



Braided doubles and rational Cherednik algebras

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Abstract

We introduce and study a large class of algebras with triangular decomposition which we call *braided doubles*. Braided doubles provide a unifying framework for classical and quantum universal enveloping algebras and rational Cherednik algebras. We classify braided doubles in terms of *quasi-Yetter–Drinfeld (QYD) modules* over Hopf algebras which turn out to be a generalisation of the ordinary Yetter–Drinfeld modules. To each braiding (a solution to the braid equation) we associate a QYD-module and the corresponding *braided Heisenberg double*—this is a quantum deformation of the Weyl algebra where the role of polynomial algebras is played by Nichols–Woronowicz algebras. Our main result is that any rational Cherednik algebra canonically embeds in the braided Heisenberg double attached to the corresponding complex reflection group.

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0. Introduction

In the present paper we introduce and study a large class of algebras with triangular decomposition which we call *braided doubles*. Our approach is motivated by two recent developments in representation theory and quantum algebra:

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- The discovery, by Etingof and Ginzburg [12], of rational Cherednik algebras $H_{t,c}(W)$ for an arbitrary complex reflection group W . Similarly to enveloping algebras and their quantum deformations, rational Cherednik algebras admit a triangular decomposition $H_{t,c}(W) = S(\mathfrak{h}) \otimes \mathbb{C}W \otimes S(\mathfrak{h}^*)$ (here \mathfrak{h} is the reflection representation of W).
- The emergence of the Fomin–Kirillov algebra as a noncommutative model for the cohomology of the flag manifold [13], and its interpretation by Majid [35] in terms of a Nichols–Woronowicz algebra \mathcal{B}_{S_n} attached to the symmetric group. The Fomin–Kirillov model was later generalised to all Coxeter groups W by the first author [6], as a W -equivariant homomorphism $S(\mathfrak{h}) \rightarrow \mathcal{B}_W$, where \mathcal{B}_W is the Nichols–Woronowicz algebra attached to W .

Our first principal result, Theorem 7.20, extends the above homomorphism $S(\mathfrak{h}) \rightarrow \mathcal{B}_W$ to an embedding of the restricted Cherednik algebra $\overline{H}_{0,c}(W)$ in what we call a *braided Heisenberg double* \mathcal{H}_W , which also has triangular decomposition $\mathcal{H}_W = \mathcal{B}_W \otimes \mathbb{C}W \otimes \mathcal{B}_W$. For nonzero t , such an embedding of $H_{t,c}(W)$ is obtained by replacing \mathcal{B}_W with its deformation $\mathcal{B}_{W,t}$. We thus find a new, quantum group-like realisation of each rational Cherednik algebra.

The above has prompted us to look for a framework in which both the enveloping algebras (and quantum groups) and the rational Cherednik algebras could be uniformly treated. This is precisely the framework of braided doubles, where the aforementioned objects fit into a general class of algebras with triangular decomposition $A = U^- \otimes H \otimes U^+$ over a Hopf algebra H , such that the algebras U^- , U^+ are generated by dually paired H -modules V , V^* and the commutator of V and V^* in A lies in H .

Surprisingly, we have been able to completely classify (Theorem 3.3) all *free* braided doubles in terms of *quasi-Yetter–Drinfeld modules*, which are a generalisation of Yetter–Drinfeld modules [52,29]. Our quasi-Yetter–Drinfeld modules turn out to have a natural interpretation in terms of monoidal categories. Using a variant of the Tannaka–Krein duality, we prove in Section 2 that a set Π of *compatible braidings* on a vector space V turns V into a quasi-YD module over a certain Hopf algebra H_Π , hence yields a free braided double of the form $T(V) \otimes H_\Pi \otimes T(V^*)$.

Braided doubles are such quotients of free braided doubles that still admit triangular decomposition. The most interesting are the *minimal doubles*. For a quasi-Yetter–Drinfeld module V over a Hopf algebra H , we describe (Theorem 4.11) the relations in the corresponding minimal double $\bar{A}(V)$, implicitly as kernels of *quasibraided factorials* on $T(V)$ and $T(V^*)$ given in terms of the quasi-YD structure. If V is a Yetter–Drinfeld module, $\bar{A}(V)$ is a braided Heisenberg double, which factorises into H and two dually paired Nichols–Woronowicz algebras (Theorem 5.4). Prominent examples of minimal doubles are the universal enveloping algebra $U(\mathfrak{g})$ and its quantisation $U_q(\mathfrak{g})$; the relations in minimal doubles are therefore a (vast) generalisation of the Serre relations.

