Berenstein-Zelevinsky triangles (see [BZ3] and Remark 2.4).

Finally, it is interesting to note the following connection between our transformations $s_i^{(\pm 1)}, \ 1 \le i \le n-1$, restricted on the integral points set of the convex polytope K^{λ} (see Section 1), and the crystal graph corresponding to the irreducible representation V_{λ} of the Lie algebra $\mathfrak{gl}_n$ with the highest weight λ, [Ka1],[Ka2], [KN].

Namely, let us denote by f_i (respectively e_i) the restriction of the map $s_i^{(-1)}$ (resp. $s_i^{(+1)}$) on the GT-polytope K^{λ} [Remark: the transformations $s_i^{(\pm 1)}$ does't conserve GT-cone K_n, so for $x \in K_n$ we define $f_i(x) = s_i^{(-1)}(x)$, if $s_i^{(-1)}(x) \in K_n$ and $f_i(x) = 0$, if $s_i^{(-1)}(x) \notin K_n$]. Let further $(L(\lambda), \ B(\lambda))$ be the crystal base of an irreducible $\mathfrak{gl}_n$-module V_{λ} with highest weight λ and $\widetilde{F}_i, \ \widetilde{E}_i, \ \ 1 \le i \le n-1$, be the deformed generators of the Hopf algebra $U_q(\mathfrak{gl}_n)$ as defined by M. Kashiwara [Ka1], [Ka2]. Then there exist a bijection

$$N \ : \ B(\lambda) \to K^{\lambda} \cap (X_n)_{\mathbf{Z}}$$

such that

$$(i) \quad b \in B(\lambda)_\beta := B(\lambda) \cap V_\lambda(\beta), \quad \text{iff } N(b) \in K^\lambda(\beta) \cap (X_n)_{\mathbf{Z}},$$
$$(ii) \ \text{if } \widetilde{F}_i(b) \in B \ (\Longleftrightarrow \widetilde{F}_i(b) \neq 0), \ \text{then } N(\widetilde{F}_i(b)) = s_i^{(-1)}(N(b)),$$
$$(iii) \ \text{if } \widetilde{E}_i(b) \in B \ (\Longleftrightarrow \widetilde{E}_i(b) \neq 0), \ \ \text{then } N(\widetilde{E}_i(b)) = s_i^{(+1)}(N(b)).$$

The classical representation theory of the symmetric and general linear groups is based on different combinatorial constructions among which the ones with Young tableaux play an essential role. The main reason is that Young tableaux in a natural way parametrize a basis of irreducible representation of the symmetric or general linear groups [Ru],[JK],[St]. In some sense the choice of a concrete realization of an irreducible representation predetermine the corresponding combinatorial structures. If we consider the realization of irreducible representations of the general linear group in the space of the Gelfand-Tsetlin patterns, [GZ1], then we deal with combinatorics of convex polytopes of a special kind. If we consider the realization of representations of the symmetric (or general linear) group by means of Specht (or Weyl) module, [JK], then we deal with combinatorics of Young tableaux. While the combinatorics of Young tableaux is extensivly developed (see e.g. [Sa]), the combinatorics of the Gelfand-Tsetlin patterns seems only began to be studied [GZ1],[GZ2],[BZ1].

The main goal of our paper is to show that the majority of combinatorial constructions over Young tableaux admits an "algebraization" and may be defined on the space of triangles. Of course we did not exhaust the subject and we believe that other combinatorial constructions, e.g. Robinson-Schensted correspondence (e.g. [Sch2]), also admit natural continuous analogs.

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§1. Groups acting on the space of triangles.

Let n be a positive integer. In this section we define the group G_n generated by involutions and $\widetilde{G}_n$ its extension by means of $\mathbf{R}^1$, which acts on a space of triangles X_n. By definition the space X_n consists of all sequences

$$x = (x^{(n)}, x^{(n-1)}, \cdots, x^{(1)}),$$

where $x^{(j)} = (x_{1j}, \cdots, x_{jj}) \in \mathbf{R}^j$. As a vector space $X_n \simeq \mathbf{R}^{\frac{n(n+1)}{2}}$. We will call the vector $x^{(n)} \in \mathbf{R}^n$ the highest weight of the triangle $x \in X_n$ and denote it by $\lambda(x) := x^{(n)}$. Let us define a weight $\beta := \beta(x)$ of a triangle $x \in X_n$ as a vector $\beta = (\beta_1, \cdots, \beta_n) \in \mathbf{R}^n$ such that

$$\beta_j = |x^{(j)}| - |x^{(j-1)}|, \;\; 1 \le j \le n, \tag{1.1}$$

where $|x^{(j)}| := \sum_{i=1}^{j} x_{ij}, \; x^{(0)} = \phi$. For given vectors $\lambda \in \mathbf{R}^n$ and $\beta \in \mathbf{R}^n$ we define the following subspaces of the space X_n:

$$\begin{aligned} X^\lambda &= \{x \in X_n \;\mid\; \lambda(x) = \lambda\}, \\ X^\lambda(\beta) &= \{x \in X^\lambda \;\mid\; \beta(x) = \beta\}. \end{aligned} \tag{1.2}$$

Now we define the Gelfand-Tsetlin patterns [GZ1],[GZ2],[BZ1],[BZ2]. By definition a triangle $x \in X_n$ is called a Gelfand-Tsetlin pattern (GT-pattern) iff the following inequalities are satisfied

$$\begin{aligned} &x_{ij} \ge x_{i+1,j+1}, \;\; 1 \le i \le j \le n-1, \\ &x_{ij} \ge x_{i,j-1}, \;\; 1 \le i < j \le n, \\ &x_{ij} \ge 0, \;\; 1 \le i \le j \le n. \end{aligned} \tag{1.3}$$

We denote by the set of all GT-patterns $K = K_n \subset X_n$ and those which lie in the subspaces (1.2) correspondingly by K^λ and $K^\lambda(\beta)$. It is clear that K^λ and $K^\lambda(\beta)$ are convex, compact polytopes in the space $\mathbf{R}_+^{\frac{n(n+1)}{2}}$. It is well known (see §2, or [GZ2]), that if λ is a partition, and β is a composition, then the number of integral points in the convex polytope K^λ (respectively in $K^\lambda(\beta)$) is equal to the dimension of the irreducible representation V_λ of the Lie algebra $\mathfrak{gl}_n$ with the highest weight λ (respectively, the dimension of the subspace $V_\lambda(\beta) \subset V_\lambda$ of the weight β). On the other side, as is also well known, the dimension of the weight subspace $V_\lambda(\beta) \subset V_\lambda$ admits a pure combinatorial description as the number of standard Young tableaux of shape λ and content β :

$$|K^\lambda(\beta) \cap \mathbf{Z}^{\frac{n(n+1)}{2}}| = dim V_\lambda(\beta) = |STY(\lambda, \beta)|. \tag{1.4}$$

The equality (1.4) is a starting point of our investigations. We will try to translate the combinatorial information about the set of standard Young tableaux $STY(\lambda, \beta)$ into the language of GT-patterns and vice versa.

Now let us begin the construction of the main objects of this note: the group G_n and its extension $\widetilde{G}_n$. At first we define the "elementary" transformations t_j of the space of triangles X_n. For this purpose, let us fix a positive integer $j,\ 1 \le j < n$, and introduce the sequences of numbers $a_1, \cdots, a_j$ and $b_1, \cdots, b_j$. Given a triangle $x = (x^{(n)}, \cdots, x^{(1)}) \in X_n$, we define

$$\begin{aligned} &a_1 = x_{1,j+1},\ \ a_i = min(x_{i,j+1}, x_{i-1,j-1}),\ \ 2 \le i \le j,\\ &b_j = x_{j+1,j+1},\ \ b_i = max(x_{i,j-1}, x_{i+1,j+1}),\ \ 1 \le i \le j-1. \end{aligned} \tag{1.5}$$

Definition 1.1. The transformation $t_j : X_n \to X_n$ is given by the following formulae

$$\begin{aligned} &t_j(x) = \widetilde{x},\ \ \text{where}\\ &\widetilde{x}_{ij} = a_i + b_i - x_{ij},\ \ 1 \le i \le j\\ &\widetilde{x}_{kl} = x_{kl},\ \ \text{if}\ \ l \ne j. \end{aligned} \tag{1.6}$$

Proposition 1.1. We have

$$a)\ \ t_j^2 = 1,\ \ 1 \le j \le n-1, \tag{1.7}$$

$$b)\ \ t_i t_j = t_j t_i,\ \ \text{if}\ \ \ |i-j| \ge 2, \tag{1.8}$$

$$c)\ \ \beta(t_j(x)) = (j, j+1) \cdot \beta(x),\ \ 1 \le j \le n-1, \tag{1.9}$$

where the action of transposition $(j, j+1)$ on the weight space $\mathbf{R}^n$ is given by $(j, j+1)(\beta_1, \cdots, \beta_n) = (\beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_n)$.

Proof. The assertions $a)$ and $b)$ follow directly from the definition (1.6) of the transformation t_j. As for $c)$, let us remark, that

$$b_i + a_{i+1} = x_{i,j-1} + x_{i+1,j+1},\ \ 1 \le i \le j-1.$$

So

$$|\widetilde{x}^{(j)}| = \sum_i (a_i + b_i) - |x^{(j)}| = |x^{(j+1)}| - |x^{(j)}| + |x^{(j-1)}|.$$

Consequently,

$$\beta_{j+1}(\widetilde{x}) = |x^{(j+1)}| - |\widetilde{x}^{(j)}| = |x^{(j)}| - |x^{(j-1)}| = \beta_j(x)$$

and

$$\beta_j(\widetilde{x}) = |\widetilde{x}^{(j)}| - |x^{(j-1)}| = |x^{(j+1)}| - |x^{(j)}| = \beta_{j+1}(x).$$

∎

Now let us consider the restriction of the action of t_j on the Gelfand-Tsetlin polytopes. Fix vectors λ and β in the space $\mathbf{R}^n$.

Proposition 1.2. We have for $1 \leq j \leq n-1$

$$a)\ t_j : \ K_n \to K_n, \tag{1.10}$$

$$b)\ t_j : \ K^\lambda \to K^\lambda, \ t_j : \ K^\lambda(\beta) \to K^\lambda((j, j+1) \cdot \beta). \tag{1.11}$$

Proof. It is sufficient to show that the involutions t_j conserve the inequalities (1.3). Consider "the elementary neighborhood" of $x \in K_n$:

$$\begin{array}{ccccccccc} a & & & & c & & a & & & & c \\ & \geq & & \leq & & & & \geq & & \leq & \\ & & x_{ij} & & & \xrightarrow{t_j} & & & min(a,b)+max(c,d)-x_{ij} & & \\ & \geq & & \leq & & & & \geq & & \leq & \\ b & & & & d & & b & & & & d \end{array}$$

It is clear that $(x := x_{ij})$ $min(a,b) \geq x \geq max(c,d)$. So

$$a \geq min(a,b) \geq min(a,b) - (x - max(c,d)),$$
$$(min(a,b) - x) + max(c,d) \geq c.$$

So all necessary inequalities are satisfied. ■

We denote by G_n the group generated by the involutions $t_1, \cdots, t_{n-1}$:

$$G_n =< t_1, \cdots, t_{n-1} > . \tag{1.12}$$

This group acts on the space of triangles X_n and for any fixed vector $\lambda \in \mathbf{R}^n_+$ transforms the convex polytope K^λ into itself and interchanges the polytopes $K^\lambda(\beta)$.

The group G_n may be embedded in a bigger group $\widetilde{G}_n$. In order to construct such an extension, we define a transformation T_ϵ of the space X_n in the following way: assume $\epsilon \in \mathbf{R}^1$ and $x \in X_n$, then let us put $T_\epsilon(x) = \widetilde{x}$, where $\widetilde{x}^{(j)} = x^{(j)}$, if $2 \leq j \leq n$, and $\widetilde{x}^{(1)} = x_{11} - \epsilon$. We denote by $\widetilde{G}_n$ the group generated by G_n and $T_\epsilon, \ \epsilon \in \mathbf{R}^1$:

$$\widetilde{G}_n =< t_1, \cdots, t_{n-1}, T_\epsilon, \epsilon \in \mathbf{R}^1 > . \tag{1.13}$$

The next proposition gives the description of some relations between the generators $t_i, \ 1 \leq i \leq n-1$, and $T_\epsilon, \ \epsilon \in \mathbf{R}^1$.

Proposition 1.3. We have

$$\begin{aligned} &a)\ t_1 T_\epsilon = T_{-\epsilon} t_1, \ \text{or equivalently}, \ (t_1 T_\epsilon)^2 = 1, \\ &b)\ t_i T_\epsilon = T_\epsilon t_i, \ \ 3 \leq i \leq n-1, \\ &c)\ (t_1 t_2)^6 = 1, \\ &d)\ [t_i, t_j] = 0, \ \text{if} \ |i-j| \geq 2, \\ &e)\ t_2 T_\epsilon t_2 T_{-\epsilon+\delta} t_2 T_{-\delta} = T_\delta t_2 T_{-\delta+\epsilon} t_2 T_{-\epsilon} t_2. \\ &f)\ (T_\epsilon t_2 t_1 T_\delta t_2 t_1)^3 = 1, \ \epsilon, \delta \in \mathbf{R}^1. \end{aligned} \tag{1.14}$$

Proof. The assertions $a)$, $b)$ and $d)$ are evident.

c) It is sufficient to show that

$$(t_1t_2)^3 = (t_2t_1)^3 \quad : X_3 \to X_3.$$

By direct computation we find $t_2t_1(x) = \widetilde{x}$, where

$$\begin{aligned}
&\widetilde{x}_{12} = x_{13} - x_{12} + max(x_{23}, \beta_2(x)),\\
&\widetilde{x}_{22} = x_{33} - x_{22} + min(x_{23}, \beta_2(x)),\\
&\beta_2(\widetilde{x}) = \beta_3(x), \ \ \widetilde{x}_{11} = \beta_2(x),\\
&\widetilde{x}_{13} = x_{13}, \ \ \widetilde{x}_{23} = x_{23}, \ \ \widetilde{x}_{33} = x_{33}.
\end{aligned}$$

Consequently, $(t_2t_1)^3(x) = \widetilde{\widetilde{x}}$, where $\widetilde{\widetilde{x}}_{\alpha\beta} = x_{\alpha,\beta}$, if $(\alpha,\beta) \neq (1,2)$ or (2,2), and

$$\begin{aligned}
&\widetilde{\widetilde{x}}_{12} = x_{13} - x_{12} - max(x_{23}, \beta_3(x)) + max(x_{23}, \beta_2(x)) + max(x_{23}, \beta_1(x)),\\
&\widetilde{\widetilde{x}}_{22} = x_{33} - x_{22} - min(x_{23}, \beta_3(x)) + min(x_{23}, \beta_2(x)) + min(x_{23}, \beta_1(x)). \qquad (1.15)
\end{aligned}$$

Similarly, we find $t_1t_2(x) = \overline{x}$, where

$$\begin{aligned}
&\overline{x}_{12} = x_{13} - x_{12} + max(x_{23}, \beta_1(x)),\\
&\overline{x}_{22} = x_{32} - x_{22} + min(x_{23}, \beta_1(x)),\\
&\overline{x}_{11} = \beta_3(x), \ \ \beta_3(\overline{x}) = \beta_2(x),\\
&\overline{x}_{\alpha 3} = x_{\alpha 3}, \ \ \alpha = 1,2,3.
\end{aligned}$$

Using these formulae, it is easy to see that for $(t_1t_2)^3(x)$ we obtain the same expression (1.15).

e) Let us define $t_2^{\epsilon} := T_{\epsilon}t_2T_{-\epsilon}t_2, \ \epsilon \in \mathbf{R}^1$. It is sufficient to show that

$$t_2^{\epsilon}t_2^{\delta} = t_2^{\delta}t_2^{\epsilon} : X_3 \to X_3,$$

for any $\epsilon, \delta \in \mathbf{R}^1$. Given $x \in X_3$, the following relations hold

$$\begin{aligned}
&x_{\alpha,\beta}(t_2^{\epsilon}(x)) = x_{\alpha,\beta} \text{ if } (\alpha,\beta) \neq (1,2) \text{ or } (2,2),\\
&x_{12}(t_2^{\epsilon}(x)) = x_{12} + max(x_{23}, x_{11} + \epsilon) - max(x_{23}, x_{11}),\\
&x_{22}(t_2^{\epsilon}(x)) = x_{22} + max(x_{23}, x_{11} + \epsilon) - max(x_{23}, x_{11}).
\end{aligned}$$

Consequently, $x_{12}(t_2^{\epsilon}t_2^{\delta}(x)) =$

$$= x_{12} + max(x_{23}, x_{11} + \epsilon) + max(x_{23}, x_{11} + \delta) - 2max(x_{23}, x_{11}) = x_{12}(t_2^{\delta}t_2^{\epsilon}(x)).$$

By the same reasoning

$$x_{22}(t_2^{\epsilon}t_2^{\delta}(x)) = x_{22}(t_2^{\delta}t_2^{\epsilon}).$$

f) It is sufficient to prove (1.14,f) for $n=3$. Let us consider a correspondence $\psi_{\epsilon,\delta}:\ X_3\to X_3$, given by

$$\psi_{\epsilon,\delta}:\quad \begin{matrix} x_{13} & & x_{23} & & x_{33} \\ & x_{12} & & x_{22} & \\ & & x_{11} & & \end{matrix} \quad\longrightarrow\quad \begin{matrix} x_{13}+\epsilon+\delta, & & x_{23}+\delta, & & x_{33} \\ & x_{12}+\epsilon+\delta, & & x_{22}+\delta & \\ & & x_{11}+\epsilon+\delta & & \end{matrix}$$

It is clear that $\psi_{\epsilon,\delta}$ is an automorphism of X_3. The following identities, which may be verified by a direct computation, reduce the proof of (1.14 f) to that of point c):

$$\psi_{\epsilon,\delta}^{-1}\ t_1T_\epsilon\ \psi_{\epsilon,\delta}=t_1,$$
$$\psi_{\epsilon,\delta}^{-1}\ T_{-\epsilon}t_2t_1T_{-\delta}t_2t_1\ \psi_{\epsilon,\delta}=(t_2t_1)^2.$$

Note that $c)$ is a particular case of $f)$. ■

One of our main results of this note, namely Theorem 1.1, gives the description of additional relations between the generators in the group $\widetilde{G}_n$. We don't know whether or not the set of relations given by (1.24) and (1.25) contains all relations, but assume that it is complete. In any case the validity of the relations (1.24) allows us to construct the section of the following exact sequence

$$1\to Ker\pi\to G_n \overset{\pi}{\rightleftharpoons} S_n\to 1,\ \ \pi(t_i)=(i,i+1)$$

and to find a subgroup (not normal subgroup !), which is isomorphic to the symmetric group S_n.

Now let us pass to the construction of the involutions s_i, which will generate the symmetric group and the transformations $s_i^{(\epsilon)}$. The transformations $s_i^{(\epsilon)}$ will satisfy the colored braid relations (Theorem 1.1).

For this purpose consider the following elements in the groups G_n and $\widetilde{G}_n$:

$$\begin{aligned} &p_i=t_it_{i-1}\cdots t_1,\\ &v_i=t_it_{i+1}\cdots t_{n-1},\\ &q_i=p_1p_2\cdots p_i,\\ &u_i=v_{n-1}v_{n-2}\cdots v_{n-i},\\ &\sigma=t_1t_2\cdots t_{n-1}:=p_{n-1}^{-1}=v_1,\\ &s_i=\sigma^{i-1}t_1\sigma^{1-i},\ \ s_i^{(\epsilon)}=\sigma^{i-1}T_\epsilon\sigma^{1-i}, \end{aligned}\tag{1.16}$$

where $2\le i\le n-1$ and put $p_1=q_1=s_1=t_1,\ v_n=1.$

Proposition 1.4. We have

$a)$ $q_i=q_{i-1}p_i=p_i^{-1}q_{i-1},\ \ u_i=u_{i-1}v_{n-i}=v_{n-i}^{-1}u_{i-1},$
consequently $q_i^2=1,\ u_i^2=1,\ 1\le i\le n-1.$

$b)$ $s_i=q_it_1q_i,\ s_i^{(\epsilon)}=q_iT_{-\epsilon}q_i.\ 1\le i\le n-1.$ (1.17) *c*) $q_{n-1}s_iq_{n-1}=s_{n-i},\ q_{n-1}s_i^{(\epsilon)}q_{n-1}=s_{n-i}^{(\epsilon)}\ \ 1\le i\le n-1.$

d) $(s_{i-1}s_i)^3=q_{i-1}t_1p_it_1(t_2t_1)^6t_1p_i^{-1}t_1q_{i-1},\ \ 2\le i\le n-1.$

e) *Crucial Lemma.* Assume $x\in X_n,\ \ \beta(x)=(\beta_1,\cdots,\beta_n),$ then

$$s_i(x)=s_i^{(\beta_i-\beta_{i+1})}(x). \tag{1.18}$$

f) $s_i\cdot s_i^{(\epsilon)}=s_i^{(-\epsilon)}\cdot s_i,\ \ 1\le i\le n-1.$

g) Assume that $s_it_j=t_js_i$ for all $j<i-1,$ then

$$s_i=t_{i-1}t_is_{i-1}t_it_{i-1}. \tag{1.19}$$

h) Formulas for the weights

$$\beta(s_i(x))=(i,i+1)\beta(x),$$
$$\beta(s_i^{(\epsilon)}(x))=(\beta_1,\cdots,\beta_{i-1},\beta_i-\epsilon,\beta_{i+1}+\epsilon,\beta_{i+2},\cdots,\beta_n),$$
$$\beta(\sigma(x))=(n,n-1,\cdots,2,1)\beta(x),$$
$$\beta(q_{n-1}(x))=\overleftarrow{\beta}(x):=w_0\beta(x),$$

where $w_0\in S_n$ is the element of the maximal length in the symmetric group S_n.

Proof. *a*) We must prove that $p_{i+1}q_ip_{i+1}=q_i$. Let us use induction:

$$p_{i+1}q_ip_{i+1}=t_{i+1}p_iq_{i-1}p_it_{i+1}p_i=t_{i+1}q_{i-1}t_{i+1}p_i=q_{i-1}p_i=q_i,$$

because $[t_{i+1},q_{i-1}]=0$. Consequently,

$$q_i^2=q_iq_i=q_{i-1}p_i^{-1}p_iq_{i-1}=q_{i-1}^2=\cdots=q_1^2=t_1^2=1.$$

Similarly we may prove that

$$v_{n-i}u_{i-1}v_{n-i}=u_{i-1}\ \text{ and }\ u_i^2=1.$$

b) By induction, it is easy to see that

$$\sigma^{i-1}=q_iu_{i+1}v_1^{-1}v_2^{-1}t_1,\ \ 2\le i\le n-1,$$
$$\sigma^{n-1}=q_{n-1}u_{n-1}v_1^{-1},$$
$$\sigma^n=q_{n-1}u_{n-1}. \tag{1.20}$$

So we have

$$s_i=\sigma^{i-1}t_1\sigma^{-(i-1)}=q_i(u_{i+1}v_1^{-1}v_2^{-1}t_1v_2v_1u_{i+1}^{-1})q_i=q_it_1q_i,$$

because $[u_iv_1^{-1}v_2^{-1},t_1]=0$, if $i\ge 2$.

c) It is sufficient to show that

$$[q_{n-i}q_{n-1}q_i,\ \ t_1]=0,\ \ [q_{n-i}q_{n-1}q_i,\ \ T_\epsilon]=0,\ \ 1\le i\le n-1. \tag{1.21}$$

Let us assume that $2\le i\le n-2$. For $i=1$ or $i=n-1$ the equalities (1.21) are clear. From *a*) by induction we find

$$q_i=q_jp_{j+1}\cdots p_i=p_i^{-1}\cdots p_{j+1}^{-1}q_j,\quad \text{if}\quad i\ge j. \tag{1.22}$$

Consequently,

$$q_{n-i}q_{n-1} = p_{n-i+1}\cdots p_{n-1}.$$

Now it is easy to see by induction that

$$p_{n-i+k-1} = v_{n-i+k}\cdot v_{k+1}^{-1}\cdot p_k,\ \ 2\le k\le i.$$

So

$$p_{n-i+1}\cdots p_{n-1} = (\prod_{k=2}^{i} v_{n-i+k}v_{k+1}^{-1})\cdot t_1 q_i.$$

Thus we have

$$q_{n-i}q_{n-1}q_i = (\prod_{k=2}^{i} v_{n-i+k}v_{k+1}^{-1})\cdot t_1.$$

But the product $\prod_{k=2}^{i} v_{n-i+k}v_{k+1}^{-1}$ does not contain the involutions t_1 and t_2 and therefore commutes with t_1.

d) We use the identity (1.22). Let $i-j\ge 1$, then

$$s_is_j = q_it_1q_iq_jt_1q_j = q_j(p_{j+1}\cdots p_it_1p_i^{-1}\cdots p_{j+1}^{-1}t_1)q_j.$$

Now let us take $j=i-1$. Then

$$\begin{aligned}(s_{j+1}s_j)^3 &= q_j(p_{j+1}t_1p_{j+1}^{-1}t_1)^3q_j = q_jt_1p_{j+1}t_1(t_1p_{j+1}^{-1}t_1p_{j+1})^3t_1p_{j+1}^{-1}t_1q_j =\\ &= q_jt_1p_{j+1}t_1(t_2t_1)^6t_1p_{j+1}^{-1}t_1q_j,\end{aligned}$$

since

$$t_1p_{j+1}^{-1}t_1p_{j+1} = t_1t_1\cdots t_{j+1}t_1t_{j+1}\cdots t_1 = t_2t_1t_2t_1.$$

e) It is sufficient to show (see (1.17) of Proposition 1.4) that if $\beta(x) = (\beta_1,\cdots,\beta_n)$ then

$$t_1(q_i(x)) = T_{\beta_{i+1}-\beta_i}(q_i(x)).$$

We note that

$$\beta(q_i(x)) = (\beta_{i+1},\beta_i,\cdots,\beta_1,\beta_{i+2},\cdots,\beta_n),\ 1\le i\le n-1.$$

Consequently,

$$\begin{aligned}&x_{11}(t_1(q_i(x)) = \beta_i,\\ &x_{11}(T_{\beta_{i+1}-\beta_i}(q_i(x))) = \beta_{i+1}-(\beta_{i+1}-\beta_i) = \beta_i.\end{aligned}$$

Crucial Lemma allows us to reduce any statment about s_i to that about s_i^{ϵ}. For example, the relation (1.24,*b*) follows from that (1.25,*b*) if $\epsilon = \beta_{i+1}-\beta_{i+2}$ and $\delta = \beta_i-\beta_{i+1}$.

g) In fact, by (1.17) and part *a*), we have

$$s_i = q_i t_1 q_i = p_i^{-1} q_{i-1} t_1 q_{i-1} p_i = p_i^{-1} s_{i-1} p_i =$$
$$= t_1 \cdots t_{i-2} (t_{i-1} t_i s_{i-1} t_i t_{i-1}) t_{i-2} \cdots t_1.$$

However, by assumption we have $[\widetilde{s}_i, t_1] = \cdots = [\widetilde{s}_i, t_{i-2}] = 0$, where $\widetilde{s}_i$ is given by (1.19). So $s_i = \widetilde{s}_i = t_{i-1} t_i s_{i-1} t_i t_{i-1}$.

f) Follows from Proposition 1.3, point *a*). ■

Note that from (1.17) we obtain the formula for s_i as a product of $(i^2 + i - 1)$ simple involutions t_j, whereas formula (1.19) gives the expression for s_i as a products of $4i - 3$ involutions. For example

$$s_1 = t_1, \ \ q_1 = t_1, \ \ s_2 = t_1 t_2 t_1 t_2 t_1, \ \ q_2 = t_1 t_2 t_1, \ \ q_3 = t_1 t_2 t_1 t_3 t_2 t_1,$$
$$s_3 = t_1 t_2 t_1 t_3 t_2 t_1 t_2 t_3 t_1 t_2 t_1 = t_2 t_3 t_1 t_2 t_1 t_2 t_1 t_3 t_2. \tag{1.23}$$

Note that equality (1.23) for s_3 is equivalent to the following relation in the group G_n, $n \geq 4$ (see Corollary 1.1): $(t_1 q_3)^4 = 1$.

Now let us give a formulation of our main result of this Section.

Theorem 1.1. 1^0. The involutions s_i, $1 \leq i \leq n-1$, satisfy the relations of the symmetric group S_n, i.e.

$$\begin{aligned} &a) \ \ s_i^2 = 1, \ \ i = 1, \cdots, n-1, \\ &b) \ \ (s_i s_{i+1})^3 = 1, \ \ i = 1, \cdots, n-2, \\ &c) \ \ s_i s_j = s_j s_i, \ \ \text{if} \ \ 1 \leq i, j \leq n-1, \ \ |i-j| \geq 2. \end{aligned} \tag{1.24}$$

2^0. The transformations $s_i^{(\epsilon)}$ satisfy the colored braid relations, i.e. for any $\epsilon, \delta \in \mathbf{R}^1$ we have

$$\begin{aligned} &a) \ \ s_i^{(\epsilon)} \cdot s_i^{(\delta)} = s_i^{(\epsilon+\delta)}, \text{in particular } s_i^{(-\epsilon)} = (s_i^{(\epsilon)})^{-1}, \\ &b) \ \ s_i^{(\epsilon)} \cdot s_{i+1}^{(\epsilon+\delta)} \cdot s_i^{(\delta)} = s_{i+1}^{(\delta)} \cdot s_i^{(\epsilon+\delta)} \cdot s_{i+1}^{(\epsilon)}, \ \ 1 \leq i \leq n-2, \\ &c) \ \ s_i^{(\epsilon)} \cdot s_j^{(\delta)} = s_j^{(\delta)} \cdot s_i^{(\epsilon)}, \ \ \text{if} \ \ |i-j| \geq 2. \end{aligned} \tag{1.25}$$

Corollary 1.1 (of the Theorem 1.1) 1^0. We have the following relations between the generators t_i in the group G_n

$$\begin{aligned} &1) \ \ t_i^2 = 1, \ \ t_i t_j = t_j t_i, \ \ \text{if} \ \ |i-j| \geq 2, \\ &2) \ \ (t_1 t_2)^6 = 1, \\ &3) \ \ (t_1 q_i)^4 = 1, \ \ \text{if} \ \ 3 \leq i \leq n-1, \end{aligned}$$

where $q_i = p_1 p_2 \cdots p_i = t_1 \cdot \underbrace{t_2 t_1} \cdot \underbrace{t_3 t_2 t_1} \cdot \cdots \cdot \underbrace{t_i t_{i-1} \cdots t_2 t_1}$.

2^0. Besides the relations 1) - 3) of Corollary 1.1, part 1^0, we have the following relations between the generators t_i, and T_ϵ in the group $\widetilde{G}_n$

1) $T_\epsilon \cdot T_\delta = T_{\epsilon+\delta}$,
2) $t_1 T_\epsilon = T_{-\epsilon} t_1, \quad t_i T_\epsilon = T_\epsilon t_i$, if $i \geq 3$,
3) $(T_\epsilon t_2 t_1 T_\delta t_2 t_1)^3 = 1$, for any $\epsilon, \delta \in \mathbf{R}^1$,
4) $T_\epsilon t_2 T_{-\epsilon+\delta} t_2 T_{-\delta} t_2 = t_2 T_\delta t_2 T_{-\delta+\epsilon} t_2 T_{-\epsilon}$,
5) $T_\epsilon\ q_j T_\delta\ q_j = q_j\ T_\delta\ q_j\ T_\epsilon,\ 3 \leq j \leq n-1$,
6) $(t_1 T_\epsilon\ q_j t_1 T_\delta\ q_j)^2 = 1,\ 3 \leq j \leq n-1$.

Proof. 1^0 The assertions 1) and 2) are proved in Proposition 1.3. On the other hand, if $i - j \geq 2$, then $s_j s_i = \sigma^{j-1}(t_1 s_{i-j+1})\sigma^{-(j-1)}$ and $(s_j s_i)^2 = 1$ iff $(t_1 s_{i-j+1})^2 = 1$. But if $i \geq 3$, then $(t_1 s_i)^2 = (t_1 q_i)^4$.

2^0. The properties 1) - 4) follow from Proposition 1.3. The identity 5) follows from the fact that the transformations $s_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute, if $|i-j| \geq 2$ (see Theorem 1.1). The last identity 6) follows from an observation that the elements $s_i s_i^{(\epsilon)}$ and $s_j s_j^{(\delta)}$ commute, if $|i-j| \geq 2$, which is a consequence of Corollary 1.4. ∎

Corollary 1.2 (of the Theorem 1.1). Let us define

$$s_0 := s_0^{(\epsilon)} = s_{n-1} s_{n-2} \cdots s_2 s_1 T_\epsilon s_2 \cdots s_{n-1}.$$

Then $s_0, s_1, \cdots s_{n-1}$ are the standard generators of the affine Weyl group of the type $A_{n-1}^{(1)}$, and $\beta(s_0^{(\epsilon)}(x)) = s_0^{(\epsilon)}(\beta(x)) = (\beta_n + \epsilon,\ \beta_2, \cdots, \beta_{n-1},\ \beta_1 - \epsilon)$.

Proof. It is sufficient to show that $(s_0 s_1)^3 = 1$ and $(s_0 s_{n-1})^3 = 1$. We have the following chains of the equivalent statements

i) $(s_0 s_1)^3 = 1 \Longleftrightarrow (s_2 s_1 T_\epsilon s_2 s_1)^3 = 1 \Longleftrightarrow (T_\epsilon s_1 s_2)^3 = 1$.

The last relation is equivalent to the following one $(T_\epsilon t_2 t_1 t_2 t_1)^3 = 1$, which is a particular case of (1.14,f) when $\delta = 0$.

ii) $(s_0 s_{n-1})^3 = 1 \Longleftrightarrow [\,(s_2 s_3 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_3 s_2) s_1 T_\epsilon\,]^3 = 1$.

The last relation is equivalent to one of the form $(s_2 s_1 T_\epsilon)^3 = 1$, which is also a particular case of (1.14,f). It is also clear, that $s_0^2 = 1$ and $s_0 s_j = s_j s_0$, if $2 \leq j \leq n-2$. ∎

Corollary 1.3. The elements $s_i s_i^{(\epsilon)},\ 1 \leq i \leq n,\ \epsilon \in \mathbf{R}^1$ satisfy the following relations

1) $(s_i s_i^{(\epsilon)})^2 = 1, \quad 1 \leq i \leq n-1$;
2) $(s_i s_i^{(\epsilon)} s_{i+1} s_{i+1}^{(\delta)})^3 = 1, \quad 1 \leq i \leq n-2, \quad \epsilon,\ \delta \in \mathbf{R}^1$,
3) $s_i s_i^{(\epsilon)} s_j s_j^{(\delta)} = s_j s_j^{(\delta)} s_i s_i^{(\epsilon)}, \quad \text{if} \quad |i-j| \geq 2, \quad \epsilon,\ \delta \in \mathbf{R}^1$.

Proof. We must prove 2) only when $i = 1$, i.e. $(s_1 T_\epsilon s_2 s_2^{(\delta)})^3 = 1$. But this relation is exactly (1.14,f). The statement 3) follows from Corollary 1.4. ∎

Using the same arguments as in the proof of Corollary 1.2, we may prove that if

$$\widetilde{s}_0 := s_0^{(\delta,\epsilon)} = s_{n-1}s_{n-1}^{(\delta)}s_{n-2}s_{n-2}^{(\delta)}\cdots s_2s_2^{(\delta)}s_1s_1^{(\delta)}\,T_\epsilon\,s_2s_2^{(\delta)}\cdots s_{n-1}s_{n-1}^{(\delta)},$$

then $\widetilde{s}_0,\ s_1s_1^{(\delta)},\cdots,s_{n-1}s_{n-1}^{(\delta)}$ are the standard generators of the affine Weyl group of the type $A_{n-1}^{(1)}$ and

$$\beta(s_0^{(\delta,\epsilon)}(x)) = (\beta_n + n\delta + \epsilon,\ \beta_2,\cdots,\beta_{n-1},\ \beta_1 - n\delta - \epsilon).$$

Let us give some comments.

i) Part 1^0 of Theorem 1.1 follows from part 2^0 and the equality (1.18).

ii) The assertion *b*) of part 2^0 follows from Proposition 1.3, part *e*). In fact, formula (1.25, *b*) is equivalent to the following one

$$T_\epsilon \cdot s_2^{(\epsilon+\delta)} \cdot T_\delta = s_2^{(\delta)} \cdot T_{\epsilon+\delta} \cdot s_2^{(\epsilon)},$$

which in turn coincides with (1.14).

iii) Assertion *c*) of part 2^0 is equivalent to the following identity

$$T_\epsilon \cdot s_{j-i+1}^{(\delta)} = s_{j-i+1}^{(\delta)} \cdot T_\epsilon, \text{ if } j-i \geq 2. \tag{1.26}$$

In fact, using (1.16) we find that if $j \geq i$, then

$$s_i^{(\epsilon)} \cdot s_j^{(\delta)} = \sigma^{i-1}(T_\epsilon \cdot s_{j-i+1}^{(\delta)})\sigma^{1-i}.$$

iv) We assume that relations between the generators t_i, $1 \leq i \leq n-1$, (respectively between t_i and T_ϵ) as described in Corollary1.1, are the defining relations for the group G_n (respectively for the group $\widetilde{G}_n$).

v) Let us say a few words about the group G_n for $n = 3$ and $n = 4$.

$$G_3 = \{\ t_1, t_2 \mid t_1^2 = t_2^2 = (t_1t_2)^6 = 1\ \},$$
$$G_4 = \{\ t_1, t_2, t_3 \mid t_1^2 = t_2^2 = t_3^2 = (t_1t_3)^2 = (t_1t_2)^6 = (t_1t_2t_1t_3t_2)^4 = 1\ \}$$

The group G_3 is finite of the order 12, contains the normal subgroup $N := < 1,\ (t_1t_2)^2 >$ of the order 3 with the factor group $G/N \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

The group G_4 seems to be very interesting. For example, for arbitrary n the symmetric group S_n is a factor of the group G_4. Namely, let us define an epimorphism $\rho := \rho_n : G_4 \to S_n$ by setting

$$\rho(t_1) = s_1,\ \rho(t_2) = s_2s_4s_6\cdots,\ \rho(t_3) = s_3s_5s_7\cdots,$$

where $s_i = (i, i+1) \in S_n,\ \ 1 \leq i \leq n-1$, is a simple transpositions.

In fact, one can show that ρ is agree with the relations in the group G_4 and $\rho(t_1),\ \rho(t_2)$ and $\rho(t_3)$ are really generate the symmetric group S_n. Consequently, the group G_n is infinite, if $n \geq 4$.

The main difficulties in the demonstration of Theorem 1.1 appear to be to prove that the generators $s_i^{(\epsilon)}$ and $s_i^{(\delta)}$ commute, if $|i-j| \geq 2$, or equivalently, to verify identity (1.26). Our strategy is to prove more precise result about the action of transformations $s_i^{(\epsilon)}$ on the triangles. Before stating the theorems about the structure of the mapping $s_i^{(\epsilon)}$, let us define some additional operators acting on the space of triangles X_n.

Definition 1.2. For each triple of integers (ijk), $1 \le i < j < k \le n$, let us give a transformation $R_{ijk} : X_n \to X_n$ by the following formulae

$$R_{ijk}(x) = \widetilde{x}, \ x \in X_n, \ \text{where}$$

$$\begin{aligned} &\widetilde{x}_{ij} = x_{ij} + x_{ik} - x_{i,k-1} - min(x_{jk} - x_{j,k-1} \ , \ x_{ij} - x_{i,j-1}), \\ &\widetilde{x}_{jj} = x_{jj} - x_{ik} + x_{i,k-1} + min(x_{jk} - x_{j,k-1} \ , \ x_{ij} - x_{i,j-1}), \\ &\widetilde{x}_{\alpha\beta} = x_{\alpha\beta}, \ \text{if } (\alpha,\beta) \ne (i,j) \text{ or } (j,j). \end{aligned} \tag{1.27}$$

Proposition 1.5. The operators R_{ijk} satisfy the following relations:

a) $(R_{ijk})^2 = 1$,
b) $R_{ijk} \cdot R_{i'j'k'} = R_{i'j'k'} \cdot R_{ijk}$, if $|(ijk) \cap (i'j'k')| \ne 2$,
c) $(R_{ijk}R_{ijl}R_{ikl}R_{jkl})^2 = 1$,
d) $(R_{ijl}R_{ikl})^3 = 1$,
e) $\beta(R_{ijk}(x)) = \beta(x), \ x \in X_n$.

Proof. The assertions *a*), *b*) and *e*) are almost evident, and *c*), *d*) may be checked by direct computation.

Remark 1.1.
i) The assertions *a*) $-$ *c*) are essentially due to G. Lusztig [Lu2].
ii) It seems plausible that relations *a*) $-$ *d*) are the defining ones for the group L_n, generated by all R_{ijk} with $1 \le i < j < k \le n$.

To go further, let us define the transformations $T_\epsilon^{(i)}$ and φ_i acting on the space X_n by the formulae:

$$\begin{aligned} &T_\epsilon^{(i)}(x) = \widetilde{x}, \ \varphi_i(x) = \widetilde{\widetilde{x}}, \ x \in X_n, \ \text{where} \\ &\widetilde{x}_{ii} = x_{ii} - \epsilon, \ \widetilde{\widetilde{x}}_{ii} = x_{ii} + \beta_{i+1}(x) - \beta_i(x), \end{aligned}$$

both $\widetilde{x}_{\alpha\beta}$ and $\widetilde{\widetilde{x}}_{\alpha\beta}$ are equal to $x_{\alpha\beta}$, iff $(\alpha,\beta) \ne (i,i)$.

It is clear from the definitions that if $x \in X_n, \ \beta = \beta(x) = (\beta_1, \cdots, \beta_n)$, then $\varphi_i(x) = T_{\beta_i - \beta_{i+1}}(x)$, and

$$\begin{aligned} &\beta(T_\epsilon^{(i)}(x)) = (\beta_1 \cdots, \beta_i - \epsilon, \beta_{i+1} + \epsilon, \cdots, \beta_n), \\ &\beta(\varphi_i(x)) = (i, i+1) \cdot \beta(x). \end{aligned}$$

Now we are ready to state our result about the structure of the transformations s_i and $s_i^{(\epsilon)}$.

Theorem 1.2. The following equalities are fulfilled

$$s_i = R_{i-1,i}R_{i-2,i} \cdots R_{1,i} \cdot \varphi_i \cdot R_{1,i} \cdots R_{i-2,i}R_{i-1,i}, \tag{1.28a}$$

$$s_i^{(\epsilon)} = R_{i-1,i}R_{i-2,i} \cdots R_{1,i} \cdot T_\epsilon^{(i)} R_{1,i} \cdots R_{i-2,i}R_{i-1,i}, \tag{1.28b}$$

where $1 \le i \le n-1$ and $R_{jk} := R_{jkk+1}, \ 1 \le j < k \le n-1$.

The proof of Theorem 1.2 will be given in Appendix.

Corollary 1.4. Let $x \in X_n,\ x = (x^{(n)}, \cdots, x^{(1)})$. Then

1^0. $s_i(x) = (x^{(n)}, \cdots, x^{(i+1)}, \widetilde{x}^{(i)}, x^{(i-1)}, \cdots x^{(1)}),\ 1 \le i \le n-1$, and a vector $\widetilde{x}^{(i)} \in \mathbf{R}^i$ depends only on the components of the vectors $x^{(i-1)}, x^{(i)}$ and $x^{(i+1)}$.

2^0. $s_i^{(\epsilon)}(x) = (x^{(n)}, \cdots, x^{(i+1)}, \widetilde{\widetilde{x}}^{(i)}, x^{(i-1)}, \cdots, x^{(1)})$, and a vector $\widetilde{\widetilde{x}}^{(i)}$ depends only on ϵ and the components of vectors $x^{(i-1)}, x^{(i)}$ and $x^{(i+1)}$.

In particular, from Corollary 1.2, part 1^0, follows that $s_i t_j = t_j s_i$ if $1 \le j < i-1$ and consequently, $s_i s_j = s_j s_i$, if $i - j \ge 2$. This proves the part 1^0 of Theorem 1.1 and also the recurrence relation (1.19) for s_i.

Similarly, from Corollary 1.2, part 2^0, it follows that $T_\epsilon \cdot s_i^{(\delta)} = s_i^{(\delta)} \cdot T_\epsilon$ for any $\epsilon, \delta \in \mathbf{R}$ and $3 \le i \le n-1$. This proves part 2^0 of Theorem 1.1 (see (1.26)).

As another application of Theorem 1.2 we will deduce a recurrence relation for the transformations $s_i^{(\epsilon)}$, but before doing so it is necessary to introduce some additional notations.

Definition 1.3. Let us give a mapping $[i, i+1] : X_n \to X_n,\ 1 \le i \le n-1$, by

$$[i, i+1](x) = \widetilde{x},\ x \in X_n,$$

where

$$\begin{aligned} &\widetilde{x}_{ki} = x_{k,i+1} + x_{k,i-1} - x_{ki},\ \text{if}\ k < i, \\ &\widetilde{x}_{ii} = x_{i,i+1} + x_{i+1,i+1} - x_{ii}, \\ &\widetilde{x}_{ik} = x_{i+1,k},\ \text{and}\ \widetilde{x}_{i+1,k} = x_{i,k},\ \text{if}\ k > i, \end{aligned}$$

and for all remaining elements $\widetilde{x}_{\alpha\beta} = x_{\alpha\beta}$.