Finally, we discover that any quasi-YD module can be obtained as a certain sub-quotient of a Yetter–Drinfeld module (Theorem 6.9). In interesting cases, this allows us to embed a minimal double in a braided Heisenberg double. We put this observation to use when we classify braided doubles $U^- \otimes \mathbb{k}G \otimes U^+$ over group algebras, where U^+ and U^- are commutative. The outcome of the classification is rational Cherednik algebras; this is how the motivating results, described in the beginning of this Introduction, naturally re-emerge in the braided doubles setup. An immediate consequence of the theory is the *PBW theorem* for rational Cherednik algebras over an arbitrary field—a crucial property which has so far been known only in characteristic zero.

1. Overview of main results

In this section we state and discuss the main results of the paper. Details and proofs will be given in Sections 2–7. We assume that the reader is familiar with the basics of the theory of Hopf algebras, for which [47] is one of the standard references.

Notation

Throughout the paper, \mathbb{k} is the ground field (of arbitrary characteristic). Vector spaces, tensor products, (bi)algebras and Hopf algebras are over \mathbb{k} . The tensor algebra of a vector space V is denoted by $T(V)$; it has grading $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$. We use Sweedler-type notation (without the summation sign [38, 1.4.2]): if H is a bialgebra, the coproduct of $h \in H$ is denoted by $h_{(1)} \otimes h_{(2)} \in H \otimes H$; a left coaction $\delta: V \rightarrow H \otimes V$ of H on a space V is denoted by $v \mapsto v^{(-1)} \otimes v^{(0)}$. By writing $\delta(v) = v^{[-1]} \otimes v^{[0]}$, we imply that δ is not a coaction but just a linear map $V \mapsto H \otimes V$ (referred to in the paper as *quasicoaction*). The symbols \triangleright and \triangleleft mean left, respectively right, action of a bialgebra. The counit of a bialgebra H is denoted by $\epsilon: H \rightarrow \mathbb{k}$.

If U^- (respectively U^+) is a left (respectively right) module algebra for H , the corresponding semidirect product is denoted by $U^- \rtimes H$ (respectively $H \rtimes U^+$), see [47, Section 7.2].

Finally, if H is a Hopf algebra, then $S: H \rightarrow H$ denotes the antipode of H . All Hopf algebras are assumed to have *bijective* antipode. Theorem A below holds when H is any bialgebra; in Theorems B–G, we assume H to be a Hopf algebra.

1.1. A problem in deformation theory

Let V be a finite-dimensional space over \mathbb{k} with left action, $\triangleright: H \otimes V \rightarrow V$, of a bialgebra H . The dual vector space V^* is canonically a right H -module, with right action $\triangleleft: V^* \otimes H \rightarrow V^*$ defined by

$$\langle f \triangleleft h, v \rangle = \langle f, h \triangleright v \rangle, \quad f \in V^*, \quad h \in H, \quad v \in V,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between V^* and V .

To every linear map $\beta: V^* \otimes V \rightarrow H$ (a *bialgebra-valued pairing*) there corresponds an associative algebra \tilde{A}_β , generated by the spaces V, V^* and the algebra H subject to the relations

$$fh = h_{(1)}(f \triangleleft h_{(2)}), \quad hv = (h_{(1)} \triangleright v)h_{(2)}, \quad [f, v] = \beta(f, v) \in H,$$

for all $f \in V^*, h \in H, v \in V$. Here and below, $[f, v]$ denotes the commutator $fv - vf$. It is clear from the defining relations that the map

$$m_\beta: T(V) \otimes H \otimes T(V^*) \rightarrow \tilde{A}_\beta,$$

of vector spaces, given by multiplication of generators in \tilde{A}_β , is surjective. We say that \tilde{A}_β has *triangular decomposition over H* , if m_β is one-to-one. We will indicate this by writing

$$\tilde{A}_\beta = T(V) \rtimes H \rtimes T(V^*);$$

observe that the subalgebras $T(V) \rtimes H$ and $H \rtimes T(V^*)$ of \tilde{A}_β are indeed semidirect products with respect to the action of H , which extends from V to $T(V)$ (respectively from V^* to $T(V^*)$) via the coproduct in H .