Lemma 1.1. The mappings $[i, i+1],\ 1 \le i \le n-1$, satisfy the relations of the symmetric group S_n, i.e.

a) $[i, i+1]^2 = 1$,

b) $[i, i+1] \cdot [i+1, i+2] \cdot [i, i+1] = [i+1, i+2] \cdot [i, i+1] \cdot [i+1, i+2]$,

c) $[i, i+1] \cdot [j, j+1] = [j, j+1] \cdot [i, i+1]$, if $|i-j| \ge 2$,

d) $\beta([i, i+1](x)) = (i, i+1)\beta(x),\ x \in X_n$.

It is not difficult to show that the group generated by all $[i, i+1],\ 1 \le i \le n-1$, is really isomorphic to the symmetric group S_n. ■

Proposition 1.6. Let us take $i \ge 2$. Then

$$\begin{aligned} s_i &= R_{i-1,i}[i-1, i][i, i+1]s_{i-1}[i, i+1][i-1, i]R_{i-1,i} \\ s_i^{(\epsilon)} &= R_{i-1,i}[i-1, i][i, i+1]s_{i-1}^{(\epsilon)}[i, i+1][i-1, i]R_{i-1,i}. \end{aligned} \tag{1.29}$$

Proof. It is easy to see, that

$$\begin{aligned} R_{k,i} &= [i-1, i][i, i+1]R_{k,i-1}[i, i+1][i-1, i], \\ T_\epsilon^{(i)} &= [i-1, i][i, i+1]T_\epsilon^{(i)}[i, i+1][i-1, i], \end{aligned} \tag{1.30}$$

where $1 \le k < i \le n-1$. Identity (1.29) follows by induction from (1.19), (1.30) and Theorem 1.2, parts 1^0 and 2^0. ■

The recurrence relations (1.19) and (1.29) are used as an induction base in a first proof of Theorem 1.2. However, it is possible to solve these recurrence relations in an explicit form and consequently to obtain a second proof of Theorem 1.2. Before describing the solutions of (1.19) and (1.29), let us give the appropriate definitions. It is convenient to use the notations

$$(a)_+ = max(a,0),\ (a)_- = min(a,0),\ a \in \mathbf{R}.$$

At first we define the sequence of piece-wise linear functions $Q_n^k(a_1,\cdots,a_n)$, $1 \le k \le n$, inductively, in the following way:

$$\begin{aligned} &Q_1^1(a_1) := -a_1,\\ &Q_n^1(a_1,\cdots,a_n) := (Q_{n-1}^1(a_1,\cdots,a_n))_- + (Q_{n-1}^1(a_2,\cdots,a_n))_+,\\ &Q_n^k(a_1,\cdots,a_n) := Q_n^1(a_k,\cdots,a_n,a_1,a_2,\cdots,a_{k-1}),\ 1 \le k \le n. \end{aligned} \tag{1.31}$$

Secondly, we define the linear functionals

$$\varphi_{ij} := \varphi_{ij}^{(n)} : X_n \to \mathbf{R},\ 1 \le i \le j \le n-1,$$

on the space of triangles X_n by the formulae:

$$\begin{aligned} &\varphi_{ij}(x) := x_{i-1,j} + x_{ij} - x_{i-1,j-1} - x_{i,j+1},\ \text{if } 2 \le i \le j \le n-1, \\ &\varphi_{11}(x) := 0,\ \varphi_{1,j}(x) := x_{1,j} + x_{j,j} - x_{1,j+1} - x_{j+1,j+1},\ \text{if } 1 < j \le n-1. \end{aligned} \tag{1.32}$$

Theorem 1.3. Let us fix a positive integer k, $2 \le k \le n-1$, and a triangle $x \in X_n$. Assume that

$$s_k(x) = \widetilde{x},\ s_k^{(\epsilon)}(x) = \widetilde{\widetilde{x}}.$$

Then

$$\widetilde{x}_{ik} = x_{ik} + Q_k^i(\varphi_{2k}(x),\cdots,\varphi_{kk}(x),\varphi_{1k}(x)); \tag{1.33}$$

$$\widetilde{\widetilde{x}}_{ik} = x_{ik} + Q_k^i(\varphi_{2k}(x),\cdots,\varphi_{kk}(x),\varphi_{1k}(x)+\beta_{k+1}(x)-\beta_k(x)+\epsilon). \tag{1.34}$$

The proof of Theorem 1.3 will be given elsewere.We shall give the exact formulae for s_k, if $k \le 3$, in Section 3.

Remark 1.2. We know (see (1.20)) that $\sigma^n = q_{n-1}u_{n-1}$. Assume additionally that $(q_{n-1}u_{n-1})^h = 1$, h may be equal to ∞. Consider the subgroup $\Sigma_n \subset G_n$ generated by the elements $s_i = \sigma^{i-1}t_1\sigma^{1-i}$, $1 \le i < nh$. Then from Corollary 1.1 it follows that

$$\begin{aligned} &a)\ s_i^2 = 1,\ 1 \le i < nh,\\ &b)\ (s_is_{i+1})^3 = 1,\ 1 \le i \le nh-2,\\ &c)\ s_is_j = s_js_i\ \text{ if } 2 \le |i-j| \le n-2. \end{aligned}$$

Proof. From the definition it is easy to see that $s_i s_j = \sigma^{-i}(t_1 s_{j-i+1})\sigma^i$, if $j \geq i$. So $(s_i s_{i+1})^3 = 1$ iff $(t_1 s_2)^3 = 1$. But $t_1 s_2 = (t_2 t_1)^2$ and by Proposition 1.3 part c) we know that $(t_1 t_2)^6 = 1$. Similarly $(s_i s_j)^2 = 1$ iff $(t_1 s_{j-i+1})^2 = 1$ and under the assumption $2 \leq |i-j| \leq n-2$ the assertion c) follows from Theorem 1.1 part 1^0). ∎

In particular, for any $1 \leq a \leq (n-1)h$ the involutions $s_a, \cdots, s_{a+n-2}$ generate a subgroup of G_n which is isomorphic to the symmetric group S_n.

Remark 1.3. The involutions q_i, $1 \leq i \leq n-1$, also give a system of generators for the group G_n, because we have

$$t_1 = q_1, \quad t_i = q_{i-1} q_i q_{i-1} q_{i-2}, \quad \text{if} \quad i \geq 2, \quad (q_0 := 1). \tag{1.35}$$

The proof of (1.35) follows from Proposition 1.4, point a). In fact, we have

$$q_{i-1} q_i q_{i-1} q_{i-2} = q_{i-1} q_{i-1} p_i p_{i-1}^{-1} q_{i-2} q_{i-2} = p_i p_{i-1}^{-1} = t_i, \quad \text{if} \quad i \geq 2.$$

∎

The relations between the generators q_i, $1 \leq i \leq n-1$, follow from Corollary 1.1 and have the following form

$$\begin{aligned} &1) \ q_i^2 = 1, \ 1 \leq i \leq n-1, \\ &2) \ (q_1 q_2)^6 = 1, \ (q_1 q_i)^4 = 1, \ \text{if} \ 3 \leq i \leq n-1, \\ &3) \ [\, q_{i+1}, \ q_i q_{i-1} q_i \,] = 0, \ \text{if} \ 3 \leq i \leq n-2, \\ &4) \ [\, q_i q_{i+1} q_i q_{i-1}, \ q_j q_{j+1} q_j q_{j-1} \,] = 0, \ \text{if} \ |i-j| \geq 2. \end{aligned} \tag{1.36}$$

In Section 2 we will show that in the case when $\lambda \in \mathbf{Z}_+^n$ is a partition, the restriction of the involution q_{n-1} on the set $K_{\mathbf{Z}}^{\lambda} \simeq STY(\lambda, \leq n)$ coincides with the Schützenberger involution $\mathbf{S}$ (see e.g. [Sch1],[Sch2],[EG],[Ki1]). The similarly combinatorial interpretation admits the involutions q_i, $1 \leq i \leq n-1$. So our construction gives the extension of the Schützenberger involutions q_i on the space of triangles X_n and describes the group generated by $q_1, \cdots, q_{n-1}$, i.e. the relations between $q_1, \cdots, q_{n-1}$. The next steps are Theorems 1.1 and 2.3 (see Section 2) which give us the connection between the natural action of the symmetric group $S_n =< s_1, \cdots, s_{n-1} >$ on the set $STY(\lambda, \leq n)$,which was introduced and studied by A. Lascoux and M.-P. Schützenberger [LS2],[LS3], and the Schützenberger involutions

$$s_i = q_i q_1 q_i. \tag{1.37}$$

The equality (1.37), restricted on the set $STY(\lambda, \leq n)$, is a pure combinatorial assertion and may be deduced directly from the properties of the plactic monoid (e.g. [LS2]) and the Robinson-Schensted correspondence. Details will appear elsewhere.

Remark 1.4. In a particular case that the a weight $\beta \in \mathbf{R}_+^n$ has the form $\beta = (a^n)$, i.e. $\beta_1 = \beta_2 = \cdots = \beta_n = a$, the group G_n conserves the convex polytope $K^\lambda(\beta)$ for any highest weight $\lambda \in \mathbf{R}_+^n$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. It is easy to see that a restriction of t_1 to the polytope $K^\lambda(\beta)$ becomes the identical map and consequently for all $1 \leq i \leq n-1$ we have

$$s_i|_{K^\lambda(\beta)} = Id_{K^\lambda(\beta)}.$$

The following example

$$\begin{array}{ccccccc} 7 & & 6 & & 3 & & 0 \\ & 6 & & 5 & & 1 & \\ & & 5 & & 3 & & \\ & & & 4 & & & \end{array}$$

shows that $(t_3t_2)^6 x \neq x$ (in fact, for this example $(t_3t_2)^{24} x = x$) and it is not clear whether or not it is possible to find the subgroup in G_n which is isomorphic to the symmetric group S_N (for some N) after the restriction of the action of G_n on the polytope $K^\lambda(\beta)$.

We finish this section by introducing some additional transformations of the space X_n and by considering of the stable behavior of the involutions s_i. First, consider the map I: $X_n \to X_n$ which is defined in the following way: let $x \in X_n$, then $I(x) = \widetilde{x}$, where

$$\widetilde{x}_{ij} = x_{1n} - x_{j-i+1,j}, \;\; 1 \leq i \leq j \leq n. \tag{1.38}$$

Proposition 1.7. We have

1) $\beta_i(I(x)) = x_{1n} - \beta_i(x)$.
2) The map I commutes with the actions of s_i and q_i, i.e.
$Is_i = s_iI$ and $q_iI = Iq_i$, $1 \leq i \leq n-1$,
3) $I \cdot T_\epsilon = T_{-\epsilon} \cdot I$, $I \cdot s_i^{(\epsilon)} = s_i^{(-\epsilon)} \cdot I$, $1 \leq i \leq n-1$.

Proof. Using Definition 1.1 for the involution t_j we find that for $x \in X_n$,

$$x_{ij}(t_j(Ix)) = x_{1n} - x_{j-i+1,j}(t_j(x)).$$

Consequently, $It_j = t_jI$, i.e. the actions of t_j and I commute. ■

Secondly, for a partition λ we consider the restrictions of the maps $s_i^{(\pm 1)}$ to the Gelfand-Tsetlin polytope K^λ:

$$e_i := pr \cdot s_i^{(1)}, \;\; f_i := pr \cdot s_i^{(-1)}, \;\; 1 \leq i \leq n-1, \tag{1.39}$$

where $pr : X_n \to K^\lambda$ is the projection map.

Proposition 1.8. We have

$$s_i f_i s_i = e_i, \ \ 1 \le i \le n.$$

Proof. The assertion follows from definitions (1.16), Proposition 1.4 point f) and (1.39). ■

Now we consider the stable behavior of the involutions s_i and q_i. Given the space of triangles X_{n-1}, we define the following embeddings $\varphi_\alpha : X_{n-1} \hookrightarrow X_n, \ \ \alpha = 1, 2$: given $x = (x^{(n-1)}, \cdots, x^{(1)}) \in X_{n-1}$, then

$$\varphi_1(x) = (x^{(n)}, x^{(n-1)}, \cdots, x^{(1)}), \tag{1.40a}$$

where $x^{(n)} = (x_{1,n-1}, \cdots, x_{n-1,n-1}, 0)$; and

$$\varphi_2(x) = (\widetilde{x}^{(n)}, \cdots, \widetilde{x}^{(1)}), \tag{1.40b}$$

where $\widetilde{x}^{(1)} = 0, \ \ \widetilde{x}^{(i+1)} = (x_{1,i}, \cdots, x_{ii}, 0), \ \ 1 \le i \le n-1.$

Proposition 1.9. We have

$$\begin{aligned} &1) \ \ s_i(\varphi_1(x)) = s_i(x), \ \text{ if } \ 1 \le i \le n-2, \\ &\quad\ s_{n-1}(\varphi_1(x)) = (x_{1,n-1}, \cdots, x_{n-2,n-2}, 0). \qquad (1.41) \\ &2) \ \ t_i(\varphi_2(x)) = \varphi_2(t_{i-1}(x)), \\ &\quad\ q_i(\varphi_2(x)) = \varphi_2(q_{i-1}(x)), \\ &\quad\ s_i(\varphi_2(x)) = \varphi_2(s_{i-1}(x)), \ \text{ where } \ \ 1 \le i \le n-1. \qquad (1.42) \end{aligned}$$

Remark 1.5. All our main results (including those dealing with a cocharge $\overline{c}$ (see Section 3)) after small modification are still valid for the space of truncated triangles $X_{n,m}, \ \ 0 \le m \le n$ [BZ1]. By definition the space $X_{n,m}$ consists of all sequences

$$x = (x^{(n)}, \cdots, x^{(m)}),$$

where $x^{(j)} = (x_{1,j}, \cdots, x_{j,j}) \in \mathbf{R}^j$. In what follows, we always consider the space $X_{n,m}$ as the subspace in X_n under the constraint

$$x_{i,j} = x_{i,j+1}, \ \ 1 \le i \le j \le m-1. \tag{1.43}$$

Given a truncated triangle $x \in X_{n,m}$, it is convenient to use notations $\lambda(x) := x^{(n)}, \ \ \nu(x) := x^{(m)}$ and to define a weight $\beta := \beta(x)$ of a truncated triangle $x \in X_{n,m}$ as a vector $\beta = (\beta_{m+1}, \cdots, \beta_n) \in \mathbf{R}^m$ such that $\beta_j = |x^{(j)}| - |x^{(j-1)}|$, $m < j \le n$. For given vectors $\lambda \in \mathbf{R}^n, \ \ \nu \in \mathbf{R}^m$ and $\beta \in \mathbf{R}^{n-m}$ we consider the following subspaces of the space $X_{n,m}$

$$\begin{aligned} X_{n,m}^{\lambda} &= \{x \in X_{n,m} \mid \lambda(x) = \lambda\}; \\ X_{n,m}^{\lambda\backslash\nu} &= \{x \in X_{n,m}^{\lambda} \mid \nu(x) = \nu\}; \qquad (1.44) \\ X_{n,m}^{\lambda\backslash\nu}(\beta) &= \{x \in X_{n,m}^{\lambda\backslash\nu} \mid \beta(x) = \beta\}. \end{aligned}$$

By definition, a truncated triangle $x \in X_{n,m}$ is called a truncated Gelfand-Tsetlin pattern iff x belongs to the cone K_n. We denote by $K_{n,m}$ the cone of all truncated GT-patterns and its intersections with subspaces (1.44) correspondently by $K^{\lambda}_{n,m}$, $K^{\lambda\setminus\nu}$ and $K^{\lambda\setminus\nu}(\beta)$. It is clear that $K^{\lambda\setminus\nu}$ and $K^{\lambda\setminus\nu}(\beta)$ are the convex, compact polytopes in the space $\mathbf{R}^{c_{n,m}}_{+}$, where $c_{n,m} = \frac{1}{2}(n-m)(n+m+1)$. It is well-known (see Section 2 or [BZ1]), that if λ and ν be the partitions, $\lambda \supseteq \nu$, $|\lambda \setminus \nu| = p$ and β is a composition of the same integer p, then the number of an integral points in the convex polytope $K^{\lambda\setminus\nu}$ (respectively in $K^{\lambda\setminus\nu}(\beta)$) is equal to the dimension of the representation $V_{\lambda\setminus\nu}$ of the Lie algebra $\mathfrak{gl}_{n-m}$ (see Section 2)(respectively the dimension of the subspace $V(\lambda\setminus\nu,\beta) \subset V_{\lambda\setminus\nu}$ of the weight β). On the other hand, as is also well known, the dimension of the weight subspace $V(\lambda\setminus\nu,\beta) \subset V_{\lambda\setminus\nu}$ admits a pure combinatorial description as the number of (skew) standard Young tableaux of the shape λ and content β

$$|K^{\lambda\setminus\nu}_{\mathbf{Z}}(\beta)| := |K^{\lambda\setminus\nu}(\beta) \cap \mathbf{Z}^{c_{n,m}}| = dim V(\lambda\setminus\nu,\beta) = |STY(\lambda\setminus\nu,\beta)|. \tag{1.45}$$

Using the constraint (1.43), it is possible to define the action of the symmetric group S_{n-m} on the space of truncated triangles $X_{n,m}$ by setting

$$\sigma_i = s_{m+i}, \quad 1 \le i \le n-m-1. \tag{1.46}$$

The fact that the involutions σ_i really generate the symmetric group S_{n-m} follows from Theorem 1.1 and Corollary 1.2.

Exercise 1.1. Fix real number ϵ and integers i, j s.t. $1 \le i \le j \le n$. Let us define a transformation $T^{(i,j)}_{\epsilon} : X_n \to X_n$ in the following way

$$T^{(i,j)}_{\epsilon}(x) := \widetilde{x}, \quad x \in X_n$$

where $\widetilde{x}_{ij} = x_{ij} - \epsilon$ and $\widetilde{x}_{kl} = x_{kl}$, if $(k,l) \ne (i,j)$. Show that

(*i*) $(t_j T^{(i,j)}_{\epsilon})^2 = 1$,
(*ii*) $t_k T^{(i,j)}_{\epsilon} = T^{(i,j)}_{\epsilon} t_k$, if $|k-j| \ge 2$,
(*iii*) $t_k T^{(i,j)}_{\delta} t_k T^{(i,j)}_{\epsilon-\delta} t_k T^{(i,j)}_{-\epsilon} = T^{(i,j)}_{\epsilon} t_k T^{(i,j)}_{-\epsilon+\delta} t_k T^{(i,j)}_{-\delta} t_k$, if $|k-j| = 1$, $\epsilon, \delta \in \mathbf{R}$,

which is a generalization of (1.14e).

Exercise 1.2. Fix real number q and integer j, $1 \le j \le n$. Let us define a transformation $\widetilde{t}_j : X_n \to X_n$:

$$\begin{aligned} &\widetilde{t}_j := t_j[q](x) = \widetilde{x}, \quad \text{where} \\ &\widetilde{x}_{i,k} = x_{i,k}, \quad \text{if} \quad k \ne j, \\ &\widetilde{x}_{i,j} = min(x_{i,j+1},\ q x_{i-1,j-1}) + max(x_{i+1,j+1},\ q x_{i,j-1}) - q x_{i,j}. \end{aligned}$$

Here we presuppose that $x_{0,j} := +\infty$, $x_{j,j-1} := -\infty$, $1 \le j \le n-1$. Show that

(*i*) $\widetilde{t}_j^2 = (1-q)\widetilde{t}_j + q \cdot Id_{X_n}$, $\widetilde{t}_i \widetilde{t}_j = \widetilde{t}_j \widetilde{t}_i$, if $|i-j| \ge 2$,
(*ii*) $\beta(\widetilde{t}_j(x)) = (\beta_1, \cdots, \beta_{j-1}, \beta_{j+1} + (1-q)\beta_j,\ q\beta_j, \beta_{j+1}, \cdots, \beta_n)$.

where $\beta(x) := (\beta_1, \cdots, \beta_n)$ is the weight of triangle x. Thus we obtain a representation of the Hecke algebra $H_n(q)$ on the space of weights $\mathbf{R}^n$.

Problems. (i) To find the defining relations between the transformations $\widetilde{t}_j$.

(ii) Is it possible to extend this representation of the Hecke algebra $H_n(q)$ to the whole space of triangles X_n?

(iii) Is it possible to construct a *cpl*-representation of the braid group B_n on the space of triangles X_n?

§2 Combinatorial description of the basic transformations.

In this section we give a combinatorial description of the restrictions of the transformations constructed in the previous section to the set of standard Young tableaux of a given shape λ and content β. We will denote this last set by $STY(\lambda, \beta)$. Also we exploit the notations $STY(\lambda \setminus \nu, \beta)$ for the set of skew standard Young tableaux of a skew shape $\lambda \setminus \nu$ and content β, and $STY(\lambda \setminus \nu, \leq n)$ for the set of all skew standard Young tableaux of the shape $\lambda \setminus \nu$ with all entries not exceeding n. We will assume in the sequel that $l(\lambda) \leq n$, $l(\nu) \leq m$, $l(\beta) \leq n$ for some fixed positive integers $m \leq n$.

At first we remind the well-known bijection (e.g.[GZ1],[GZ2],[BZ1]) between the set $STY(\lambda \setminus \nu, \beta)$ and the set of integral points in the convex polytope $K^{\lambda \setminus \nu}(\beta)$. So, take $T \in STY(\lambda \setminus \nu, \beta)$. As it is well known (e.g. [Ma]), one may consider the tableaux T as a sequence of Young diagrams

$$\nu = \lambda^{(m)} \subset \lambda^{m+1} \subset \cdots \subset \lambda^{(n)} = \lambda, \tag{2.1}$$

such that all skew diagrams $\lambda^{(i)} \setminus \lambda^{(i-1)}$, $m < i \leq n$ are a horizontal strip. Let us define the triangle $x = x(T) = (x^{(n)}, \cdots, x^{(m)})$, where $x^{(i)}$ is the shape of the diagram $\lambda^{(i)}$. Then we have

$$T \in STY(\lambda \setminus \nu, \beta) \ \text{ iff } \ x(T) \in K_{\mathbf{Z}}^{\lambda \setminus \nu}(\beta).$$

Let us construct the inverse map $K_{\mathbf{Z}}^{\lambda \setminus \nu}(\beta) \to STY(\lambda \setminus \nu, \beta)$. Given a point $x \in K_{\mathbf{Z}}^{\lambda \setminus \nu}(\beta)$, we are filling the shape $\lambda \setminus \nu$ by the numbers $1, \cdots, n$ according to the following rule: in the i-th row $(\lambda \setminus \nu)_i$ of the skew diagram $\lambda \setminus \nu$ we write exactly $x_{k,i+m} - x_{k,i+m+1}$ numbers equal to k, starting from $k = 1$.

Let us consider an explanatory example. Assume

$$T := \begin{array}{ccccc} & & & 1 & 4 \\ & 2 & 2 & 3 & 5 \\ 1 & 3 & 4 & 4 & \end{array}, \quad \lambda = (5,5,4), \ \ \nu = (3,1), \ \ \mu = (2,2,2,3,1).$$

Then we have the following sequence of shapes for (2.1):

$$\nu = (3,1) \subset (4,1,1) \subset (4,3,1) \subset (4,4,2) \subset (5,4,4) \subset (5,5,4) = \lambda.$$

Consequently,

$$X_{7,2} \ni x(T) := \begin{array}{ccccccccccccc} 5 & & 5 & & 4 & & 0 & & 0 & & 0 & & 0 \\ & 5 & & 4 & & 4 & & 0 & & 0 & & 0 & \\ & & 4 & & 4 & & 2 & & 0 & & 0 & & \\ & & & 4 & & 3 & & 1 & & 0 & & & \\ & & & & 4 & & 1 & & 1 & & & & \\ & & & & & 3 & & 1 & & & & & \end{array} \in K_{\mathbf{Z}}^{\lambda\backslash\nu}(\mu).$$

Let us remind briefly the group-theoretical interpretation of the numbers $K_{\lambda\backslash\nu,\beta} := |STY(\lambda \setminus \nu, \beta)|$ and $d^{(n)}_{\lambda\backslash\nu} := |STY(\lambda \setminus \nu, \leq n)|$. Let $\mathfrak{g}_n = \mathfrak{gl}_n$. The lattice of weights P_n of the Lie algebra $\mathfrak{g}_n$ is identified in the standard way with $\mathbf{Z}^n$; the set P_n^+ of highest weights of finite-dimensional, polynomial $\mathfrak{g}_n$-modules is identified with $\{\lambda = (\lambda_1, \cdots, \lambda_n) \in P_n \mid \lambda_1 \geq \lambda_2 \geq \cdots, \geq \lambda_n \geq 0\}$. For each $\lambda \in P_n^+$ let V_λ be an irreducible $\mathfrak{g}_n$-module with highest weight λ. For all $m,\ 0 \leq m \leq n$, we imbed the Lie algebra $\mathfrak{g}_m \times \mathfrak{g}_{n-m} \subset \mathfrak{g}_n$ so that the subalgebra $\mathfrak{g}_m$ is generated by the elements $e_{ij},\ 1 \leq i, j \leq m$, where $\{e_{ij} \mid 1 \leq i, j \leq n\}$ is a standard basis of $\mathfrak{g}_n$, and $\mathfrak{g}_{n-m}$ is generated by the elements $\{e_{ij} \mid m+1 \leq i, j \leq n\}$. Accordingly we will write the weights $\beta \in P_{n-m}$ in the form $\beta = (\beta_{m+1}, \cdots, \beta_n)$. For all $\lambda \in P_n^+$ and $\nu \in P_m^+$ we define (see [BZ1]) the skew $\mathfrak{g}_{n-m}$-module $V_{\lambda\backslash\nu}$ by putting

$$V_{\lambda\backslash\nu} = Hom_{\mathfrak{g}_m}(V_\nu, V_\lambda|_{\mathfrak{g}_m}). \tag{2.2}$$

When $m = 0$ it is convenient to assume that P_m^+ consists of one element $\nu = \phi$ and that $V_{\lambda\backslash\phi}$ is the irreducible $\mathfrak{g}_n$-module V_λ.

Proposition 2.1 (see [BZ1]). The multiplicity $K_{\lambda\backslash\nu,\beta}$ of the weight β in the skew $\mathfrak{g}_{n-m}$ -module $V_{\lambda\backslash\nu}$ is equal to the number of all truncated Gelfand-Tsetlin patterns of the highest weight $\lambda \setminus \nu$ and of weight β .

As a corollary, we obtain:

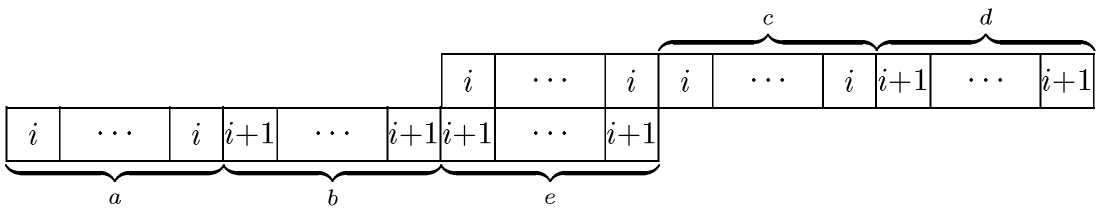
$$d^{(n)}_{\lambda\backslash\nu} := |K_{\mathbf{Z}}^{\lambda\backslash\nu}| = |STY(\lambda \setminus \nu, \leq n)| = dim V_{\lambda\backslash\nu}.$$

Note that the numbers $K_{\lambda\backslash\nu,\beta}$ admit a completely elementary description in terms of symmetric functions (e.g. [Ma]): they are the coefficients in the expression of the skew Schur function $S_{\lambda\backslash\nu}(x)$ as a linear combination of monomials $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

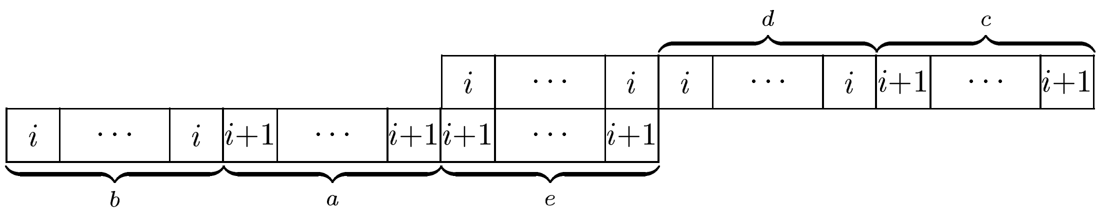
Now we are ready to give a combinatorial interpretation for the restrictions on the set $STY(\lambda \setminus \nu, \beta)$ of the transformations constructed in Section 1. We will keep the notations of Remark 1.5 and the beginning of Section 2.

A) The action of t_{i+m} on the set $STY(\lambda \setminus \nu, \beta)$.

Given a skew tableau $T \in STY(\lambda \setminus \nu, \beta)$. Consider the part T_i of the tableau T which is filled by the letters i and $i+1$. It is clear that T_i is a disjoint union of the following fragments



We replace every such fragment by the following one



The part $T \setminus T_i$ of the tableau T remains unchanged. As a result we obtain the new tableau $\widetilde{T} \in STY(\lambda \setminus \nu, (i, i+1)\beta)$.

Proposition 2.2. We have

$$t_{i+m}(T) = \widetilde{T}.$$

The proof is clear from the construction of the bijection

$$STY(\lambda \setminus \nu, \beta) \leftrightarrow K_{\mathbf{Z}}^{\lambda \setminus \nu}(\beta).$$

■

We remark that the action of the transformations t_i on the set $STY(\lambda \setminus \nu, \beta)$ as described above is well-known, see for example [BK],[SW],[Sa].

Our main observation is that the action t_i on the set of standard Young tableaux of a given shape and content admits a continuous piecewise linear prolongation on the set of triangles X_n and other well-known combinatorial operations on the set of Young tableaux may be presented as a superposition of the transformations t_i and consequently also admit a continuation on the space X_n. Anyhow, using the involutions t_i we may construct for any element σ of the symmetric group S_n a piecewise linear one-to-one mapping, defined over $\mathbf{Z}$, between the GT-polytopes $K^{\lambda \setminus \nu}(\beta)$ and $K^{\lambda \setminus \nu}(\sigma\beta)$. Note finally that for the definition of the involution t_i it is essential to have the whole tableau T , but not only the corresponding word $w(T)$, so we can not define the action of t_i on the words.

Remark 2.1. The action of the involutions t_i on the set of standard Young tableaux $STY(\lambda)$, i.e. in the case when weight $\beta = (1^n)$, $|\lambda| = n$, is studied in many papers. We mention only the work of A.Garsia and T.McLarnan [GM], which contains, among other, the interesting results concerning a connection between the Robinson-Schensted correspondence and the transformations t_i. Now let us say some words about the group G_n generated by t_i, $1 \le i \le n-1$,in the case under consideration. It is not difficult to verify that $t_1 = Id_{STY(\lambda)}$, and $(t_i t_{i+1})^6 = 1, \ 1 \le i \le n-2$. Because of $t_1 = Id_{STY(\lambda)}$, we see that $s_i = Id$ on the set $STY(\lambda)$ for all $1 \le i \le n-1$. We note that if the diagram λ is a hook, then $(t_i t_{i+1})^3 = 1, \ \ 1 \le i \le n-2$, and the group G_n considered as a subgroup of $Aut(STY(\lambda))$ is isomorphic to the symmetric group S_{n-1}.

Remark 2.2. We give another combinatorial description of the action of the trasnformation t_i on the Gelfand-Tsetlin cone K_n. For this let us describe, at first, some special triangulation of the cone K_n. We denote by

$$\mathbf{B}_n = K_n \cap \{ \ 0, 1 \ \}^{\frac{n^2+n}{2}}$$

the set of all (0,1) - patterns.

Lemma 2.1. There exist a bijection

$$\mathbf{B}_n \to \{ \ 0, 1 \ \}^n,$$

in particular, $|\mathbf{B}_n| = 2^n$.

Proof. The bijection under consideration is given by a correspondence

$$x \in \mathbf{B}_n \to \beta(x) \in \{ \ 0, 1 \ \}^n.$$

It is easy to see that the GT-pattern consisting of only from zeros and units is uniquely determined by its weight. ■

Now let us introduce on the set of triangles X_n a partial order, assuming $x \succeq \widetilde{x}$ iff $x - \widetilde{x} \in (X_n)_{\mathbf{R}_+} = (\mathbf{R}_+)^{\frac{n^2+n}{2}}$.

Proposition 2.3. The set $\mathbf{B}_n$ possesses the following properties:

i) $\mathbf{B}_n$ is a set of all generators of the cone $(K_n)_{\mathbf{Z}} = K_n \cap (X_n)_{\mathbf{Z}}$ (and, consequently, also for the cone $K_n = \mathbf{R}_+ \otimes (K_n)_{\mathbf{Z}}$).

ii) $\mathbf{B}_n$ is a lattice with respect to the partial order "$\succeq$".

iii) The length of the maximal chain in the lattice $\mathbf{B}_n$ is equal to $\frac{n(n+1)}{2}$.

iv) Assume $x \in K_n$, then there exist unique decomposition

$$x = \alpha_1 x_1 + \cdots + \alpha_m x_m, \tag{2.3}$$

where $x_1, \cdots, x_m \in \mathbf{B}_n, \ \alpha_1, \cdots, \alpha_m \in \mathbf{R}, \ \alpha_1 > 0, \cdots, \alpha_m > 0$, and $x_1 \succ x_2 \succ \cdots \succ x_m \succ 0, \ (m < \frac{n^2+n}{2})$. ■

Proposition 2.4 (The connection between the involutions t_i and the Gelfand-Tsetlin cone K_n).

i) If $x \in \mathbf{B}_n$, then $t_i(x) \in \mathbf{B}_n, \ \ 1 \le i \le n-1$.

ii) Let us assume that x is written uniquely in the form (2.3). Then

$$t_i(x) = \alpha_1 t_i(x_1) + \cdots + \alpha_m t_i(x_m).$$

■

Note that if the elements $x_1, \cdots, x_m$ lie in the same chain of the lattice $\mathbf{B}_n$, then their images $t_i(x_1), \cdots, t_i(x_m)$ may belong to the different chains.

Conjecture 2.1. Assume that $\lambda, \beta \in \mathbf{Z}^n$, then all vertices of the Gelfand-Tsetlin polytope $K^\lambda(\beta)$ belong to the set $K_n \cap (X_n)_{\mathbf{Z}}$.

B) The action of transformations p_i and p_i^{-1}.

Let λ be a partition and β be a composition, $\beta = (\beta_1, \cdots, \beta_n)$. Note at first that it is sufficient to describe the action of the transformations $\sigma = p_{n-1}^{-1}$ and $\sigma^{-1} = p_{n-1}$ on the set

$$STY(\lambda, \beta) \simeq K_{\mathbf{Z}}^\lambda(\beta) \hookrightarrow X_n.$$

In fact, we have a natural embedding $X_i \hookrightarrow X_n$, $x \to \widetilde{x}$, where for a triangle $x = (x^{(i)}, \cdots, x^{(1)}) \in X_i$ we define

$$\widetilde{x} = (\underbrace{x^{(i)}, \cdots, x^{(i)}}_{n-i}, x^{(i)}, \cdots, x^{(1)}) \in X_n.$$

On the level of Young tableaux this embedding corresponds to the consideration of the part $T_{\leq i}$ of the tableau T,which is filled by the numbers $1, \cdots, i$. It is easy to verify the commutativity of the following diagram

$$\begin{array}{ccc} X_i & \hookrightarrow & X_n \\ p_{i-1} \downarrow & & \downarrow p_{i-1} \\ X_i & \hookrightarrow & X_n \end{array}$$

This means that $p_{i-1} \in G_n$ acts non-trivially only on the tableau $T_{\leq i}$. Remark further that

$$\beta(\sigma^{-1}(x)) = (\beta_2\beta_3 \cdots \beta_n\beta_1) = (1, 2, \cdots, n)\beta(x),$$
$$\beta(\sigma(x)) = (\beta_n\beta_1 \cdots \beta_{n-1}) = (n, n-1, \cdots, 1)\beta(x).$$

Given a tableau $T \in STY(\lambda \setminus \nu, \beta)$. We may assume that $\beta_1 > 0$. We delete the most upperleft box filled with number 1. After this we apply a "jeu de taquin" (e.g. [Sch1],[Sch2]) in oder to obtain a standard tableau T' which has the same left border strip and one box less than T. Repeating this procedure we eliminate all ones from the tableau T. As a result we obtain a new tableau T'' of the content $(0, \beta_2, \cdots, \beta_n)$ and a horizontal strip θ_n, $|\theta_n| = \beta_1$. We fill the strip θ_n by the number n. Now let us subtract one from all numbers situated in the boxes of T''. We obtain a tableau $\widetilde{T}$ of the content $\widetilde{\beta} = (\widetilde{\beta}_1, \cdots, \widetilde{\beta}_{n-1})$, where $\widetilde{\beta}_i = \beta_{i+1}$, $1 \leq i \leq n-1$. The tableau $\widetilde{T}$ and the horizontal strip θ_n define a tableau $\overline{T} \in STY(\lambda, \overline{\beta})$, where $\overline{\beta} = (\beta_2, \cdots, \beta_n, \beta_1)$.

Proposition 2.5. We have (in the case $\nu = \phi$)

$$\sigma^{-1}(T) = p_{n-1}(T) = \overline{T}.$$

Similarly we may describe the action of the transformation $\sigma = p_{n-1}^{-1}$. However in this case it is more convenient to consider the action of σ on the set of skew Young tableaux. So, let ν be a partition, $\nu \subseteq \lambda$. Given a tableau $T \in STY(\lambda \setminus \nu, \beta)$, we may assume that $\beta_n > 0$. Let us delete the most right and bottom box in T which occupied by the number n. After this we apply a "jeu de taquin" (e.g. [Sch2],[Sa]) in order to obtain a standard tableau $T' \in STY(\lambda \setminus \widetilde{\nu}, \widetilde{\beta})$ with the same right border strip, which has one box less than T (in fact $\widetilde{\beta} = (\beta_1, \cdots, \beta_{n-1}, \beta_n - 1)$, $|\widetilde{\nu}| = |\nu| + 1$). Repeating this procedure we eliminate all numbers equal to n from the tableau T. As a result we obtain a new tableau T'' of the content $\overline{\beta} = (\overline{\beta}_1, \cdots \overline{\beta}_n)$, where $\overline{\beta}_1 = 0$, $\overline{\beta}_i = \beta_{i-1}$, $2 \le i \le n$. Further, let us fill up the strip θ_1 by the number 1. The tableau $\overline{T}$ and the horizontal strip θ_1 define a tableau $\widehat{T} \in STY(\lambda \setminus \nu, \widehat{\beta})$ where $\widehat{\beta} = (\beta_n, \beta_1, \beta_2, \cdots, \beta_{n-1})$.

Proposition 2.6. We have (in the case $\nu = \phi$)

$$\sigma(T) = p_{n-1}^{-1}(T) = \widehat{T}.$$

The proof of Propositions 2.5 and 2.6 is based on the work of M.-P. Schützenberger [Sch2] and Proposition 2.2 . Details will appear elsewhere.

C) The action of the involutions $q_i \in G_n$, $1 \le i \le n$.

Let a tableau $T \in STY(\lambda, \beta)$ be given. Note at the beginning, that involution q_{i-1} acts nontrivially only on the tableau $T_{\le i}$ (see Section B), so it is sufficient to describe the action of the transformation q_{n-1} on the set

$$STY(\lambda, \beta) \simeq K_{\mathbf{Z}}^{\lambda}(\beta) \subset X_n.$$

Remark 2.3. We use a term involution for the restriction of q_i to the set $STY(\lambda, \beta)$ in spite of the fact that in general the transformation q_i does't conserve this set, but, of course, $q_i^2 = Id_{STY(\lambda,\beta)}$.

Theorem 2.1. The restriction of the involution q_{n-1} to the set $STY(\lambda, \beta)$ coinsides with the Schützenberger involution $\mathbf{S}$.

Let us recall the corresponding definitions. Let λ, ν, β and n be as above. Let be given a tableau $T \in STY(\lambda \setminus \nu, \beta)$. We may define four natural operations on the set $STY(\lambda \setminus \nu, \beta)$. In this section we give a definition of the first two operations and a definition of the remaining one in the Section D). Note, that it is sufficient to consider the case when all parts of the composition β are not equal to zero. We already gave a combinatorial description of the transformations p_i and p_i^{-1}. It is a "jeu de taquin" that is the crucial step for the description of these transformations. Let us remind (see Section B) that $p_{n-1}(T) = \overline{T}$, where $T \in STY(\lambda \setminus \nu, \overline{\beta})$, $\overline{\beta} = (\beta_2, \beta_3, \cdots, \beta_n, \beta_1)$, and in fact, the tableau $\overline{T}$ is a disjoint union of the tableau $\widetilde{T}$ of content $\widetilde{\beta} = (\widetilde{\beta}_1, \cdots, \widetilde{\beta}_{n-1})$, where $\widetilde{\beta}_i = \beta_{i+1}$, $1 \le i \le n-1$, and the filled horizontal strips θ_n, $|\theta_n| = \beta_1$, with the number n in all boxes. After this we may act in two ways. In the first one, let us apply the same algorithm to the tableau $\widetilde{T}$ and so on. As a result we obtain a sequence of horizontal strips $\theta_1, \theta_2, \cdots, \theta_n$ such that $|\theta_i| = \beta_{n-i+1}$, $1 \le i \le n$, which defines (e.g. [Ma]) a standard Young tableau

$T_0 \in STY(\lambda \setminus \nu, \overleftarrow{\beta})$ where $\overleftarrow{\beta} = (\beta_n, \beta_{n-1}, \cdots, \beta_1)$. Therefore there are maps

$$\begin{aligned} \mathbf{S}: \ & STY(\lambda \setminus \nu, \beta) \to STY(\lambda \setminus \nu, \overleftarrow{\beta}), \ T \to T_0, \\ \mathbf{S}: \ & K_{\mathbf{Z}}^{\lambda\setminus\nu} \to K_{\mathbf{Z}}^{\lambda\setminus\nu}. \end{aligned} \tag{2.4}$$

The mapping $\mathbf{S}$ is called the Schützenberger involution. If the weight β has equal parts, then we obtain the involution $\mathbf{S}: \ STY(\lambda, \beta) \to STY(\lambda, \beta)$.

The second way is to apply the transformation p_{n-1} to the tableau T n times, i.e. consider a tableau $p_{n-1}^n(T)$. It is interesting to remark that transformation $\mathbf{U} := p_{n-1}^n \cdot \mathbf{S}$ is also an involution, which will be described in Section D . From our description of the Schützenberger involution $\mathbf{S}$ it is obvious that $q_{n-1}|_{STY(\lambda,\beta)} = \mathbf{S}$. This ends the proof of Theorem 2.1.

■

Remark 2.4 (The symmetries of the Littlewood-Richardson numbers). Let V_λ, V_μ, V_ν be three irreducible finite-dimensional sl_n-modules with highest weights λ, μ, ν. The multiplicity $c_{\lambda\mu}^\nu$ of V_ν in the tensor product $V_\lambda \otimes V_\mu$ is called the Littlewood-Richardson (LR) number and plays an important role in the representation theory of the symmetric and general linear groups. The combinatorial rule for computing the LR-numbers is given by Littlewood and Richardson (LR-rule), [Li] (see also [T], [Ma]). There are many various interpretations of the LR-rule:[GZ1],[GZ2], [T],[LS2],[KR],[W],[Z]. We will use one in terms of the number of standard Young tableaux of certain kind [GZ2],[KR], and another one in terms of BZ-triangles [BZ3]. Let us remind the corresponding definitions. From the beginning (see e.g. [Ma]) we observe that $c_{\lambda\mu}^\nu \neq 0$ iff both λ and μ are contained in ν and $|\nu| \equiv |\lambda| + |\mu| \ (mod\ n)$. Further, let us consider a tableau $T \in STY(\lambda, \nu \setminus \mu)$ and its descent set $D(T)$. We recall that if $T \in STY(\lambda, \mu)$ then $D(T)$ is the maximal among of those subsets $\{x_1, \cdots, x_l\} \subset T$, which satisfy the following condition: there exists a sequence of entries $y_1, \cdots, y_l$ of the tableau T, all lying in the different boxes, such that $y_i = x_i + 1$ and $n(x_i) < n(y_i)$, $1 \le i \le l$. Here $n(x)$ is an index of the row which contains x. So, let $D(T) = \{x_1, \cdots, x_l\}$ be the descent set of T. We define a set of exponents for the tableau T as a collection of the integers $\{d_1(T), d_2(T), \cdots, d_{n-1}(T)\}$, where

$$d_i(T) := \mu_{i+1} - |\{x_j \in D(T) \mid x_j = i\}|, \ \ 1 \le i \le n-1.$$

Proposition 2.7 [GZ2],[KR]. The LR-number $c_{\lambda\mu}^\nu$ is equal to the number of standard Young tableaux T of the shape λ and content $\nu \setminus \mu$, such that

$$d_i(T) \le \mu_i - \mu_{i+1}, \ \ 1 \le i \le n-1.$$

Let us denote the set of Young tableaux mentioned in Proposition 2.7 by $STY(\lambda, \nu \setminus \mu \parallel \mu)$.

Proposition 2.8. The involution q_{n-1} defines a bijection

$$q_{n-1} : STY(\lambda, \nu \setminus \mu \parallel \mu) \to STY(\lambda, \mu^* \setminus \nu^* \parallel \nu^*), \tag{2.5}$$

where for a partition $\mu = (\mu_1, \cdots, \mu_{n-1})$ we put $\mu^* := (\mu_1, \mu_1 - \mu_{n-1}, \cdots, \mu_1 - \mu_2)$.