The algebra \tilde{A}_0 can be shown to have triangular decomposition. Algebras \tilde{A}_β may be viewed as deformations of \tilde{A}_0 , with parameter β which takes values in $\text{Hom}_{\mathbb{k}}(V^* \otimes V, H)$. Triangular decomposition means that \tilde{A}_β is a flat deformation of \tilde{A}_0 . Our first principal result (which appears as Theorem 3.3 in Section 3) describes all values of β for which the deformation is flat:

Theorem A. *The algebra \tilde{A}_β has triangular decomposition over the bialgebra H , if and only if the H -valued pairing $\beta: V^* \otimes V \rightarrow H$ satisfies*

$$h_{(1)}\beta(f \triangleleft h_{(2)}, v) = \beta(f, h_{(1)} \triangleright v)h_{(2)} \tag{A}$$

for all $f \in V^*$, $v \in V$ and $h \in H$.

Remark. Observe that Eq. (A) is necessary for \tilde{A}_β to have triangular decomposition, because of obstruction in degree 3; the product fhv of three generators $f \in V^*$, $h \in H$ and $v \in V$, can be expanded in two ways which must coincide:

$$0 = (fh)v - f(hv) = h_{(1)}\beta(f \triangleleft h_{(2)}, v) - \beta(f, h_{(1)} \triangleright v)h_{(2)} \in H \hookrightarrow \tilde{A}_\beta.$$

Remark. When $H = \mathbb{k}G$ is a group algebra of a group G , Eq. (A) means that $\beta: V^* \otimes V \rightarrow \mathbb{k}G$ is a G -equivariant map, where the action of $g \in G$ on $V^* \otimes V$ is given by $g(f \otimes v) := f \triangleleft g^{-1} \otimes g \triangleright v$, and the G -action on $\mathbb{k}G$ is the adjoint one. In other words, Eq. (A) is precisely what allows us to extend the action of G from each of the factors $T(V)$, $\mathbb{k}G$, $T(V^*)$ in the triangular decomposition to a covariant G -action on the whole algebra \tilde{A}_β .

For a Hopf algebra H , one shows that (under mild technical assumptions) the H -action extends in this way to a covariant H -action on the algebra \tilde{A}_β if and only if H is cocommutative. We would thus like to warn the reader that in general, algebras \tilde{A}_β have no natural covariant action of H and *cannot* be viewed as algebras in the category of H -modules.

We will now make the above deformation problem harder by assuming additional relations, not necessarily quadratic, between the elements of V (respectively V^*). Let $I^- \subset T^{>0}(V)$, $I^+ \subset T^{>0}(V^*)$ be two-sided ideals. The algebra $\tilde{A}_\beta / \langle I^-, I^+ \rangle$ is said to have triangular decomposition over H , if the natural linear map

$$T(V)/I^- \otimes H \otimes T(V^*)/I^+ \rightarrow \tilde{A}_\beta / \langle I^-, I^+ \rangle$$

is bijective. (Angular brackets denote a two-sided ideal with given generators.)

Once again, algebras $\tilde{A}_\beta / \langle I^-, I^+ \rangle$ with triangular decomposition are flat deformations of $\tilde{A}_0 / \langle I^-, I^+ \rangle$. But now, instead of looking for the values of $\beta \in \text{Hom}_{\mathbb{k}}(V^* \otimes V, H)$ which guarantee flatness, we pose an inverse problem:

Problem. For a given bialgebra-valued pairing $\beta: V^* \otimes V \rightarrow H$, describe all possible ideals $I^- \subset T^{>0}(V)$, $I^+ \subset T^{>0}(V^*)$ of relations such that the algebra $\tilde{A}_\beta / \langle I^-, I^+ \rangle$ has triangular decomposition over H .

To attack this deformation problem, we introduce and study quasi-Yetter–Drinfeld modules.

1.2. *Quasi-Yetter–Drinfeld modules*

The following observation is crucial for the theory of braided doubles developed in the present paper: Eq. (A) appears in the definition of a *Yetter–Drinfeld module* over a bialgebra H . There is, however, an extra ingredient in that definition, which we do not have in our picture.

We will now define finite-dimensional Yetter–Drinfeld modules in a way different from (but equivalent to) what is usually seen in the quantum groups literature, and will introduce their generalisation called *quasi-Yetter–Drinfeld modules*. Note that the space $V^* \otimes V$ has a standard structure of a coalgebra, dual to the algebra $\text{End}(V) \cong V \otimes V^*$.

Definition. A *quasi-Yetter–Drinfeld module* over a bialgebra H is a finite-dimensional space V with the following structure:

- left H -action \triangleright on V ;
- linear map $\beta : V^* \otimes V \rightarrow H$, which satisfies (A).

Definition. A *Yetter–Drinfeld module* over H is a quasi-Yetter–Drinfeld module where the map β is a morphism of coalgebras.

Yetter–Drinfeld modules over a bialgebra H were introduced by Yetter in [52] as “crossed bimodules,” and were shown by Majid [29] to be the same as modules over the Drinfeld quantum double $D(H)$ when H is a finite-dimensional Hopf algebra. It can be said that Yetter–Drinfeld modules’ *raison d’être* is their relationship with *braidings*. A Yetter–Drinfeld module structure on the space V gives rise to a map

$$\Psi : V \otimes V \rightarrow V \otimes V, \quad \Psi(v \otimes w) = \beta(f^a, v) \triangleright w \otimes v_a,$$

which is a braiding, i.e., a solution to the quantum Yang–Baxter equation $\Psi_{12}\Psi_{23}\Psi_{12} = \Psi_{23}\Psi_{12}\Psi_{23}$. (Here $\{f^a\}, \{v_a\}$ denote a pair of dual bases of V^*, V ; summation over the index a is implied.) Moreover, Yetter–Drinfeld modules over a Hopf algebra form a *braided monoidal category* (see a survey in [10, 4.3]).

Traditionally, in the definition of Yetter–Drinfeld module over a bialgebra H the H -valued pairing between V^* and V is encoded by a linear map V to $H \otimes V$:

$$\beta : V^* \otimes V \rightarrow H \rightsquigarrow \delta = \delta_\beta : V \rightarrow H \otimes V, \quad \delta(v) = \beta(f^a, v) \otimes v_a.$$

The Yetter–Drinfeld condition translates in terms of δ into a formula with two levels of Sweedler notation, see Definition 2.1. Moreover, β is a coalgebra morphism if and only if δ is a coaction of H . Dropping the coaction condition leads to the class of quasi-Yetter–Drinfeld modules. We will think of quasi-YD modules for a bialgebra H as pairs (V, δ) , where V is an H -module and $\delta \in \text{Hom}_{\mathbb{k}}(V, H \otimes V)$ is a *Yetter–Drinfeld quasicoaction*.

The original motivation for the quasi-YD modules was the deformation problem given above. However, Section 2 of the present paper treats them from a categorical viewpoint, drawing a parallel with Yetter–Drinfeld modules. In particular, quasi-YD modules over a Hopf algebra form what we call a *semibraided monoidal category*. A converse is also true: a given semibraided category can be realised as quasi-YD modules over some Hopf algebra, which (the Hopf algebra) is reconstructed from the category. We present the reconstruction process as a form of the Tannaka–Krein duality.

Unlike for braidings, there is no canonical notion of a semibraiding on a single vector space (as opposed to a whole category), which would lead to a realisation of such space as a quasi-YD module. Nevertheless, we show in 2.8 that if V is equipped with a finite set Π of braidings which are pairwise *compatible*, then V is canonically an object in a semibraided category, hence a quasi-Yetter–Drinfeld module for a certain Hopf algebra H_Π .

A basic example of a set of compatible braidings and a quasi-YD module is as follows. Let $(V, \beta : V^* \otimes V \rightarrow H)$ be a Yetter–Drinfeld module over a cocommutative Hopf algebra H , with induced braiding Ψ . Then $\Pi = \{\Psi, \tau\}$ is a set of compatible braidings, where $\tau(v \otimes w) = w \otimes v$ is the trivial braiding on V . The space V can be made a quasi-Yetter–Drinfeld module over H via the new H -valued pairing $\beta_{\Psi, \lambda\tau} : V^* \otimes V \rightarrow H$, defined by $\beta_{\Psi, \lambda\tau}(f, v) = \beta(f, v) + \lambda\langle f, v \rangle$ for any scalar λ .

1.3. Braided doubles

We are now ready to give the

Definition. In the notation as above, an algebra $\tilde{A}_\beta / \langle I^-, I^+ \rangle$ with triangular decomposition $T(V)/I^- \rtimes H \rtimes T(V^*)/I^+$ over the bialgebra H is called a *braided double*.

Thus, by definition, braided doubles are the same as solutions to the deformation theory problem posed in 1.1.

The algebras \tilde{A}_β with triangular decomposition will now be referred to as *free braided doubles*. Theorem A means that free braided doubles are parametrised by quasi-Yetter–Drinfeld modules. Instead of \tilde{A}_β we write $\tilde{A}(V, \delta)$ for a free braided double associated to the quasi-YD module (V, δ) .