Now we want to extend the bijection (2.5) to one between some convex polytopes in the GT-patterns cone K_n, but before let us give the necessary definitions. Given a triangle $x \in X_n$, let us put

$$d_j^{(i)}(x) = \sum_{1 \le k < j} (x_{k,i+1} - 2x_{k,i} + x_{k,i-1}) + x_{j,i+1} - x_{j,i},$$

where $1 \le j \le i \le n-1$. Following [GZ1], we will call the numbers $d_j^{(i)}$ as the exponents of the triangle $x \in X_n$. Note [KR], that if λ is a partition, β is a composition and $x \in K_{\mathbf{Z}}^{\lambda}(\beta)$, then

$$max\{ d_j^{(i)}(x) \mid 1 \le j \le i \} = d_i(x(T)),\ 1 \le i \le n-1.$$

At last, for $\lambda, \beta, \gamma \in \mathbf{R}^n$ let us define (see [GZ1]) a convex polytope

$$K^{\lambda}(\beta, \gamma) = \{x \in K^{\lambda}(\beta) \mid d_j^{(i)}(x) \le \gamma_i - \gamma_{i+1},\ 1 \le j \le i \le n\}. \tag{2.6}$$

Proposition 2.9. Let $V_\lambda,\ V_\mu,\ V_\nu$ be three irreducible finite-dimensional sl_n-modules with the highest weights λ, μ, ν. Then

i) (Gelfand-Zelevinsky's theorem)

$$c_{\lambda\mu}^{\nu} = |K^{\lambda}(\nu \setminus \mu, \mu) \cap (X_n)_{\mathbf{Z}}|$$

ii) The involution q_{n-1} defines a bijection

$$q_{n-1}\ :\ K^{\lambda}(\nu \setminus \mu, \mu) \to K^{\lambda}(\mu^* \setminus \nu^*, \nu^*). \tag{2.7}$$

The bijection (2.5) gives a combinatorial explanation of the well known equality $c_{\lambda\mu}^{\nu} = c_{\lambda\nu^*}^{\mu^*}$. In oder to understand better the other symmetries of the LR-numbers, it is convenient to use an interpretation of the LR-rule in terms of Berenstein-Zelevinsky's triangles (BZ-triangles) [BZ3]. Let us review briefly the corresponding construction from [BZ3],[C]. Denote by $T_n,\ n \ge 3$, the set of vertices of a regular triangular lattice filling the regular triangle with vertices $(2n-3,\ 0,\ 0)$, $(0,\ 2n-3,\ 0)$ and $(0,\ 0,\ 2n-3)$; this triangle is decomposed into the union of elementary triangles having all three vertices in a set Q_n, and of elementary hexagons centered at points H_n, where

$$H_n := \{(i,j,k) \in T_n \mid \text{all } i,j,k \text{ are odd}\},$$
$$Q_n = T_n \setminus H_n.$$

Let us consider the vector space L consisting of families $(z(\xi)),\ \xi \in Q_n$ of real numbers such that for any elementary hexagon the sums of summits of opposite sides are equal; let also $\mathcal{K} = \mathcal{K}_n := L \cap \mathbf{R}_+^{Q_n}$ and $\mathcal{K}_{\mathbf{Z}} := L \cap \mathbf{Z}_+^{Q_n}$. We define a linear projection $pr\ :\ L \to \mathbf{R}^{3n-3}$ by the formulas

$$pr(z) = (l_1, \cdots, l_{n-1}; m_1, \cdots, m_{n-1}; k_1, \cdots, k_{n-1}),$$

where

$$\begin{aligned} l_p &= z(2(n-p)-1,\ 2p-2,\ 0) + z(2(n-1-p),\ 2p-1,\ 0); \\ k_p &= z(0,\ 2(n-p)-1,\ 2p-2) + z(0,\ 2(n-1-p),\ 2p-1); \\ m_p &= z(2p-2,\ 0,\ 2(n-p)-1) + z(2p-1,\ 0,\ 2(n-1-p)). \end{aligned} \tag{2.8}$$

Now let V_λ, V_μ, V_ν be three irreducible finite dimensional sl_n-modules with the highest weights λ, μ, and ν. We recall that the triple multiplicity $c_{\lambda\mu\nu}$ is defined as $dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathfrak{g}}$. Evidently we have $c_{\lambda\mu\nu} = c^{\nu^*}_{\lambda\mu}$, where ν^* is the highest weight of the module V^*_ν dual to V_ν (see also Proposition 2.8). Let us denote by $BZ(\lambda,\mu,\nu)$ a set of families $z(\xi) \in \mathcal{K}_n$ such that $pr(z(\xi)) = (l,m,k)$ where $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$ and $\nu_i = k_i - k_{i+1}$ for all i, $1 \le i \le n-1$ (we assume that $l_n = m_n = k_n = 0$). We define a mapping

$$\theta_{\lambda\mu\nu}:\ K^\lambda(\nu^* \setminus \mu, \mu) \to BZ(\lambda,\mu,\nu) \tag{2.9}$$

by the following way: given a triangle $x \in K^\lambda(\nu^* \setminus \mu, \mu)$, let us put $\theta_{\lambda\mu\nu}(x) = z(\xi)$, where

$$z(2i-2,\ 2(j-i)-1,\ 2(n-j)) = x_{ij} - x_{i,j-1},\ 1 \le i \le j \le n,$$

and observe that the value of $z(\xi)$ in all other points $\xi \in Q_n$ are uniquely determined from the condition $z(\xi) \in BZ(\lambda,\mu,\nu)$ (see [BZ3]).

Proposition 2.10. Assume V_λ, V_μ, V_ν as before.

i) (Berenstein-Zelevinsky [BZ3]). The mapping (2.9) is a bijection. In particular,

$$c_{\lambda\mu\nu} = |BZ(\lambda,\mu,\nu)|. \tag{2.10}$$

ii) A transformation $\widetilde{q}_{n-1} := \theta^{-1}_{\lambda\nu\mu} \cdot q_{n-1} \cdot \theta_{\lambda\mu\nu}$ defines a bijection

$$\begin{array}{ccc} BZ(\lambda,\mu,\nu) & \xrightarrow{\widetilde{q}_{n-1}} & BZ(\lambda,\nu,\mu) \\ \big\downarrow \theta_{\lambda\mu\nu} & & \big\downarrow \theta_{\lambda\nu\mu} \\ K^\lambda(\nu^* \setminus \mu, \mu) & \xrightarrow{q_{n-1}} & K^\lambda(\mu^* \setminus \nu, \nu) \end{array} \tag{2.11}$$

Now let as summarize our discussion of the symmetries of the Littlewood-Richardson numbers, or, that is the same, the symmetries of the triple multiplicities $c_{\lambda,\mu,\nu}$. Clearly, coefficients $c_{\lambda\mu\nu}$ must be invariant under all 6 permutations of (λ,μ,ν), and also under the replacement of (λ,μ,ν) by (λ^*,μ^*,ν^*). These transformations generate a group of 12 symmetries. Obviously, the set Q_n and H_n are invariant under all permutations of indices (i,j,k). Therefore we have a natural action of S_3 on $\mathbf{R}^{Q_n}$, and it is evident that L, $\mathcal{K}$ and $\mathcal{K}_{\mathbf{Z}}$ are invariant under this action. Formulas (2.8) imply at once that if $pr(z) = (\lambda,\mu,\nu)$ then $pr(s_1(z)) = (\lambda^*,\nu^*,\mu^*)$ and $pr(s_2(z)) = (\mu^*,\lambda^*,\nu^*)$, where $z := z(\xi) \in BZ(\lambda,\mu,\nu)$, and $s_1 = (1,2)$ and

$s_2 = (2,3)$ be two standard generators of S_3. We see that the expression (2.10) of triple multiplicities (the Berenstein-Zelevinsky theorem [BZ3]) makes evident the following symmetries :

$$c_{\lambda\mu\nu} = c_{\lambda^*\nu^*\mu^*} = c_{\mu^*\lambda^*\nu^*} = c_{\nu\lambda\mu} = c_{\mu\nu\lambda} = c_{\nu^*\mu^*\lambda^*}. \tag{2.12}$$

The remaining symmetries can be derived from Proposition 2.10, *ii*), namely, the transformation $\widetilde{q}_{n-1}$ (see (2.11)) defines the bijection $BZ(\lambda,\mu,\nu) \to BZ(\lambda,\nu,\mu)$, which gives a combinatorial proof of the equality $c_{\lambda\mu\nu} = c_{\lambda\nu\mu}$. All other symmetries follow from (2.12) for the triple (λ,ν,μ).

Finally, let us consider an explanatory example. Assume $n = 4,\ \lambda = (6,2,1)$, $\mu = (5,2,1),\ \nu = (6,5,2)$. It is easy to see that $c^{\nu}_{\lambda\mu} = 4$. Let us consider the following tableaux

$$T\ :\ \begin{array}{l} 1\,1\,2\,2\,2\,4 \\ 2\,3 \\ 3 \end{array} \in STY(\lambda, \nu\setminus\mu \| \mu), \qquad T'\ :\ \begin{array}{l} 1\,2\,2\,3\,3\,4 \\ 3\,3 \\ 4 \end{array} \in STY(\lambda, \mu^*\setminus\nu^* \| \nu^*).$$

It is easy to varify that $T' = \mathbf{S}(T)$, where $\mathbf{S}$ is the Schützenberger involution (see Remark 2.3). Futher, let us construct the corresponding GT-patterns

$$x(T)\ :\ \begin{array}{ccccccc} 6 & & 2 & & 1 & & 0 \\ & 5 & & 2 & & 1 & \\ & & 5 & & 1 & & \\ & & & 2 & & & \end{array} \in K^{\lambda}(\nu\setminus\mu,\mu), \qquad x(T')\ :\ \begin{array}{ccccccc} 6 & & 2 & & 1 & & 0 \\ & 5 & & 2 & & 0 & \\ & & 3 & & 0 & & \\ & & & 1 & & & \end{array} \in K^{\lambda}(\mu^*\setminus\nu^*,\nu^*).$$

It is easy to see that $x(T') = q_3(x(T)),\ q_3 = t_1t_2t_1t_3t_2t_1$. Finally, we construct the BZ-triangles corresponding to the GT-patterns $x(T)$ and $x(T')$. For lucidity let us give a construction in the several steps:

$$\begin{array}{ccccccc} 6 & & 2 & & 1 & & 0 \\ & 5 & & 2 & & 1 & \\ & & 5 & & 1 & & \\ & & & 2 & & & \end{array} \to \begin{array}{ccccc} 1 & & 0 & & 0 \\ & 0 & & 1 & \\ & & 3 & & \end{array} \to \begin{array}{ccccccccccc} & & & & & * & & & & & \\ & & & & 0 & & * & & & & \\ & & & * & & & & * & & & \\ & & 0 & & * & & 1 & & * & & \\ & * & & & & * & & & & * & \\ 1 & & * & & 0 & & * & & 3 & & * \end{array} \to \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & 0 & & 0 & & 1 & & 3 & & \\ & 3 & & & & 3 & & & & 1 & \\ 1 & & 0 & & 0 & & 1 & & 3 & & 0 \end{array}$$

$K^{\lambda}(\nu\setminus\mu,\mu)$ $\qquad\qquad\qquad\qquad$ $BZ(\lambda,\mu,\nu)$

$$\begin{array}{ccccccc} 6 & & 2 & & 1 & & 0 \\ & 5 & & 2 & & 0 & \\ & & 3 & & 0 & & \\ & & & 1 & & & \end{array} \to \begin{array}{ccccc} 1 & & 0 & & 1 \\ & 2 & & 2 & \\ & & 2 & & \end{array} \to \begin{array}{ccccccccccc} & & & & & * & & & & & \\ & & & & 1 & & * & & & & \\ & & & * & & & & * & & & \\ & & 0 & & * & & 2 & & * & & \\ & * & & & & * & & & & * & \\ 1 & & * & & 2 & & * & & 2 & & * \end{array} \to \begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 3 & & & & \\ & & & 1 & & & & 0 & & & \\ & & 0 & & 2 & & 2 & & 1 & & \\ & 3 & & & & 1 & & & & 1 & \\ 1 & & 0 & & 2 & & 1 & & 2 & & 0 \end{array}$$

$K^{\lambda}(\mu^*\setminus\nu^*,\nu^*)$ $\qquad\qquad\qquad\qquad$ $BZ(\lambda,\nu,\mu)$

The second triangles are composed from the differences $\{x_{i,j} - x_{i,j-1},\ 1 \le i < j \le n\}$ for the corresponding triangle $x \in X_n$.

D) The action of the involutions $u_i \in G_n$, $1 \le i \le n-1$.

Let a tableau $T \in STY(\lambda \setminus \nu, \beta)$ be given. Note that the involution u_{n-i}, $1 \le i \le n-1$ acts nontrivially only on the skew tableau $T_{\ge i}$, where $T_{\ge i}$ is the part of tableau T which is filled by the numbers $i, i+1, \cdots, n$. So it is sufficient to describe the action of the involution u_{n-1} on the set

$$STY(\lambda \setminus \nu, \beta) \subset K_{\mathbf{Z}}^{\lambda \setminus \nu}(\beta) \subset X_n.$$

Theorem 2.2. Assume $\nu = \phi$. The involution u_{n-1} on the set $STY(\lambda, \beta)$ coincides with the dual Schützenberger involution $\mathbf{U}$, where $\mathbf{U} := p_{n-1}^n \cdot \mathbf{S}$.

Let us continue the construction of natural operations on the set $STY(\lambda \setminus \nu, \beta)$ (see Section C). We remind (see Section B) that $p_{n-1}^{-1}(T) = \widehat{T}$, where $\widehat{T} \in STY(\lambda \setminus \nu, \beta)$, $\widehat{\beta} = (\beta_n, \beta_1, \cdots, \beta_{n-1})$. In fact, the tableau $\widehat{T}$ is a disjoint union of the tableau $\overline{T}$ of the content $\overline{\beta} = (\overline{\beta}_1, \cdots, \overline{\beta}_n)$, where $\overline{\beta}_1 = 0$, $\overline{\beta}_i = \beta_{i-1}$, $2 \le i \le n$, and the filled horizontal strips θ_1, $|\theta_1| = \beta_n$, with the number 1 in all boxes. As well as in Section C we may go further in two ways. In the first, let us apply the same algorithm to the tableau $\overline{T}$ and so on. As a result we obtain a sequence of horizontal strips $\theta_1, \theta_2, \cdots, \theta_n$ such that $\theta_i = \beta_{n-i+1}$, $1 \le i \le n$, which defines a standard Young tableau $T_* \in STY(\lambda \setminus \nu, \overleftarrow{\beta})$. Consequently there are maps

$$\begin{aligned} \mathbf{U}: &\ STY(\lambda \setminus \nu, \beta) \to STY(\lambda \setminus \nu, \overleftarrow{\beta}), \ T \to T_*, \\ \mathbf{U}: &\ K_{\mathbf{Z}}^{\lambda \setminus \nu} \to K_{\mathbf{Z}}^{\lambda \setminus \nu}. \end{aligned} \tag{2.13}$$

The mapping $\mathbf{U}$ is called the dual Schützenberger involution. If the weight β has equal parts, we obtain the involution $\mathbf{U}: \ STY(\lambda, \beta) \to STY(\lambda, \beta)$.

Another way is to apply the transformation p_{n-1}^{-1} to the tableau T n times, i.e. consider a tableaux $p_{n-1}^{-n}(T)$. It is possible to show that, in fact, the two definitions of the involution $\mathbf{U}$ coincide. So, $p_{n-1}^{-n} = \mathbf{SU}$. From the definition of the dual Schützenberger involution $\mathbf{U}$ it is clear that $u_{n-1}|_{STY(\lambda,\beta)} = \mathbf{U}$.T his ends the proof of Theorem 2.2.

■

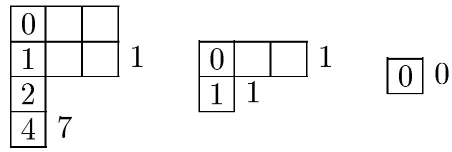
Remark 2.5. In Sections C and D we gave combinatorial definitions for transformations $\mathbf{S}, \mathbf{U}$, $\sigma^n = \mathbf{SU}$ and $\sigma^{-n} = \mathbf{US}$.

Our notations differ from one in [Sch1], [Sch2] and [EG] for the case $\beta = (1^n)$. We use words "a jeu de taquin" instead of "a promotion transformation $T \to T^\partial$" (e.g. [Sch1], [Sch2]). In fact, using the notations from [EG], we have (on the set $STY(\lambda)$) :

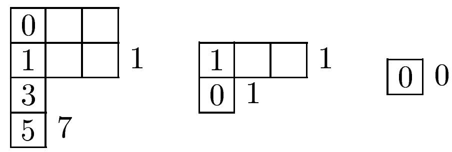
$$\mathbf{S}(T) = T^\partial + 1, \ \ \mathbf{U}(T) = T^S, \ \ \mathbf{SU} = p.$$

As an illustration of the operations $\mathbf{S}, \mathbf{U}$ σ^n, and σ^{-n}, consider the standard Young tableau

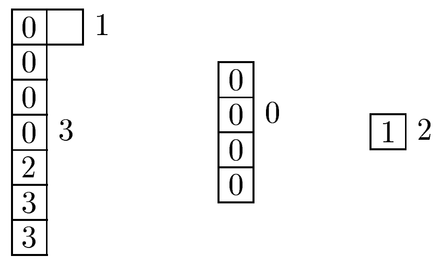
$$T \quad = \quad \begin{array}{lllll} 1 & 2 & 3 & 8 & 10 \\ 4 & 6 & 7 & 13 & \\ 5 & 9 & 12 & & \\ 11 & & & & \end{array}$$



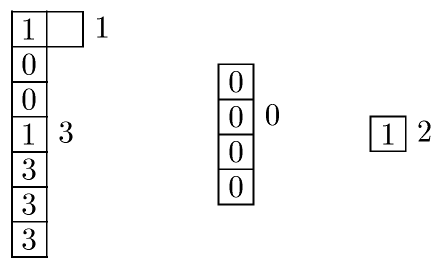
$$\mathbf{S}(T) \quad = \quad \begin{array}{lllll} 1 & 2 & 3 & 8 & 9 \\ 4 & 5 & 7 & 10 & \\ 6 & 12 & 13 & & \\ 11 & & & & \end{array}$$



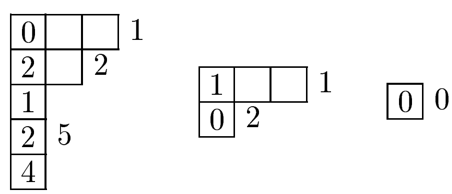
$$\mathbf{U}(T) \quad = \quad \begin{array}{lllll} 1 & 4 & 6 & 10 & 11 \\ 2 & 5 & 8 & 12 & \\ 3 & 7 & 13 & & \\ 9 & & & & \end{array}$$



$$\mathbf{SU}(T) \quad = \quad \begin{array}{lllll} 1 & 4 & 5 & 7 & 11 \\ 2 & 6 & 9 & 12 & \\ 3 & 8 & 13 & & \\ 10 & & & & \end{array}$$



$$\mathbf{US}(T) \quad = \quad \begin{array}{lllll} 1 & 2 & 4 & 5 & 11 \\ 3 & 6 & 8 & 9 & \\ 7 & 12 & 13 & & \\ 10 & & & & \end{array}$$



Note that the tableaux T and $\mathbf{S}(T)$ have the same configuration and the complementary quantum numbers, i.e.

$$J_{\alpha,n}^{(k)}(T) + J_{m_n(\nu^{(k)})-\alpha+1,n}^{(k)}(\mathbf{S}(T)) = P_n^{(k)}(\nu,\beta),$$

see Remark 2.8. It is a general property of the Schützenberger involution, [Ki1].

Remark 2.6. The involutions $\mathbf{S}$ and $\mathbf{U}$ in general do not commute. It is an interesting combinatorial task to find the order of the transformation $\mathbf{SU}$ on the set $STY(\lambda,\beta)$. In some special cases the answer is known (e.g. [Sch1], [EG]). Using this answer and formula (1.20), we find

Proposition 2.11.

a) If λ is a rectangular diagram, $l(\lambda) \leq n$ then

$$\sigma^n|_{K^\lambda_{\mathbf{Z}}} = \mathbf{SU} = Id_{K^\lambda_{\mathbf{Z}}}. \tag{2.14}$$

b) If $\lambda = (n-1, \cdots, 1, 0)$ be a staircase diagram, then

$$\sigma^{2n}|_{STY(\lambda)} = (\mathbf{SU})^2 = Id_{STY(\lambda)}. \tag{2.15}$$

In fact, if $T \in STY(\lambda)$, then $\sigma^n(T)$ is the conjugating of T.

Assertion *a*) was proved by M.-P. Schützenberger [Sch1] and later rediscovered by T. Miwa [Mi] in a context of the theory of crystal base (e.g. [Ka1]). The assertion *b*) was proved by P. Edelman and C. Greene [EG] in the context of the theory of balanced tableaux. Both the proof in [Sch1] and that in [EG] depend on the consideration of the "geometrical" properties of the involutions **S** and **U**. It is desirable to find the algebraic proofs of (2.14) and (2.15). Note that we have also another distinguished involution $\mathbf{W}_0 \in G_n$, where w_0 is the element of maximal length in the symmetric group S_n

$$\mathbf{W}_0 = s_1 \underbrace{s_2 s_1} \underbrace{s_3 s_2 s_1} \cdots \underbrace{s_{n-1} s_{n-2} \cdots s_2 s_1},$$

$$\mathbf{W}_0 : K^\lambda \to K^\lambda, \;\; \beta(W_0(x)) = w_0 \beta(x) = \overleftarrow{\beta}(x).$$

It seems plausible that the involutions **S** and $\mathbf{W}_0$ are commuted on the set $K^\lambda_{\mathbf{Z}}$ (but not on all X_n !). In any case it is interesting to understand the structure of the group generated by $\mathbf{S}, \mathbf{U}$ and $\mathbf{W}_0$.

E) The action of the symmetric group S_n on the set $STY(\lambda, \beta)$ (compare with [LS2],[LS3]).

Given a tableau $T \in STY(\lambda, \beta)$ and involution $s_i \in G_n$. Consider a word $w(T)$ which corresponds to the tableau T . Recall that by definition the word $w(T)$ may be obtained from the tableau T in the following way : let us read the elements of the tableau T consecutively from the right to the left, starting from the upper row. We obtain the word $w(T)$ under consideration (see e.g. [Ma]). For convenience we will denote i by a and $i+1$ by b . First, extract from we word $w(T)$ a subword w' which contains only letters a and b . Secondly, in the word w' we subtract consecutively all pairs ab . As a result we obtain a subword of type $b^m a^n$. Replace it by the word $b^n a^m$ and after this recover all subtracted pairs and all letters which differ from a and b . As a result we obtain a new word $\widetilde{w}$. To this word corresponds some Young tableau $\widetilde{T}$. Let us denote this tableau by $\sigma_i(T) := \widetilde{T}$.

Theorem 2.3. We have

$$s_i(T) = \sigma_i(T), \;\; 1 \leq i \leq n-1.$$

Remark 2.7. The fact that the involutions σ_i, $1 \leq i \leq n-1$, generate the symmetric group was proved by A. Lascoux and M.-P. Schützenberger [LS2] using a delicate analysis of the properties of the plactic monoid and the Robinson-Schensted correspondence. The same action was rediscovered by M. Kashiwara [Ka2] in the context of the theory of crystal base, as an action of the Weyl group W (in our case $S_n \simeq W$) on the crystal base, corresponding to an irreducible representation (irrep) V_λ with the highest weight λ of the Lie algebra $\mathfrak{gl}_n$. Our definition of the involution s_i also appears as an attempt to understand the action of the Weyl group W but on the Gelfand-Tsetlin basis (e.g. [GZ3]) of the irrep V_λ. In fact, it is possible to show that if $s_i \in W \simeq S_n,\ 1 \le i \le n-1$, then for $x \in K_{\mathbf{Z}}^{\lambda}$ we have

$$s_i \cdot |x> = |s_i(x)> + \sum_{y<x} \alpha_{xy} |y>,$$

with respect to some ordering on the set $K_{\mathbf{Z}}^{\lambda}$. Here for $x \in K_{\mathbf{Z}}^{\lambda}$ the symbol $|x>$ means a base vector in the space V_λ which corresponds to GT-pattern x.

Remark 2.8 Let us give another description for the action of the symmetric group on the set of standard Young tableaux, based on the concept of rigged configurations, [Ki1],[Ki2]. From the beginning, we remind some definitions from [Ki1]. Given a partition λ and composition β, $l(\beta) \le N$, a configuration $\{\nu\}$ of the type (λ, β) is, by definition, a collection of partitions $\nu^{(1)}, \nu^{(2)}, \cdots$ such that

$$(i)\quad |\nu^{(k)}| = \sum_{j \ge k+1} \lambda_j;$$

$$(ii)\quad P_n^{(k)}(\nu, \beta) := Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) \ge 0, \quad \text{for all} \quad k, n \ge 1,$$

where $\nu^{(0)} := \beta$, and for any composition γ we put $Q_n(\gamma) := \sum_k min(n,k)\gamma_k$.

According to the ultimate traditions of the Bethe ansatz [KR], the numbers $P_n^{(k)}(\nu, \beta)$ are called by the vacancy numbers. We define a cocharge $\overline{c}(\nu)$ of the configuration $\{\nu\}$ in the following manner:

$$\overline{c}(\nu) = \sum_{n \ge 1} \binom{(\nu^{(1)})'_n + 1}{2} + \sum_{k,n \ge 1} \binom{(\nu^{(k+1)})'_n - (\nu^{(k)})'_n}{2}. \tag{2.16}$$

Now we want to define a set of rigged configurations $QM(\lambda, \beta)$ of the type (λ, β). By definition, a rigged comfiguration of the type (λ, β) is a configuration of the type (λ, μ) together with a set of integers $J_{\alpha,n}^{(k)}, \quad k, n \ge 1, \quad 1 \le \alpha \le s, \quad s := m_n(\nu^{(k)})$, which satisfy the following inequalities

$$0 \le J_{1,n}^{(k)} \le J_{2,n}^{(k)} \le \cdots \le J_{s,n}^{(k)} \le P_n^{(k)}(\nu, \beta),$$

where $m_n(\nu^{(k)}) = (\nu^{(k)})'_n - (\nu^{(k)})'_{n+1}$.

The main property of rigged configurations is that they give another way to describe the set of Young tableaux, namely, there exist a bijection [KR],[Ki1]:

$$\Phi_\beta \ :\ STY(\lambda, \beta) \longrightarrow QM(\lambda, \beta).$$

This bijection has many interesting properties [KR],[Ki1],[Ki2], but just now let us remark that the set $QM(\lambda,\beta)$ depends only on the partition, corresponding to the composition β. Thus we may define an action of the symmetric group S_N on the set of standard Young tableaux $STY(\lambda,\leq N)$ using the commutative diagram ($\sigma\in S_N$):

$$\begin{array}{ccc} STY(\lambda,\beta) & \stackrel{\Phi_\beta}{\longrightarrow} & QM(\lambda,\beta) \\ \Phi_\sigma \Big\downarrow & & \Big\| \\ STY(\lambda,\sigma\beta) & \stackrel{\Phi_{\sigma\beta}^{-1}}{\longleftarrow} & QM(\lambda,\sigma\beta) \end{array} \tag{2.17}$$

Proposition 2.12. Let λ be a partition, $l(\lambda)\leq N$, and $T\in STY(\lambda,\leq N)$. Then

$$s_i(T)=\Phi_{(i,i+1)}(T),$$

where the map $\Phi_{(i,i+1)}$ corresponds to the vertical arrow in (2.17) when $\sigma=(i,i+1)$ is a simple transposition.

From Proposition 2.12 follows that all Young tableaux lying in the same orbit w.r.t. the action of the symmetric group generated by the involutions s_i (see (1.16)), $1\leq i\leq N-1$, have the same rigged configuration. In particular, the cocharge of Young tableau (see (2.16)) is an invariant of the action of the symmetric group.

F) Combinatorial interpretation of the transformations f_i and e_i and the crystal graph (e.g. [Ka1],[Ka2]).

Let us recall at first the construction of the action of the symmetric group S_n on the set $K_{\mathbf{Z}}^{\lambda}$ (see Section E or [LS2]). Given tableau $T\in STY(\lambda,\beta)$, consider the word $w(T)$ and the corresponding reduced subword $b^m\ a^n$. But now replace this subword by $b^{m+1}\ a^{n-1}$, if $n\geq 1$, or 0, if $n=0$ (respectively by $b^{m-1}\ a^{n+1}$ if $m\geq 1$, or 0, if $m=0$) and after this recover all subtracted pairs ab and all letters which differ from a and b. We obtain a new word $\widetilde{w}$ (corr. a word $\overline{w}$). To this word there corresponds some Young tableau $\widetilde{T}$ (resp. $\overline{T}$).

Theorem 2.4. We have for $T\in STY(\lambda,\leq n)$:

$$f_iT=\widetilde{T},\ e_iT=\overline{T},\ 1\leq i\leq n-1.$$

Now let us consider a colored graph $\mathbf{\Gamma_Z}(\lambda)$. The vertices of this graph correspond to all standard Young tableaux T of the shape λ with all entries $\leq n$. Two vertices T_1 and T_2 are connected by an edge of color i, $T_1\stackrel{i}{\longrightarrow}T_2$ iff $T_2=f_iT_1$ (or equivalent, $T_1=e_iT_2$).

Theorem 2.5. The graph $\mathbf{\Gamma_Z}(\lambda)$ coincides with the crystal graph corresponding to the irreducible representation V_λ of the Lie algebra $\mathfrak{gl}_n$ with the highest weight λ.

Remark 2.9. The description of the edges of the crystal graph $\mathbf{B}(V_\lambda)$ was obtained at first by M. Kashiwara and T. Nakashima [KN],[N] in pure combinatorial terms. We gave in fact the same description. The difference is that M. Kashiwara and T. Nakashima in [KN] give a description of the action of the generators $\widetilde{f}_i$ and $\widetilde{e}_i$ on the set $STY(\lambda, \leq n)$. Our result gives an opportunity to extend this actiono to the Gelfand-Tsetlin polytope K^λ and obtain some subdivision of the convex polytope K^λ on coloring parts.

Remark 2.10. Let $U = U_q(sl(n))$ is the q-analog of the universal enveloping algebra of the Lie algebra $sl(n)$. The algebra U is the algebra over $\mathbf{Q}(q)$ generated by E_i, F_i and $K_i^{\pm 1}$, $1 \leq i \leq n-1$ (see e.g. [J],[Ka1]). We denote by U_-^* a multiplicative monoid in U generated by $F_1, \cdots, F_{n-1}$. Let G^* be a multiplicative monoid in $Aut(X_n))$ generated by all transformations $s_1^{(1)}, \cdots, s_{n-1}^{(1)}$ (see (1.16)).

Conjecture 2.2. The correspondence $F_i \longrightarrow s_i^{(1)}$, $1 \leq i \leq n-1$ defines an isomorphism $U_-^* \simeq G^*$ of the multiplicative monoids.

Let us remark, that for any $k, l \in \mathbf{Z}_+$ we have the following relation (e.g. [Lu2]):

$$F_i^k\ F_{i+1}^{k+l}\ F_i^l = F_{i+1}^l\ F_i^{l+k}\ F_{i+1}^k, \quad 1 \leq i \leq n-2.$$

§3 Cocharge and Gelfand-Tsetlin patterns.

First we remind the definition of the charge and cocharge of a tableau, according to A. Lascoux and M.-P. Schützenberger [LS1],[Ma]. Let λ and μ be partitions, $T \in STY(\lambda, \mu)$. Consider the word $w(T)$, which corresponds to tableau T (e.g. [Ma]). We define the charge $c(T)$ of tableau T as the charge of corresponding word $w(T)$. Now let us define the charge of a word w. Remind that the weight μ of a word w is a sequence $\mu = (\mu_1, \mu_2, \cdots)$, where μ_i is the number of i appearing in the word w. We assume that the weight μ of the word w is dominant, i.e. $\mu_1 \geq \mu_2 \geq \cdots$.

(i) We first assume that w is a standard word, i.e. its weight is $\mu = (1^N)$. Let us index all elements of w as follows: the index of 1 is equal to 0, and if the index of k is i, the index $k+1$ is either i or $i+1$ according to the location of $k+1$ either to the right or to the left of k. The charge $c(w)$ of w is then the sum of all its indices.

(ii) Assume now that w is a word of weight μ and μ is a partition. We extract a standard subword out of w in the following way. Reading w from left to right we choose the first entry of 1, then the first entry of 2 to the right of the 1 chosen and so on. If at some step there is no $s+1$ to the right of the s chosen before, we come back to the beginning of the word. This operation extracts from w a standard subword w_1 out of w. Let us delete the word w_1 from w and repeat the operation, thus obtaining w_2, etc..

The charge of w is defined to be the sum of the charges of the standard subwords obtained in this way : $c(w) = \sum c(w_i)$. We note that the charge of w is zero iff w is a lattice word.

Now let λ and μ be partitions and $T \in STY(\lambda, \mu)$. We define the cocharge of the tableau T as

$$\overline{c}(T) = n(\mu) - n(\lambda) - c(T). \tag{3.1}$$

We give another method for computing of the cocharge of a tableau $T \in STY(\lambda, \mu)$. First assume that $T \in STY(\lambda)$. For any $x \in T$ let us denote by $n(x)$ an index of the row which contains x. For any $x \in T$ we define $\overline{c}(x)$ by induction : $\overline{c}(1) = 0$,

$$\overline{c}(x+1) = \overline{c}(x) + \begin{cases} n(x) - n(x+1), & \text{if } n(x) \geq n(x+1) \\ n(x) - n(x+1) + 1, & \text{if } n(x) < n(x+1) \end{cases} \tag{3.2}$$

and put $\overline{c}(T) = \sum_{x \in T} \overline{c}(x)$.

Proposition 3.1. Assume that $T \in STY(\lambda)$. Then
(*i*) $\overline{c}(x) \geq 0$ for all $x \in T$.
(*ii*) $\overline{c}(T)$ is equal to the cocharge of the tableau T, as it was defined by (3.1).
(*iii*) Let us define the descent set $D(T)$ of a tableau T as

$$D(T) = \{x \in T \mid n(x) < n(x+1)\}, \tag{3.3}$$

and put

$$des(T) = \sum_{x \in D(T)} x, \quad p = \sum_{x \in D(T)} 1.$$

Then

$$\overline{c}(T) = pN - des(T) - n(\lambda),$$
$$c(T) = \binom{N}{2} - pN + des(T), \text{ where } N = |\lambda|.$$

It is convenient to consider a tableau $\overline{C}(T)$, which obtained from the tableau T by replacement each entries $x \in T$ by $\overline{c}(x)$. It is possible to show that $\overline{C}(T)$ is, in fact a reverse plane partition.

Let us consider a clarifying example.

$$T := \begin{matrix} 1 & 2 & 4 & 10 \\ 3 & 5 & 8 & 12 \\ 6 & 7 & 14 & \\ 9 & 11 & & \\ 13 & & & \end{matrix} \qquad \overline{C}(T) := \begin{matrix} 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 4 & \\ 1 & 2 & & \\ 2 & & & \end{matrix}$$

In our case we have $D(T) = \{2, 4, 5, 8, 10, 12\}$, $p(T) = 6$, $des(T) = 41$, $n(\lambda) = 20$ and

$$\overline{c}(T) := \sum_{c \in \overline{C}(T)} c = 23 = 6 \cdot 14 - 41 - 20.$$

Secondly, if $T \in STY(\lambda, \mu)$, μ be a partition, we act as in step (*ii*) of Lascoux-Schützenberger's algorithm : consider the most left and upper 1 in the tableau T and compute $\overline{c}(1) = n(1) - 1$, after this consider the most left 2 such that

$n(2) > n(1)$ and compute $\overline{c}(2)$ according to (3.2) and so on. If at some step there is no $x+1$ with condition $n(x+1) > n(x)$, then consider the most left and upper $x+1 \in T$, and compute $\overline{c}(x+1)$(using (3.2)). This operation extracts from the tableau T a subset w_1. Let us delete this subset w_1 from the tableau T and repeat the previous operation, thus obtaining w_2 etc.. The cocharge of the tableau T is define to be the sum

$$\overline{c}(T) = \sum_i \sum_{x \in w_i} \overline{c}(x).$$

We note that some of the numbers $\overline{c}(x)$, $x \in T$, in general, may be negative.

Thus we define the charge and cocharge of a tableau $T \in STY(\lambda, \mu)$, when μ is a partition. We know (see Section 1, or [GZ1],[BZ1]) that $STY(\lambda, \mu)$ is a set of integral points in the convex polytope $K^\lambda(\mu) \subset K^\lambda$, so it is natural to ask is it possible to continue the cocharge (or charge) from the set $STY(\lambda, \mu)$, μ is a partition, on all convex polytope K^λ in a natural manner? It is easy to see that the charge does't possess a good properties with respect to the action of the symmetric group S_n on the Gelfand-Tseitlin polytope K^λ.

The main result of this Section, Theorem 3.1, asserts that such continuation really exists and has the nice properties (at least for $n \leq 5$). Thus we want to construct a function $\overline{c}_n : X_n \to \mathbf{R}$, which coincides with cocharge on the integral points set of the cone K_n (at least for $n \leq 5$ and hypothetically for all n). To start, let us define an imbedding of the symmetric group S_n into the set $(K_n)_{\mathbf{Z}}$:

$$p_n : S_n \hookrightarrow (K_n)_{\mathbf{Z}}.$$

Definition 3.1. Let us denote by $\mathbf{P}_n$ a set of all Gelfand-Tsetlin patterns $p := (p_{ij}) \in (K_n)_{\mathbf{Z}}$, which satisfy the following conditions

$$\begin{aligned} &i)\ p_{n-1,n-1} = p_{n,n} = 0; \\ &ii)\ p_{i,j} - p_{i+1,j+1} \leq 1,\ \ 1 \leq i \leq j \leq n-1; \\ &iii)\ p_{i,n} - 1 \leq p_{i,i} \leq p_{i+1,n},\ \ 1 \leq i \leq n-1. \end{aligned} \tag{3.4}$$

Proposition 3.2. There exists a bijection

$$\pi : \mathbf{P}_n \simeq [0, n-1] \times [0, n-2] \times \cdots \times [0,1] \times [0,0].$$

In particular, $|\mathbf{P}_n| = n!$.

Proof. Let $p = (p^{(n)}, \cdots, p^{(1)}) \in \mathbf{P}_n$. We will construct a vector $k = (k_1, \cdots, k_n)$ by the following rule. For the beginning, let us observe that for given i, $1 \leq i \leq n$ from inequality $0 \leq p_{i,n} - p_{i,i} \leq 1$ follows an existence of integer j, $0 \leq j \leq n-i$, such that

$$p_{i,i} = p_{i,i+1} = \cdots = p_{i,i+j} < p_{i,i+j+1} - \cdots = p_{i,n}.$$

In this case we put $k_i = j$. The correspondence $p \to k$ is desired bijection.

∎

As an illustrating example, let us take $n = 4$ and $k = (3,1,0,0)$. At first we observe that $p_{3,4} = 1$. Secondly, from inequalities (3.4) we find that $p_{3,4} \le p_{2,3} = p_{3,3} \le p_{3,4}$. Consequently, $p_{2,3} = p_{3,3} = p_{3,4} = 1$ and $p_{2,4} = 2$. Further, using the same trick, we obtain that $p_{2,4} \le p_{1,4} = p_{1,3} = p_{1,2} = p_{1,1} \le p_{2,4}$. Thus

$$p = \begin{matrix} 2 & & 2 & & 1 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{matrix}$$

Let us continue and consider the following linear functionals on the space of triangles X_{n+1}:

i) $\varphi_{0,n} : X_{n+1} \to \mathbf{R}$, given by a formula

$$\varphi_{0,n}(x) = \sum_{j=1}^{n-1} (n-j)(x_{j,n+1} - x_{j,n}). \tag{3.5}$$

ii) For every $p \in \mathbf{P}_n$, $\varphi_p : X_{n+1} \to \mathbf{R}$, given by a formula

$$\varphi_p(x) = \sum_{1 \le i \le j \le n} p^T_{i,j} \cdot \varphi_{n-j+1,n-i+1}(x), \tag{3.6}$$

where if $p = (p_{ij}) \in \mathbf{P}_n$, then $p^T := (p^T_{i,j})$, $p^T_{i,j} = p_{n-j+1,n-i+1}$, and the linear maps $\varphi_{ij} : X_{n+1} \to \mathbf{R}$ are defined by (1.32).

At last, we define a function $\overline{c}_{n+1} : X_{n+1} \to \mathbf{R}$ by the following inductive formula

$$\begin{aligned} &i) \ \overline{c}_1(x) = 0, \ \text{if } x \in X_1, \\ &ii) \ \overline{c}_{n+1}(x) = \overline{c}_n(x) + \varphi_{0,n}(x) + min\{\varphi_p(x) | p \in \mathbf{P}_n\}, \end{aligned} \tag{3.7}$$

where for any triangle $x \in X_{n+1}$, $x = (x^{(n+1)}, \widetilde{x})$, $\widetilde{x} \in X_n$, we set by definition, that $\overline{c}_n(x) = \overline{c}_n(\widetilde{x})$.

Now we are ready to formulate our main result of this Section.

Theorem 3.1. The function $\overline{c}_n : X_n \to \mathbf{R}$ is a continuous, piecewise linear and satisfies the following properties

1) invariance:

$$\begin{aligned} &(i) \ \overline{c}_n(\sigma x) = \overline{c}_n(x) \ \text{ for any } \sigma \in S_n; \\ &(ii) \ \overline{c}_n(Ix) = \overline{c}_n(x), \ \text{ where the map } I \text{ is defined in (1.38)}; \\ &(iii) \ \overline{c}_n(T_{\pm 1}x) = \overline{c}_n(x), \ \text{ if } T_{+1}x \ne 0 \text{ (corr. } T_{-1}x \ne 0). \end{aligned}$$

2) stability: if $\varphi_\alpha : X_{n-1} \to X_n$, $\alpha = 1,2$, are the embeddings given by (1.40), then

$$\overline{c}_n(x) = \overline{c}_{n-1}(\widetilde{x})$$

3) positiveness:

$$\overline{c}_n(x) \geq 0 \text{ for all } x \in K_n.$$

We assume that the function $\overline{c}_n(x)$ given by (3.7) possesses an additional property:

4) **Main Conjecture:** if λ and μ be the partitions $l(\lambda) \leq n$, $l(\mu) \leq n$, $T \in STY(\lambda, \mu)$ and $x(T)$ be its image (see Section 2) in the set $K_{\mathbf{Z}}^{\lambda}$, then

$$\overline{c}_n(x(T)) = \overline{c}_n(T).$$

Corollary 3.1. (of the Main Conjecture) Let λ and μ be the dominant weights of the Lie algebra sl_n, and $\overline{K}_{\lambda,\mu}(q)$ be the Kazhdan-Lusztig polynomial for the affine Hecke algebra (e.g. [Lu1],[Ma]) corresponding to λ and μ. Then

$$\overline{K}_{\lambda,\mu}(q) = \sum_{x \in K^{\lambda}(\mu) \cap (X_n)_{\mathbf{Z}}} q^{\overline{c}_n(x)}. \tag{3.8}$$

Corollary 3.2. (of Theorem 3.1) Let λ be a dominant weight and β be any (integral) weight of the Lie algebra sl_n. Let us define a polynomial

$$\overline{K}_{\lambda,\beta}(q) := \sum_{x \in K^{\lambda}(\beta) \cap (X_n)_{\mathbf{Z}}} q^{\overline{c}_n(x)}. \tag{3.9}$$

Then, $\overline{K}_{\lambda,\beta}(q) = \overline{K}_{\lambda,w(\beta)}(q)$ any $w \in S_n$.

Problem 3.1. Assume $\lambda,\ \beta \in \mathbf{R}^n$. To compute the following integral

$$K_{\lambda,\beta}^{cont}(q) := \int_{K^{\lambda}(\beta)} exp(h \cdot \overline{c}_n(x)) d\mu(x), \ \ q = exp(h), \tag{3.10}$$

where $\mu(x)$ is the Lebesque measure on $K^{\lambda}(\beta)$ induced from $\mathbf{R}^{\frac{n(n+1)}{2}}$.

It is clear that $K_{\lambda,\beta}^{cont}(1) = Vol(K^{\lambda}(\beta))$. We consider the functions $K_{\lambda,\beta}^{cont}(q)$ as a continuons analog of the Kostka-Foulkes polynomials.

We postpone the proof of Theorem 3.1 to the end of Section 3 (see Remark 3.1), while right now let us consider in more details the particular cases $n = 3$ and $n = 4$. Our goal is to prove for these cases the Main Conjecture.

Let us start with the case $n = 3$. Given a triangle $x \in X_3$, we recall that

$$\varphi_{22}(x) := x_{12} + x_{22} - x_{11} - x_{23}$$
$$\varphi_{12}(x) := x_{12} + x_{22} - x_{13} - x_{33}.$$

It is easy to see that

$$x_{12}(s_2(x)) = x_{12} - (\varphi_{12})_- - (\varphi_{22})_+,$$
$$x_{22}(s_2(x)) = x_{22} - (\varphi_{12})_+ - (\varphi_{22})_-,$$
$$\varphi_{12}(s_2(x)) = -\varphi_{22}(x), \ \ \varphi_{12}(s_2(x)) = -\varphi_{12}(x),$$
$$\varphi_{22}(s_1(x)) = x_{11} - x_{23}, \ \ \varphi_{22}(s_1(x)) = \varphi_{22}(x).$$

Lemma 3.1. In the case $n = 3$ we have

$$\overline{c}_3(x) = min(x_{13} - x_{12}, x_{22} - x_{33}), \tag{3.11}$$
$$\overline{c}_3(q_2(x)) = min(x_{23} - x_{22}, x_{12} - x_{11}) + (x_{11} - x_{23})_-, \tag{3.12}$$

where $q_2 = t_1 t_2 t_1 \in G_3$, $x \in X_3$ and $(a)_+ := max(x, 0)$, $(a)_- := min(x, 0)$.