The following example demonstrates how one-dimensional quasi-Yetter–Drinfeld modules lead to interesting algebraic objects already at the level of free braided doubles.

Example. We show in 3.2 that all one-dimensional quasi-YD modules over a cocommutative Hopf algebra H are of the form $V_{\alpha, p}$, where $\alpha : H \rightarrow \mathbb{k}$ is any algebra homomorphism and p is any central element of H ; one has $h \triangleright v = \alpha(h)v$ and $\delta(v) = p \otimes v$ for $h \in H, v \in V_{\alpha, p}$. Let $H = S(\mathfrak{h})$ be the algebra of polynomials over a vector space \mathfrak{h} , with Hopf structure given by coproduct $\Delta h = h \otimes 1 + 1 \otimes h$ for $h \in \mathfrak{h}$. Consider any quasi-YD module (V, δ) which is a direct sum of one-dimensional quasi-YD modules: $V = V_{\alpha_1, p_1} \oplus \dots \oplus V_{\alpha_m, p_m}$ where $\alpha_i \in \mathfrak{h}^*$, and p_i are arbitrary polynomials in $S(\mathfrak{h})$. Let $\{f_i\}, \{e_i\}$ be dual bases of V^*, V such that $e_i \in V_{\alpha_i, p_i}$. The free braided double $\tilde{A}(V, \delta)$ is given by generators and relations

$$[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \quad h \in \mathfrak{h}; \quad [f_i, e_i] = p_i.$$

If the p_i are chosen to be in \mathfrak{h} , and the α_i are a basis of \mathfrak{h}^* related to the p_i via a generalised Cartan matrix, the algebras with triangular decomposition thus obtained are $\tilde{U}(\mathfrak{g})$, the Kac–Moody universal enveloping algebras before quotienting by the Serre relations. The Serre relations arise in the context of minimal doubles (see below).

Lifting the restriction on α_i and p_i leads to new algebras. In the simplest case $m = \dim \mathfrak{h} = 1$, we obtain a Smith algebra (a “polynomial deformation of $sl(2)$ ”), considered in [45] (and earlier in a different form in [18]). These algebras have a notion of highest weight modules and an

analogue of the BGG category \mathcal{O} ; those are, of course, consequences of the triangular decomposition. Polynomial deformations of $sl(2)$ have physical applications in quantum mechanics, conformal field theory, Yang–Mills-type gauge theories, inverse scattering, quantum optics [7].

After this example, let us move on to the case of nonzero ideals I^\pm .

1.4. Minimal doubles

A minimal double $\bar{A}(V, \delta)$ is a quotient of the free double $\tilde{A}(V, \delta)$ by largest ideals $I(V, \delta) \subset T^{>0}(V)$ and $I^*(V, \delta) \subset T^{>0}(V^*)$ such that the quotient still has the triangular decomposition property.

Minimal doubles are the most interesting braided doubles; they have the largest set of relations. Results of Section 4 of the present paper imply

Theorem B.

1. Any braided double has a unique minimal double as a quotient double.
2. The ideals $I(V, \delta) \subset T(V)$ and $I(V^*, \delta) \subset T(V^*)$ are graded, and are given by

$$I(V, \delta) = \bigoplus_{n \geq 1} \ker [\tilde{n}]!_\delta, \quad I(V^*, \delta) = \bigoplus_{n \geq 1} \ker [\tilde{n}]!_{\delta_r}$$

where $[\tilde{n}]!_\delta : V^{\otimes n} \rightarrow (H \otimes V)^{\otimes n}$ and $[\tilde{n}]!_{\delta_r} : V^{*\otimes n} \rightarrow (V^* \otimes H)^{\otimes n}$ are quasibraided factorials, which arise from the quasi-Yetter–Drinfeld structure δ on V .

To each quasi-Yetter–Drinfeld module (V, δ) over a Hopf algebra H , Theorem B associates two graded algebras, generated in degree one:

$$U(V, \delta) = T(V)/I(V, \delta), \quad U(V^*, \delta) = T(V^*)/I(V^*, \delta),$$

such that the minimal double has triangular decomposition

$$\bar{A}(V, \delta) = U(V, \delta) \rtimes H \rtimes U(V^*, \delta).$$

Theorem B is formally an answer to the deformation problem posed in 1.1, however, the relations in the algebras $U(V, \delta)$, $U(V^*, \delta)$ are given only implicitly by the kernels of quasibraided factorials (introduced in Definition 4.9). The latter might not be well suited for computational purposes: these operators may have values in infinite-dimensional spaces.