Proof. According to Definition 3.1, we have

$$\mathbf{P}_2 = \left\{ \begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}, \quad \begin{smallmatrix} 0 & & 0 \\ & 0 & \end{smallmatrix} \right\}, \quad \pi(\mathbf{P}_2) = \{ (0,0),\ (1,0) \},$$
$$\varphi_{\begin{smallmatrix} 1 & 0 \\ 0 & \end{smallmatrix}}(x) = p_{12}^T \cdot \varphi_{12}(x) = \varphi_{12}(x),$$
$$\varphi_{\begin{smallmatrix} 0 & 0 \\ 0 & \end{smallmatrix}}(x) = 0, \quad \varphi_{02}(x) = x_{13} - x_{12}.$$

Consequently,

$$\overline{c}_3(x) = x_{13} - x_{12} + (\varphi_{12})_- = min(x_{13} - x_{12},\ x_{22} - x_{33}).$$

The formula (3.12) follows from the description of Schützenberger involution q_2:

$$x_{12}(q_2(x)) = x_{13} - min(x_{11} - x_{22}, x_{12} - x_{23}),$$
$$x_{22}(q_2(x)) = x_{33} + min(x_{12} - x_{11}, x_{23} - x_{22}),$$
$$x_{11}(q_2(x)) = \beta_3(x);$$

■

The function (3.11) satisfies all properties 1) - 4) of Theorem 3.1 (see Proposition 3.3). The function (3.12) satisfies the properties 1) - 3), except $1(iii)$ and, in particular, is invariant with respect to the action of the symmetric group S_3. Further, for the dual Schützenberger's involution u_2 we have

$$x_{12}(u_2(x)) = x_{12} - (\varphi_{12})_- - (x_{11} - x_{23})_+,$$
$$x_{22}(u_2(x)) = x_{22} - (\varphi_{12})_+ - (x_{11} - x_{23})_-,$$
$$x_{11}(u_2(x)) = \beta_3(x).$$

It is easy to see that $\overline{c}_3(u_2(x)) = \overline{c}_3(x)$, and

$$q_2 \cdot u_2 = u_2 \cdot q_2. \tag{3.13}$$

The equality (3.13) is a corollary of the identity $(t_1 t_2)^6 = 1$, since $q_2 = t_1 t_2 t_1$ and $u_2 = t_2 t_1 t_2$. In general $\overline{c}_n(u_{n-1}(x)) \neq \overline{c}_n(x)$, and $\mathbf{SU} \neq \mathbf{US}$.

Now consider the case $n = 4$. Given a triangle $x \in X_4$, we recall that

$$\varphi_{13}(x) := x_{13} + x_{33} - x_{14} - x_{44},$$
$$\varphi_{23}(x) := x_{13} + x_{23} - x_{12} - x_{24},$$
$$\varphi_{33}(x) := x_{23} + x_{33} - x_{22} - x_{24}.$$

It is possible to check (see Theorem 1.3), that

$$
\begin{aligned}
x_{13}(s_3(x)) &= x_{13} - (\varphi_{23} + (\varphi_{33})_-)_+ - (\varphi_{13} + (\varphi_{33})_+)_-, \\
x_{23}(s_3(x)) &= x_{23} - (\varphi_{33} + (\varphi_{13})_-)_+ - (\varphi_{23} + (\varphi_{13})_+)_-, \\
x_{33}(s_3(x)) &= x_{33} - (\varphi_{13} + (\varphi_{23})_-)_+ - (\varphi_{33} + (\varphi_{23})_+)_-.
\end{aligned}
\tag{3.14}
$$

Proposition 3.3. Assume $x \in X_4$, then the function

$$
\begin{aligned}
\overline{c}_4(x) &= min(x_{13} - x_{12},\ x_{22} - x_{33}) + x_{14} - x_{13} + x_{33} - x_{44} + \\
&+ min(x_{23} - x_{34},\ x_{24} - x_{23},\ x_{13} - x_{12},\ x_{22} - x_{33},\ \beta_4(x) - x_{34},\ x_{24} - \beta_4(x))
\end{aligned}
\tag{3.15}
$$

satisfies the all properties 1) - 4) of Theorem 3.1.

Proof. First of all, let us check that our previous definition (3.7) of the cocharge $\overline{c}_n$ coincides with (3.15). According to Definition 3.1, we have

$$
\mathbf{P}_3 = \left\{ \begin{matrix} 1\ 1\ 0 \\ 1\ 0 \\ 0 \end{matrix},\ \begin{matrix} 1\ 0\ 0 \\ 1\ 0 \\ 0 \end{matrix},\ \begin{matrix} 1\ 0\ 0 \\ 0\ 0 \\ 0 \end{matrix},\ \begin{matrix} 2\ 1\ 0 \\ 1\ 0 \\ 1 \end{matrix},\ \begin{matrix} 1\ 1\ 0 \\ 1\ 0 \\ 1 \end{matrix},\ \begin{matrix} 0\ 0\ 0 \\ 0\ 0 \\ 0 \end{matrix} \right\},
$$

$$
\pi(\mathbf{P}_3) = \{\ (000),\ (010),\ (110),\ (100),\ (200),\ (210)\ \}.
$$

Using the definition (3.6), we find

$$
\begin{aligned}
&\varphi_{\substack{1\,1\,0\\1\,0\\0}}(x) = \varphi_{12} + \varphi_{13} + \varphi_{23}, && \varphi_{\substack{1\,0\,0\\1\,0\\0}}(x) = \varphi_{13} + \varphi_{23}, \\
&\varphi_{\substack{2\,1\,0\\1\,0\\1}}(x) = \varphi_{12} + 2\varphi_{13} + \varphi_{23} + \varphi_{33}, && \varphi_{\substack{1\,0\,0\\0\,0\\0}}(x) = \varphi_{13}, \\
&\varphi_{\substack{1\,1\,0\\1\,0\\1}}(x) = \varphi_{12} + \varphi_{13} + \varphi_{23} + \varphi_{33}, && \varphi_{\substack{0\,0\,0\\0\,0\\0}}(x) = 0.
\end{aligned}
$$

Furthermore, it follows from (3.5) that

$$
\begin{aligned}
\varphi_{03}(x) &= 2(x_{14} - x_{13}) + (x_{24} - x_{23}) = \\
&= x_{14} - x_{13} + x_{33} - x_{44} + x_{24} - x_{23} - \varphi_{13}.
\end{aligned}
$$

After some not so complicated calculations we can obtain from (3.7) the formula (3.15).

Secondly, let us check that the function $\overline{c}_4(x)$ is an invariant with respect to the action of the symmetric group S_4. Note that the invariance under the action of generators s_1 and s_2 is clear from (3.5). In order to prove the s_3-invariance of the $\overline{c}_4(x)$ we make use of a few lemmas.

Lemma 3.2. Assume $x \in X_4$, then

$$(\varphi_{\alpha 3}(s_3(x)))_- = x_{\alpha 3}(s_3(x)) - x_{\alpha 3} + (\varphi_{\alpha 3}(x))_-, \ \ \alpha = 1,2,3, \tag{3.16}$$

where for any $a \in \mathbf{R}$, $(a)_- := min(a,0)$, $(a)_+ := max(a,0)$.

Proof. Using the formulas (3.14) for the action of s_3 on the space of triangles X_4, we find

$$\widetilde{\varphi}_{13} = -\varphi_{23} - \varphi_{33} + (\varphi_{23} + (\varphi_{13})_+)_- + (\varphi_{33} + (\varphi_{13})_-)_+,$$

where for any function φ on the space of triangles we exploit a notation $\widetilde{\varphi} := \varphi(s_3(x))$. In order to calculate $(\widetilde{\varphi}_{13})_-$, we make use of the following identity, which may be varified by direct computation

$$[-a - b + (a + (c)_-)_+ + (b + (c)_+)_-]_- = (c)_- - (b + (a)_-)_+ - (c + (a)_+)_-.$$

Consequantly,

$$(\widetilde{\varphi}_{13})_- = (\varphi_{13})_- - (\varphi_{23} + (\varphi_{33})_-)_+ - (\varphi_{13} + (\varphi_{33})_+)_- = \widetilde{x}_{13} - x_{13} + (\varphi_{13})_-.$$

Similarly we may prove the corresponding formulas for $\alpha = 2$ or 3.

■

Corollary 3.3 (of Lemma 3.2). The following functions are s_3-invariance

$$\begin{aligned} &min(x_{14} - x_{13},\ x_{33} - x_{44}),\\ &min(x_{13} - x_{12},\ x_{24} - x_{23}),\\ &min(x_{23} - x_{22},\ x_{34} - x_{33}). \end{aligned} \tag{3.17}$$

Lemma 3.3. The expression for $\overline{c}_4(x)$, given below, is equivalent to (3.15):

$$\overline{c}_4(x) = x_{14} - x_{13} - x_{44} + \overline{c}_3(x) + \overline{c}_3(s_3x) + \widetilde{x}_{33} + (\varphi_{33} + (\varphi_{23})_-)_-.$$

The proof is based on a direct computation. ■

Thus, we reduce the proof of s_3-invariance for the cocharge $\overline{c}_4(x)$ to that for the function

$$(\varphi_{33} + (\varphi_{13})_-)_- - x_{13} - x_{33}.$$

The s_3-invariance of the last function may be easily deduced from Lemma 3.2.

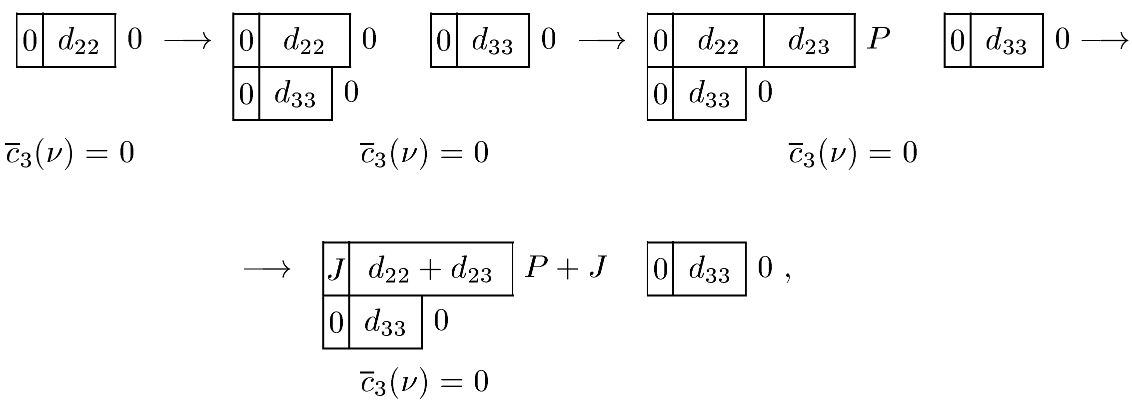
Finally, we must prove that if $T \in STY(\lambda, \mu)$ then $\overline{c}_4(x(T))$ coinsides with the cocharge of tableau T.

For this purpose we use a description of Young tableaux by means of rigged configurations [Ki1],[Ki2]. More precisely, let T be a standard Young tableau

$T :=$

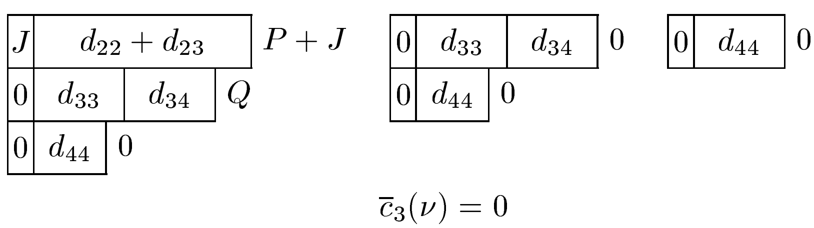
d_{11}	d_{12}	d_{13}	d_{14}
d_{22}	d_{23}	d_{24}	
d_{33}	d_{34}		
d_{44}			

where d_{ij} is the number of j's lying in the i-th row of T. We shall construct the corresponding rigged configuration step by step:



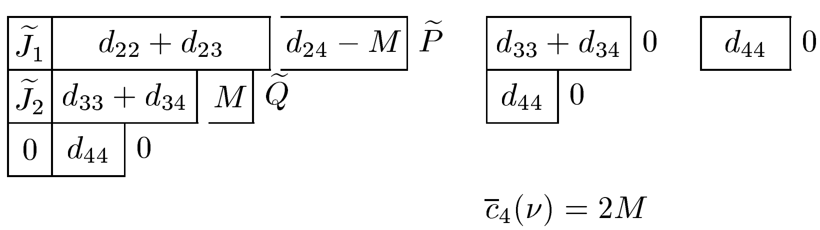
where $P = min(d_{11} - d_{22},\ d_{23}) + (d_{12} - d_{23})_-,\ \ J = min(d_{13},\ d_{22} - d_{33})$.

Consequently, $\overline{c}_3(T) = \overline{c}(\nu) + J = min(x_{13} - x_{12},\ x_{22} - x_{33}) = \overline{c}_3(x(T))$. Let's go further and start to join successively the parts $\boxed{d_{44}}$ and $\boxed{d_{34}}$.



where $Q = min(d_{11} - d_{33},\ d_{34}) + min(d_{12} + d_{22},\ d_{33} + d_{34}) - 3d_{34}$.

Finally, let us add the parts $\boxed{d_{24}}$ and $\boxed{d_{14}}$. As a result we obtain the following rigged configuration:



where

$$
\begin{aligned}
M &= min(d_{22} + d_{23} - d_{33} - d_{34},\ min(d_{13},\ d_{22} - d_{33})),\\
\widetilde{J}_1 &= min(d_{14},\ d_{22} + d_{23} - d_{34} - d_{44} - M) + [min(d_{13},\ d_{22} - d_{33}) - d_{24}] \cdot \epsilon,\\
\widetilde{J}_2 &= min(d_{33} - d_{44},\ d_{14} + d_{24} - M) + min(d_{13},\ d_{22} - d_{33}) - M - \qquad (3.18)\\
&\qquad - [min(d_{13},\ d_{22} - d_{33}) - d_{24}] \cdot \epsilon,
\end{aligned}
$$

and $\epsilon := \epsilon(T)$ is equal to 1, if $d_{24} \leq min(d_{22}+d_{23}-d_{33}-d_{34},\ d_{22}-d_{33},\ d_{13})$, and equals to 0 otherwise.

The exact values of the vacancies numbers $\widetilde{P}$ and $\widetilde{Q}$ are not essential for a computation of the cocharge. Consequently, $\overline{c}_4(T) =$

$$= \overline{c}(\nu) + \widetilde{J}_1 + \widetilde{J}_2 = \overline{c}_3(x) + d_{14} + d_{33} - d_{34} + min(M,\ d_{14}+d_{24}-d_{33}+d_{44}) + \\ + min(M,\ d_{22}+d_{23}-d_{14}-d_{34}-d_{44}) - M.$$

It is easy to check that this expression for $\overline{c}_4(T)$ coincides with $\overline{c}_4(x(T))$. The proof of Proposition 3.3 is finished. ■

Let us write also a formula for the action of Schützenberger involution $q_3 = t_1t_2t_1t_3t_2t_1$ on the space of triangles X_4: if $x \in X_4$ and $\widetilde{x} = q_3(x)$, then

$$\begin{aligned}
\widetilde{x}_{13} &= x_{14} + x_{24} - x_{13} + (\varphi_{23} + (\varphi_{22})_+)_+, \\
\widetilde{x}_{23} &= x_{24} + x_{34} - x_{23} + (\varphi_{23} + (\varphi_{22})_+)_- + (\varphi_{33} + (\varphi_{22})_-)_+, \\
\widetilde{x}_{33} &= x_{34} + x_{44} - x_{33} + (\varphi_{33} + (\varphi_{22})_-)_-, \\
\widetilde{x}_{12} &= x_{14} + x_{24} + x_{34} - x_{13} - x_{23} + (\varphi_{33} + (\varphi_{23})_+)_+, \\
\widetilde{x}_{22} &= x_{24} + x_{34} - +x_{44} - x_{23} - x_{33} + (\varphi_{23} + (\varphi_{33})_-)_-, \\
\widetilde{x}_{11} &= \beta_4(x),\ \widetilde{x}_{\alpha 4} = x_{\alpha 4}, \quad \text{if} \quad \alpha = 1,2,3,4.
\end{aligned}$$

From the description of the action of the symmetric group S_n on the set $STY(\lambda, \leq n)$ one can deduce that all Young tableaux belonging to the same S_n-orbit have the same rigged configuration. Thus, we see that all parameters of a rigged configuration (for example, its quantum numbers or vacancies numbers) are S_n-invariant. The computations contained in Proposition 3.3 seems to be interesting, because they shed some light on the deep invariants of Young tableau, namely, its quantum numbers. At this moment we don't know whether or not the quantum numbers (3.18) considered as the functions on the space X_4 are S_4-invariants. By any way, it seems a very interesting task to find a "complete list" of the continuous, piecewise linear invariants (*cpl*-invariants) for the fixed involution s_j. In this direction we are going to prove the following result.

Theorem 3.2. Let j be an integer, $2 \leq j \leq n-1$. Then the following functions on the space X_n are s_j-invariants:

$$\psi_{1j}(x) = min(x_{1,j+1} - x_{1,j},\ x_{j,j} - x_{j+1,j+1}) \tag{3.19}$$
$$\psi_{ij}(x) = min(x_{i-1,j} - x_{i-1,j-1},\ x_{i,j+1} - x_{ij}), \quad 2 \leq i \leq j \leq n-1,$$
$$\overline{\psi}_{1j}(x) = (min(\varphi_{jj}(x), -\varphi_{1j}(x)))_+, \tag{3.20}$$
$$\overline{\psi}_{ij}(x) = (min(\varphi_{i-1j}(x), -\varphi_{ij}(x)))_+, \quad 2 \leq i \leq j \leq n-1.$$

Conjecture 3.2. The group of all continuous, piecewise linear automorphisms of the space X_n, having the same set of invariants (3.19) and (3.20), coincides with the cyclic group of the second order $< 1,\ s_j >$.

It seems plausible that the set of s_j-invariants (3.19) and (3.20) is a fundamental system of continuous, piecewise linear s_j-invariants, i.e. any *cpl*-invariant with respect to the action of s_j on the space X_n is a $min - max$ linear combination of that (3.19) and (3.20).

Proof (of the Theorem 3.2) We have to prove that for $2 \leq j \leq n-1$ all functions ψ_{ij} and $\overline{\psi}_{ij}$, $1 \leq i \leq j$, are the s_j-invariants. We shall prove it using the inductive formula (1.29) for s_j:

$$s_j = R_{j-1,j}[j-1,j][j,j+1]s_{j-1}[j,j+1][j-1,j]R_{j-1,j}. \tag{3.21}$$

The case $j = 1$ is almost evident, namely, it is necessary to check the s_1-invariance only one function $\psi_{11}(x) = min(x_{12} - x_{11},\ x_{11} - x_{22})$, but this is clear from the definition of s_1. So, we may assume that $\psi_{i,j-1}$ and $\overline{\psi}_{i,j-1}$ $(i = 1, \cdots, j-1)$ are the invariants with respect to the action of s_{j-1}. Now, using the expression (3.21), we are going to reduce proof of the s_j-invariance of the functions (3.19) and (3.20) to that for the more simple ones. At first, let us define a map $J : X_n \to X_n$ by the following manner (compare with (1.38)):

$$(J(x))_{ij} := -x_{j-i+1,j}, \quad 1 \leq i \leq j \leq n.$$

Lemma 3.4. We have

$$\begin{aligned} &(i) \quad J^2 = 1,\ t_j J = J t_j,\ s_j J = J s_j,\ 1 \leq j \leq n-1, \\ &(ii) \quad \psi_{1,j}(J(x)) = \psi_{1,j}(x), \quad \overline{\psi}_{1,j}(J(x)) = \overline{\psi}_{2,j}(x), \\ &\qquad \psi_{i,j}(J(x)) = \psi_{j-i+2,j}(x) \quad 2 \leq i \leq j, \\ &\qquad \overline{\psi}_{i,j}(J(x)) = \overline{\psi}_{j-i+3,j}(x), \quad 3 \leq i \leq j. \end{aligned}$$

The proof of this Lemma based on a direct computation. ■

Lemma 3.5. If $\psi : X_n \to \mathbf{R}$ is an s_j-invariant function (i.e. $\psi(s_j(x)) = \psi(x)$ for all $x \in X_n$), then $\widetilde{\psi}(x) = \psi(J(x))$ is also the s_j-invariant one.

■

Secondly, let us introduce the notations

$$\begin{aligned} \psi'_{ij}(x) &:= \psi_{ij}(R_{j-1,j}(x)), \\ \overline{\psi}'_{ij}(x) &:= \overline{\psi}_{ij}(R_{j-1,j}(x)), \\ \psi''_{ij}(x) &:= \psi'{}_{ij}([j,j+1][j-1,j](x)), \\ \overline{\psi}''_{ij}(x) &:= \overline{\psi}'{}_{ij}([j-1,j][j,j+1](x)), \end{aligned}$$

Accoding to the formulas (3.21) and (3.22) below, we have only to prove that the functions ψ''_{ij} and $\overline{\psi}''_{ij}$ are the s_{j-1}-invariants.

Lemma 3.6 (Reduction formulas). We have

$$
\begin{aligned}
a)\quad & \psi'_{ij} = \psi_{ij}(x) \ \text{ and} \\
& \psi''_{ij} = \psi_{ij-1}(x), \ \text{ if } \ 2 \le i \le j-2. \\
b)\quad & \overline{\psi}'_{ij}(x) = \overline{\psi}_{ij}(x) \ \text{ and} \\
& \overline{\psi}''_{ij}(x) = \overline{\psi}_{ij-1}(x), \ \text{ if } \ 3 \le i \le j-2.
\end{aligned}
$$

The proof based on a direct calculation, using the following formulas for the action of the involution $R_{j-1,j}$ (see Definition 1.2): if $\widetilde{x} := R_{j-1,j}(x)$, then

$$
\begin{aligned}
&\widetilde{x}_{j-1,j} = x_{j-1,j+1} - min(x_{j,j+1} - x_{j,j},\ x_{j-1,j} - x_{j-1,j-1}) = x_{j-1,j+1} - \psi_{jj}(x), \\
&\widetilde{x}_{j,j} = x_{j,j} - x_{j-1,j+1} + x_{j-1,j} + min(x_{j,j+1} - x_{j,j},\ x_{j-1,j} - x_{j-1,j-1}), \\
&\widetilde{x}_{\alpha\beta} = x_{\alpha\beta}, \ \text{if } (\alpha,\beta) \ne (j,j) \text{ or } (j-1,j).
\end{aligned}
$$

■

Further, using the induction assumptions and Lemma 3.6, we see, that it remains to prove the s_j-invariance only for the following functions:

$$\psi_{1j},\ \overline{\psi}_{1j},\ \overline{\psi}_{2j},\ \psi_{j-1,j}, \psi_{jj},\ \overline{\psi}_{j-1,j},\ \overline{\psi}_{jj}.$$

But according to Lemmas 3.4 and 3.5, we may reduce the proof of the s_j-invariance of the last functions to that of the following ones:

$$\psi_{1j},\ \overline{\psi}_{1j},\ \psi_{22},\ \psi_{33},\ \psi_{34},\ \overline{\psi}_{33},\ \overline{\psi}_{44},\ \overline{\psi}_{45}.$$

In order to finish the inductive step and thus to prove our Theorem, we are going to give for the functions under consideration the expressions, which are s_j-invariance either by the inductive assumption or by the evident reasons. The following Lemmas contain all necessary formulas.