In Section 4, we also point out a sufficient condition for minimality of a braided double. A braided double with triangular decomposition of the form $T(V)/I^- \rtimes H \rtimes T(V^*)/I^+$, gives rise to an H -valued Harish-Chandra pairing between the algebras $T(V^*)/I^+$ and $T(V)/I^-$: the pairing $(b, \phi)_H$ is the product $b\phi$ of $b \in T(V^*)/I^+$ and $\phi \in T(V)/I^-$ in the braided double, projected onto H . One has

Theorem C. A braided double is minimal if its Harish-Chandra pairing is non-degenerate.

For example, the universal enveloping algebra $U(\mathfrak{g})$ is a minimal double, so that the kernels of the corresponding quasibraided factorials come out as the Serre relations. The converse of Theorem C is not true (Example 4.20), and is disproved using a counterexample to third Kaplansky’s conjecture on Hopf algebras.

Non-degeneracy of the Harish-Chandra pairing is a property which strongly influences the algebra structure of a braided double. As an example of this, a simple argument in Proposition 4.19 shows that if the scalar-valued pairing $\epsilon((\cdot, \cdot)_H)$ in a braided double $\hat{A}(V)$ is non-degenerate, then any two-sided ideal in $\hat{A}(V)$ has a nontrivial projection which is an ideal in H .

An important class of braided doubles featuring this non-degeneracy property are *braided Heisenberg doubles*. These are precisely the minimal doubles $\hat{A}(V)$ which correspond to Yetter–Drinfeld modules V .

1.5. Braided Heisenberg doubles and Nichols–Woronowicz algebras

“Honest” Yetter–Drinfeld modules are obviously a distinguished class of quasi-Yetter–Drinfeld modules. Section 5 of the paper describes minimal doubles associated to this class. They are called braided Heisenberg doubles. The defining ideals in a braided Heisenberg double are expressed in terms of the braiding on the Yetter–Drinfeld module:

Theorem D. *Let V be a Yetter–Drinfeld module over a Hopf algebra H . Denote by Ψ the induced braiding on V . The minimal double associated to V has triangular decomposition of the form*

$$\mathcal{H}_V \cong \mathcal{B}(V, \Psi) \rtimes H \ltimes \mathcal{B}(V^*, \Psi^*),$$

where $\mathcal{B}(V, \Psi)$ is the Nichols–Woronowicz algebra of a braided space (V, Ψ) .

Nichols–Woronowicz algebras are a remarkable class of *braided Hopf algebras*, and are a quantum analogue of both symmetric and exterior algebras. One has $\mathcal{B}(V, \Psi) = T(V)/\ker \text{Wor}(\Psi)$ where $\text{Wor}(\Psi)$ is the Woronowicz symmetriser associated with the braiding Ψ on V . The Woronowicz symmetriser, introduced in [51], appears in our setting as a specialisation of a more general quasibraided factorial to the case of a Yetter–Drinfeld module.

The symmetric (respectively, exterior) algebra of V is $\mathcal{B}(V, \tau)$ (respectively, $\mathcal{B}(V, -\tau)$) where $\tau(v \otimes w) = w \otimes v$ is the trivial braiding on V . Note that if V is a Yetter–Drinfeld module over $H = \mathbb{k}$ with trivial action and coaction, \mathcal{H}_V is the Heisenberg–Weyl algebra $S(V) \otimes S(V^*)$.

Algebras $\mathcal{B}(V, \Psi)$ were formally introduced by Andruskiewitsch and Schneider in [2] (as ‘Nichols algebras’ honouring an earlier work of Nichols [39]) and coincide with quantum exterior algebras of Woronowicz [51]. In the present form of two dually paired algebras, they appeared in the work of Majid [31]. Nichols–Woronowicz algebras are the same as “quantum shuffle algebras” of Rosso [41]. These algebras have already been linked to a number of different areas, such as pointed Hopf algebras [3] and noncommutative differential geometry [51,35,23], and have led to a useful generalisation of root systems due to Heckenberger [17]. (In Section 5, we use Nichols–Woronowicz algebras to give a new and simple counterexample to the aforementioned third conjecture of Kaplansky.) Our approach thus leads to a surprising appearance of Nichols–Woronowicz algebras in deformation theory; the braided coproduct on $\mathcal{B}(V, \Psi)$ is now recast as the product in the graded-dual algebra $\mathcal{B}(V^*, \Psi^*)$, and the braided Hopf algebra property is encoded in the commutation relation between $\mathcal{B}(V, \Psi)$ and $\mathcal{B}(V^*, \Psi^*)$ and the associativity of multiplication in the minimal double.