Lemma 3.7 (Formula (3.22)). We have

$$
\begin{aligned}
a)\quad & \psi'_{1j}(x) = min(x_{1,j+1} - x_{1,j},\ x_{j,j+1} - x_{j+1,j+1} + x_{j-1,j} - x_{j-1,j+1}, \\
& \qquad\qquad 2x_{j-1,j} - x_{j-1,j+1} - x_{j-1,j-1} + x_{j,j} - x_{j+1,j+1}), \\
& \psi''_{1j}(x) = min(\psi_{1,j-1}(x),\ x_{j-1,j+1} - x_{j,j}). \\
b)\quad & \psi'_{jj}(x) = x_{j-1,j+1} - x_{j-1,j}, \\
& \psi''_{jj}(x) = x_{j,j} - x_{j,j+1}. \\
c)\quad & \psi'_{34} = min(x_{24} - x_{23},\ x_{34} - x_{33},\ x_{45} - x_{44}), \\
& \psi''_{34}(x) = min(\psi_{33}(x),\ x_{34} - x_{35}). \\
d)\quad & \overline{\psi}'_{jj}(x) = (min(x_{j-2,j} - x_{j-2,j-1} - x_{j-1,j} + x_{j-1,j-1}, \\
& \qquad\qquad x_{j,j+1} - x_{j,j} - x_{j-1,j} + x_{j-1,j-1}))_+, \\
& \overline{\psi}''_{jj} = (\psi_{j-1,j-1}(x) + x_{j-1,j+1} - x_{j-1,j})_+.
\end{aligned}
$$

$$e)\ \overline{\psi}_{45}(x) = (min(x_{14} - x_{13} - x_{25} + x_{24},\ x_{34} - x_{33} - x_{24} + x_{23},\ x_{45} - x_{44} - x_{24} + x_{33}))_+, \quad (3.22)$$
$$\overline{\psi}''_{45}(x) = min(\overline{\psi}_{44}(x),\ (x_{45} - x_{46} + x_{33} - x_{34})_+).$$
$$f)\ \overline{\psi}'_{1j}(x) = (min(x_{1,j+1} - x_{1,j} + x_{j-1,j+1} - x_{j,j+1},\ x_{j-1,j} - x_{j-1,j-1} - x_{j,j+1} + x_{j,j}))_+,\quad j \geq 3,$$
$$\overline{\psi}''_{1j}(x) = (\psi_{1j-1}(x) + x_{j,j} - x_{j-1,j+1})_+,\quad j \geq 3.$$
$$g)\ \overline{\psi}'_{12}(x) = \overline{\psi}_{12}(x),\ \overline{\psi}''_{12}(x) = (\overline{\psi}_{11} + x_{22} - x_{13})_+.$$

Lemma 3.8. a) The function $\overline{\psi}''_{45}(x)$ is a s_4-invariant iff $\overline{\psi}''_{45}(R_{34}([34][45])(x))$ is a s_3-invariant.

b) We have the following equality

$$\overline{\psi}''_{45}(R_{34}([34][45])(x)) = (min(\psi_{33}(x) + x_{25} - x_{24},\ x_{45} - x_{44} - x_{36} + x_{35}))_+.$$

The proofs of Lemma 3.7 and Lemma 3.8 based on a direct calculation. We need Lemma 3.8, because the s_4-invariance of $\overline{\psi}''_{45}$ does't evident from Lemma 3.7, *e*).

■

As noted above, these Lemmas complete the inductive step and so the proof of the Theorem 3.2 is over. ■

Corollary 3.4 (of Theorem 3.2). Let $Q^1_n(a_1, \cdots, a_n)$, be the piecewise linear function defined by means of the recurrence relation (1.31). Then

$$(a_1 + Q^1_n(a_1 \cdots a_n) + Q^1_n(a_2 \cdots a_n a_1))_- = (a_1)_- + Q^1_n(a_2 \cdots a_n a_1), \quad (3.24)$$

Proof. The identity (3.24) is equivalent to the s_n-invariance of the function $\psi_{1n}(x)$ (see (3.19)) on the space X_{n+1}. When $n = 3$, we obtain the following identity (see Lemma 3.2):

$$[-a - b + (a + (c)_-)_+ + (b + (c)_+)_-]_- = (c)_-(b + (a)_-)_+ - (c + (a)_+)_-,$$

where $a_1 = c,\ a_2 = a,\ a_3 = b$. Note that as a corollary of Theorem 3.2 we obtain a "geometrical" proof of the (3.24). It is an interesting task to find an algebraic proof for (3.24).

Remark 3.1. The proof of Theorem 3.1, i.e. the s_n-invariance of the cocharge $\overline{c}_{n+1}(x)$, based on the similar, but more refined, technics. The crucial step is the analogs of Lemmas 3.6 and 3.7. The details will be appear elsewhere.

Remark 3.2. Note that Theorem 3.1 gives opportunity to define the charge of a tableau $T \in STY(\lambda, \beta)$ in the case, when weight β is a composition. In fact we may define the charge $c(T)$ of tableau $T \in STY(\lambda, \beta)$ as

$$c(T) = n(\beta) - n(\lambda) - \overline{c}(T),$$

where $n(\beta)=\sum_i (i-1)\beta_i$ and $\overline{c}(T)$ is the cocharge of tableau $T\in STY(\lambda,\beta)\hookrightarrow K_{\mathbf{Z}}^{\lambda}$. Now we give a direct method for calculation of the charge, which is a generalization of the Lascoux-Schützenberger algorithm (e.g.[Ma]). Given a standard tableau $T\in STY(\lambda,\beta)$, consider the corresponding word $w(T)$ and a subdivision

$$w(T)=w_1\coprod w_2\coprod\cdots\coprod w_k$$

of the word $w(T)$ on a standard subwords w_i according to the Lascoux-Schützenberger algorithm, but now the standardness of a word means that its weight β satisfies the condition $\beta_i\le 1$ for all i, i.e. the word $w=a_1\cdots a_N$, $a_i\in[1,n]$ is called standard, if all elements a_i are different. We denote by $a(w)$ the minimal element in a word w and by $a^- := max\{\ b\in w\mid = b<a\ \}$. Now let us define the index $ind_{w_k}(a)$ of a letter $a\in w_k$ for all $k\ge 1$. At first, we want to attribute the indeces to the minimal elements $a(w_l)$ of the standard subwords w_l. We shall use an induction.

(*i*) Let us put $ind_{w_1}a(w)=a(w)-1$;

(*ii*) Assume that we already defined the indeces of all minimal elements $<a(w_k)$ (for some $k\ge 1$). Let k' be the maximal number such that $a(w_{k'})<a(w_k)$, and let us put $a:=a(w_k)$, $a':=a(w_{k'})$. In this case we define

$$ind_{w_k}(a)=\begin{cases} ind_{w_{k'}}(a')+a-a'-1, & \text{if } a' \text{ lies at the right from } a \text{ in } w;\\ ind_{w_{k'}}(a')+a-a', & \text{if } a' \text{ lies at the left from } a \text{ in } w.\end{cases}$$

Secondly, using the known values of the indeces for $a(w_k)$, we shall find all others by induction.

(*iii*) Let us assume that $a\in w_k$ and $a>a(w_k)$, then there exist $a^-\in w_k$. We define in this case

$$ind_{w_k}(a)=\begin{cases} ind_{w_k}(a^-)+(a-a^--1), & \text{if } a \text{ lies at the right from } a^- \text{ in } w_k;\\ ind_{w_k}(a^-)+(a-a^-), & \text{if } a \text{ lies at the left from } a^- \text{ in } w_k.\end{cases}$$

Finally, we define

$$c(w)=\sum_k\sum_{a\in w_k} ind_{w_k}(a).$$

Proposition 3.4. If $t\in STY(\lambda,\beta)$ then

$$c(T)+\overline{c}(T)=n(\beta)-n(\lambda).$$

■

Let us give an explanatory example. Take $\lambda=(11,9,6,2)$, $\beta=(3,4,4,4,4,4,4,4)$ and consider the the following tableau:

$$T:=\begin{matrix}1&1&1&2&2&2&4&4&6&7&8\\2&3&3&3&4&5&7&8&8\\3&4&5&6&7&8\\5&5&7\\6&6\end{matrix}\qquad\in STY(\lambda,\beta)$$

Our algorithm gives for the charge of the tableau T the following answer:

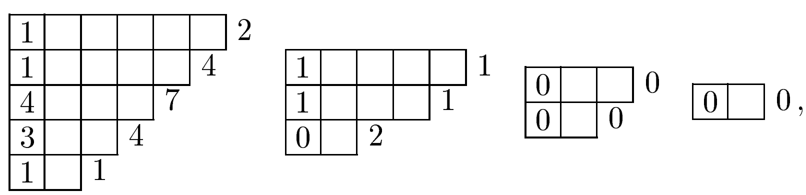
$$w_1(T) = \begin{array}{c} 8\,4\,1\,5\,2\,6\,3\,7 \\ {\scriptstyle 2\;1\;0\;1\;0\;1\;0\;1} \end{array}, \quad c(w_1) = 6,$$

$$w_2(T) = \begin{array}{c} 7\,2\,1\,8\,3\,4\,5\,6 \\ {\scriptstyle 2\;1\;0\;2\;1\;1\;1\;1} \end{array}, \quad c(w_2) = 9,$$

$$w_3(T) = \begin{array}{c} 4\,2\,1\,7\,3\,8\,5\,6 \\ {\scriptstyle 2\;1\;0\;3\;1\;3\;2\;2} \end{array}, \quad c(w_3) = 14,$$

$$w_4(T) = \begin{array}{c} 6\,2\,8\,4\,3\,7\,5 \\ {\scriptstyle 3\;1\;4\;2\;1\;3\;2} \end{array}, \quad c(w_4) = 16,$$

Consequently, $c(T) = 45$, and $\overline{c}(T) := n(\beta) - n(\lambda) - c(T) = 112 - 38 - 45 = 29$. We may use another method (see [Ki1]) for calculating the cocharge based on an application of rigged configurations. First of all, we find the rigged configuration, which corresponds to the tableau T:

$T \longleftrightarrow$ 

$$\beta = \begin{pmatrix} 3 & 3 & 4 & 4 & -2 & -1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The charge of the configuration β is equal to $c(\beta) := \sum_{i,j} \binom{\beta_{i,j}}{2} = 26$, and the charge of unique tableau $\widetilde{T}$ with the dominant weight, which lies in the S_8-orbit corresponding to the tableau T, is equal to

$$c(\widetilde{T}) := c(\beta) + \{ \text{ sum of the quantum numbers } \} = 26 + 12 = 38.$$

Consequently,

$$\overline{c}(\widetilde{T}) = n(\beta) - n(\lambda) - c(\widetilde{T}) = 105 - 38 - 38 = 29,$$

where the dominant weight $\beta = (4^7,\ 3)$. Of couse, our two methods give the same result.

Appendix. Proof of Theorem 1.2.

We give here only a proof of the statement (1.28a). The statement (1.28b) may be proved by a similar method. Let n be a fixed positive integer, consider the spaces of triangles X_n (see Section 1) and partitions D_n. By definition, a partition $d \in D_n$ is a triangular array of real numbers $d = (d_{ij}),\ \ 1 \le i < j \le n$. As a vector space $D_n \simeq \mathbf{R}^{\frac{n(n-1)}{2}}$. We define a weight of partition d as a vector $\gamma = (\gamma_1, \cdots, \gamma_n)$ with components

$$\gamma_j = -\sum_{i<j} d_{ij} + \sum_{k>j} d_{jk}, \quad 1 \le j \le n.$$

Let us define a mapping $\partial = \partial_n : X_n \to D_n$ by putting $\partial(x) = d$, where $d_{ij} = x_{ij} - x_{ij-1},\ \ 1 \le i < j \le n$.

Lemma A.1. (i) A map $(\lambda\partial) :\ x \to (\lambda(x),\ \partial(x)),\ x \in X_n$ gives a bijection of vector spaces $X_n \to \mathbf{R}^n \oplus D_n$.

$$(ii) \quad \gamma(\partial(x)) = \lambda(x) - \beta(x).$$

The bijection $(\lambda\partial)$ allows us to transfer an action of any transformation of the space X_n to the space D_n and vice versa. We will denote the operators $F :\ X_n \to X_n$ and $(\lambda\partial) \cdot F \cdot (\lambda\partial)^{-1}\ :\ \mathbf{R}^n \oplus D_n \to \mathbf{R}^n \oplus D_n$ by the same letter F if this does not lead to a confusion.

Lemma A.2. Let $[i, i+1]$ be the involution as given in Definition 1.3, then

$$\begin{aligned} & [i, i+1](\lambda, d) = (\lambda, \widetilde{d}), \quad \text{where} \\ (i) \quad & \widetilde{d}_{i,i+1} = -d_{i,i+1}, \\ & \widetilde{d}_{i,j} = d_{i+1,j}, \quad \text{if} \quad j > i+1, \\ & \widetilde{d}_{k,i+1} = d_{k,i}, \quad \text{if} \quad k < i, \end{aligned}$$

and for all remaining elements $\widetilde{x}_{\alpha\beta} = x_{\alpha\beta}$.

$$(ii) \quad \gamma(\widetilde{d}) = (i, i+1)\gamma(d).$$

It follows from Lemma 1.1 that the involutions $[i, i+1]$ generate a group which is isomorphic to the symmetric group S_n.

Let us fix a positive integer $j,\ \ 1 \le j \le n$, and define an operator $\partial^{(j)} :\ X_n \to D_n$ in the following way: $\partial^{(j)}(x) = d$, where (see Fig.1)

$$\begin{aligned} & d_{il} = x_{i,l} - x_{i,l-1}, \quad \text{if} \quad l > j \quad \text{and} \quad l > i, \\ & d_{il} = x_{i-1,l-1} - x_{i-1,l-2}, \quad \text{if} \quad 2 < i < l \le j, \\ & d_{1l} = x_{l-1,j-1} - x_{l,j}, \quad \text{if} \quad 1 < l \le j. \end{aligned}$$

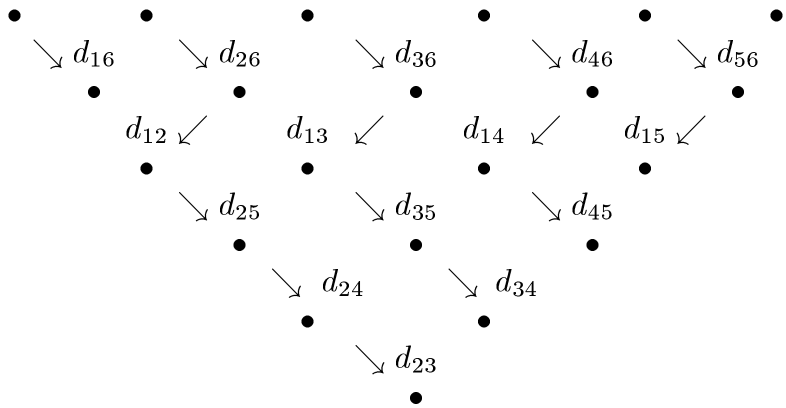


Fig.1

$$\partial^{(5)}\ :\ X_6 \to D_6.$$

It is clear that $\partial^{(1)} = \partial$. Further, let us define a mapping $(\lambda\partial^{(j)}) :\ X_n \to \mathbf{R}^n \oplus D_n$ by means of a rule

$$x \to (\lambda(x),\ \partial^{(j)}(x))$$

Lemma A.3. The map $(\lambda\partial^{(j)})$ is a vector space isomorphism.

Now we rewrite the involutions R_{ijk} and φ_i (see Definition 1.2) of the space X_n in terms of partitions.

Lemma A.4. We have (i) $R_{ijk}(\lambda, d) = (\lambda, \widetilde{d})$, where

$$\begin{aligned}
\widetilde{d}_{i,j} &= d_{i,k} + max(d_{ij} - d_{jk},\ 0),\\
\widetilde{d}_{j,k} &= d_{i,k} + max(d_{jk} - d_{ij},\ 0),\\
\widetilde{d}_{i,k} &= min(d_{ij},\ d_{jk}),
\end{aligned}$$

and for all remaining elements $\widetilde{d}_{\alpha\beta} = d_{\alpha\beta}$.

$$\begin{aligned}
(ii)\quad & \varphi_i(\lambda, d) = (\lambda, \widetilde{d}), \quad \text{where}\\
& \widetilde{d}_{i,i+1} = d_{i,i+1} + \lambda_i - \lambda_{i+1} - \gamma_i + \gamma_{i+1}.
\end{aligned}$$

For all remaining elements $\widetilde{d}_{\alpha\beta} = d_{\alpha\beta}$. Here $(\gamma_1, \cdots, \gamma_n) = \gamma(d)$ is a weight of d.

Note that the involutions $R_{ijk},\quad 1 \le i < j < k \le n$, satisfy the relations of Proposition 1.5.

At last, we define $s_i^{\vee} = (\lambda\partial)s_i(\lambda\partial)^{-1}\ :\ \mathbf{R}^n \oplus D_n \to \mathbf{R}_n \oplus D_n,\quad 1 \le i \le n-1$, where the involutions s_i are given by (1.16).

Theorem A.1. We have the following equality

$$s_i^{\vee} = R_i^{-1}\varphi_i R_i, \quad 1 \le i \le n-1,$$

where $R_i := R_{1,i,i+1} \cdot R_{2,i,i+1} \cdots R_{i-1,i,i+1}$, if $2 \le i \le n-1$, and $R_1 = Id$.

It is clear that the identity (1.28a) follows from Theorem A.1.

Proof of Theorem A.1. We use the formula (1.19) as an inductive step. Note that Theorem A.1 is evident if $i=1$. So it is sufficient to prove that

$$(\lambda\partial^{(i-1)})s_i(\lambda\partial^{(i-1)})^{-1} = R_i^{-1}\varphi_i R_i.$$

Now we use an inductive hypothesis and the formula (1.19). So we have

$$\begin{aligned}(\lambda\partial^{(i-1)})s_i(\lambda^{(i-1)})^{-1} &= \widetilde{t}_{i-1}\widetilde{t}_i\widetilde{s}_{i-1}(\widetilde{t}_i)^{-1}(\widetilde{t}_{i-1})^{-1}, \quad \text{where}\\ \widetilde{s}_{i-1} &= (\lambda\partial^{(i+1)})s_{i-1}(\lambda\partial^{(i+1)})^{-1},\\ \widetilde{t}_i &= (\lambda\partial^{(i)})t_i(\lambda\partial^{(i+1)})^{-1},\\ \widetilde{t}_{i-1} &= (\lambda\partial^{(i-1)})t_{i-1}(\lambda\partial^{(i)})^{-1}.\end{aligned} \tag{A.1}$$

Proposition A.1. (i) If Theorem A.1 is valid for the involution $s_{i-1}^{\vee}$, then

$$\begin{aligned}&\widetilde{s}_{i-1} = (\widetilde{R}_i)^{-1}\varphi_i\widetilde{R}_i,\\ (ii)\quad &\widetilde{t}_i = R^{(i+1)}, \quad \widetilde{t}_{i-1} = R^{(i)}, \quad \text{where}\\ &R^{(j)} = R_{1,2,j}\ R_{1,3,j}\ \cdots\ R_{1,j-1,j},\\ &\widetilde{R}_i = R_{2,i,i+1}\ \cdots\ R_{i-1,i,i+1} = R_{1,i,i+1}\cdot R_i.\end{aligned} \tag{A.2}$$

The statement (ii) may be proved by direct computation. Let us prove the statement (i). We have

$$\widetilde{s}_{i-1} = (\lambda\partial^{(i+1)})(\lambda\partial)^{-1}((\lambda\partial)s_{i-1}(\lambda\partial)^{-1})(\lambda\partial)(\lambda\partial^{(i+1)})^{-1}.$$

According to an inductive step we have an identity

$$(\lambda\partial)s_{i-1}(\lambda\partial)^{-1} = R_{i-1}^{-1}\varphi_{i-1}R_{i-1}.$$

The following Lemma completes the proof of the statement (i):

Lemma A.5. The following identities are valid

$$\begin{aligned}(\lambda\partial^{(i+1)})(\lambda\partial)^{-1}R_{i-1}(\lambda\partial)(\lambda\partial^{(i+1)})^{-1} &= \widetilde{R}_i,\\ (\lambda\partial^{(i+1)})(\lambda\partial)^{-1}\varphi_{i-1}(\lambda\partial)(\lambda\partial^{(i+1)})^{-1} &= \widetilde{\varphi}_i.\end{aligned}$$

This Lemma follows immediately from the following one

Lemma A.6. Let $(\lambda, d) \in \mathbf{R}^n \oplus D_n$. Then $(\lambda\partial^{(i+1)})(\lambda\partial)^{-1}(\lambda\partial) = (\lambda, \widetilde{d})$, where

$$\begin{aligned}\widetilde{d}_{k,j} &= d_{k,j} \quad \text{if} \quad j > i+1,\\ \widetilde{d}_{k,j} &= d_{k-1,j-1}, \quad \text{if} \quad j \le i+1,\ k > 1,\\ \widetilde{d}_{1,j} &= \lambda_{j-1} - d_{j-1,n} - d_{j-1,n-1} - \cdots - d_{j-1,j} -\\ &\quad - \lambda_j + d_{j,n} + d_{j,n-1} + \cdots + d_{j,j+1}, \quad \text{if} \quad j \le i+1\end{aligned}$$

We finish the prove of Theorem A.1 and thus Theorem 1.2 by means of the following relation between transformations (A.2).

Proposition A.2.

$$R^{(i)}R^{(i+1)}(\widetilde{R}_i)^{-1} = (\widetilde{R}_i)^{-1}R_{1,i,i+1}R^{(i+1)}R_{1,i,i+1}R^{(i)}. \qquad (A.3)$$

In fact, it follows from (A.1) - (A.2) that

$$(\lambda\partial^{(i-1)})s_i(\lambda\partial^{(i-1)})^{-1} = R^{(i)}R^{(i+1)}(\widetilde{R}_i)^{-1}\varphi_i(R^{(i)}R^{(i+1)}\widetilde{R}_i)^{-1}$$

Now using (A.3) we obtain $(\lambda\partial^{(i-1)})s_i(\lambda\partial^{(i-1)})^{-1} =$

$$= R_i^{-1}(R^{(i+1)}R_{1,i,i+1}R^{(i)}\varphi_i(R^{(i)}R_{1,i,i+1}R^{(i+1)})^{-1})R_i = R_i^{-1}\varphi_i R_i,$$

because both $R^{(i)}$ and $R^{(i+1)}R_{1,i,i+1}$ commute wuth φ_i.

The proof of Proposition A.2 consists of successive application of the relations c) and b) of Proposition 1.5 to the quadruples $(1,k,i,i+1)$ with $k = i-1,\ i-2,\cdots,2$. ∎

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