In Section 5 we also consider quasi-Yetter–Drinfeld modules V with structure given by compatible braidings (see 1.2). In the corresponding minimal double, the formula for the defining ideals is more involved and leads to a generalisation of Nichols–Woronowicz algebras associated to a set of compatible braidings (instead of just one braiding); but the degree 2 part of the formula is still quite manageable:

Theorem E. *Let $\delta_k: V \rightarrow H \otimes V$, $k = 1, 2, \dots, N$, be Yetter–Drinfeld coactions on an H -module V , which induce braidings Ψ_k on V . Let t_k be generic coefficients (e.g., formal parameters). Define the quasi-YD module structure on V by putting $\delta = \sum_k t_k \delta_k$. Then the defining ideals in the corresponding minimal quadratic double are*

$$I_{quad}(V) = \langle \bigcap_{k=1}^N \ker(\text{id} + \Psi_k) \rangle, \quad I_{quad}^*(V) = \langle \bigcap_{k=1}^N \ker(\text{id} + \Psi_k^*) \rangle.$$

1.6. Perfect subquotients

In Section 6, we justify our earlier claim that Yetter–Drinfeld modules are a “basic family” of solutions of the deformation problem set out in 1.1.

First of all, we define a special class of morphisms (called *subquotients*) between quasi-Yetter–Drinfeld modules V, W over a Hopf algebra H . Subquotients are diagrams $V \rightarrow W \rightarrow V$ where the arrows are H -module homomorphisms, and satisfy a certain condition of compatibility with quasi-Yetter–Drinfeld structures, δ_V on V and δ_W on W . We show that subquotients $V \rightarrow W \rightarrow V$ are the same as *triangular morphisms* between free braided doubles $\tilde{A}(V, \delta_V)$ and $\tilde{A}(W, \delta_W)$, which are the precisely the morphisms in the category \mathcal{D}_H of braided doubles over H .

One can observe that, if (W, δ_W) is a quasi-YD module for H and V is an H -module, then each pair $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ of H -module maps defines a unique quasi-Yetter–Drinfeld structure δ_V on V . In this situation we say that V is a subquotient of W via the maps μ, ν . Recall the left-side defining ideal $I(V, \delta_V) \subset T(V)$ of the minimal double associated to the quasi-Yetter–Drinfeld module V . We show (Proposition 6.5) that

$$I(V, \delta_V) \supseteq \mu^{-1}(I(W, \delta_W)).$$

If this inclusion is in fact an equality, leading to an embedding $U(V, \delta_V) \hookrightarrow U(W, \delta_W)$ of graded algebras, we say that the quasi-YD module (V, δ_V) is a *perfect subquotient* of (W, δ_W) . We prove

Theorem F. *Every quasi-YD module can be obtained as a perfect subquotient of a Yetter–Drinfeld module.*

See Theorem 6.9. Note that, given a finite-dimensional quasi-YD module V , one needs additional assumptions to guarantee that V can be realised as a subquotient of a *finite-dimensional* Yetter–Drinfeld module Y .

However, this is not yet the main problem. From the point of view of braided doubles, one would hope to find a perfect subquotient $V \xrightarrow{\mu} Y \xrightarrow{\nu} V$ such that $V^* \xrightarrow{\nu^*} Y^* \xrightarrow{\mu^*} V^*$ is also a perfect subquotient. This would yield an embedding

$$\tilde{A}(V, \delta_V) = U(V, \delta_V) \rtimes H \rtimes U(V^*, \delta_V) \hookrightarrow \mathcal{H}_Y$$

of a given minimal double into a braided Heisenberg double. But in general, we do not know what conditions (V, δ_V) should satisfy so that such two simultaneous perfect subquotients exist.

Nevertheless, we find and study a particular type of quasi-Yetter–Drinfeld modules (V, δ_V) such that $\hat{A}(V, \delta_V)$ embeds into a braided Heisenberg double. This happens for $\hat{A}(V, \delta_V)$ which are *rational Cherednik algebras*.

1.7. Rational Cherednik algebras

Let $G \leq GL(V)$ be a finite linear group over \mathbb{k} . A rational Cherednik algebra of G is a flat deformation of the semidirect product algebra $S(V \oplus V^*) \rtimes \mathbb{k}G$, obtained by replacing the right-hand side of the relation $[f, v] = 0$ ($f \in V^*, v \in V$) with some $\mathbb{k}G$ -valued pairing between V^* and V . Clearly, rational Cherednik algebras are braided doubles over $\mathbb{k}G$; as such, they become the subject of our inquiry in the last section of the paper.

The problem we pose in Section 7 is to classify braided doubles $A = U^- \rtimes \mathbb{k}G \rtimes U^+$, associated to the G -module V , such that U^\pm are commutative algebras. By Theorem B, we need to find quasi-Yetter–Drinfeld structures δ on V such that the algebras $U(V, \delta)$ and $U(V^*, \delta)$ are commutative. Such δ parametrise rational Cherednik algebras over the group G . Analysing quasi-braided factorials, we write down all such quasi-YD structures δ for an irreducible linear group G in terms of *complex reflections* (elements s such that $\text{rank}(s - 1) = 1$, otherwise called pseudoreflections) in G . A rational Cherednik algebra $H_{t,c}(G)$ has parameters $t \in \mathbb{k}$ and $c \in \mathbb{k}(\mathcal{S})^G$, where \mathcal{S} is the set of complex reflections.

Our method is independent of the characteristic of the ground field \mathbb{k} . Rational Cherednik algebras over $\mathbb{k} = \mathbb{C}$ are already known by the Etingof–Ginzburg classification [12], a new proof of which we obtain; $H_{t,c}(G)$ in positive characteristic are a relatively recent object of study (see [9,25]). In general, the *Poincaré–Birkhoff–Witt theorem* for $H_{t,c}(G)$ over a pseudoreflection group does not follow from $\mathbb{k} = \mathbb{C}$ case, and the Koszulity argument of [12] is not directly applicable. Irreducible finite pseudoreflection groups G were classified by Kantor, Wagner, Zalesskiĭ and Serežkin, see an exposition in [21]; the group algebra $\mathbb{k}G$ is, in general, not semisimple, and $H_{t,c}(G)$ may not have a \mathbb{Z} -form. The present paper gives a proof of the PBW theorem for rational Cherednik algebras in arbitrary characteristic.

We remark in passing that the representation theory of $H_{t,c}(G)$ in positive characteristic is clearly expected to differ from characteristic 0 in a number of ways, even in the non-modular case when $\mathbb{k}G$ is a semisimple algebra. For example, a family of $H_{t,c}(G)$ -modules which should be viewed as standard modules, may be finite-dimensional; in this case, there is no question of existence of finite-dimensional representations, but one is still interested in the values of parameters t, c for which the standard modules are reducible.

Going further, we apply the results on perfect subquotients obtained in Section 6 to see that all quasi-Yetter–Drinfeld structures δ on V for which the algebras $U(V, \delta)$ and $U(V^*, \delta)$ are commutative, come from perfect embeddings of V in a certain module $Y_{\mathcal{S}}(G)$ over the quantum group $D(G)$ (in our terminology, a Yetter–Drinfeld module over G). The “quantisation” $Y_{\mathcal{S}}(G)$ of V turns out to be trivial, $Y_{\mathcal{S}}(G) = V$, if G has no complex reflections; in general, $\dim Y_{\mathcal{S}}(G) = r + |\mathcal{S}|$. Using techniques developed in Section 6, in characteristic zero we obtain

Theorem G. *Let $G \leq GL(V)$ be a finite linear group over \mathbb{k} . For each value of the parameters t, c , the rational Cherednik algebra $H_{t,c}(G) \cong S(V) \rtimes \mathbb{k}G \rtimes S(V^*)$ embeds as a subdouble in the braided Heisenberg double $\mathcal{H}_{Y_{\mathcal{S}}(G)} \cong \mathcal{B}(Y_{\mathcal{S}}(G)) \rtimes \mathbb{k}G \rtimes \mathcal{B}(Y_{\mathcal{S}}(G)^*)$.*

We study the embedding given by the theorem for an irreducible complex reflection group G over \mathbb{C} . This leads to:

(1) An embedding of a restricted Cherednik algebra $\overline{H}_{0,c}(G)$ in a braided Heisenberg double \mathcal{H}_{Y_G} attached to a Yetter–Drinfeld module Y_G of dimension $|\mathcal{S}|$. This double decomposes as $\mathcal{B}(Y_G) \rtimes \mathbb{C}G \rtimes \mathcal{B}(Y_G^*)$. In particular, the coinvariant algebra S_G of G embeds in a very interesting Nichols–Woronowicz algebra $\mathcal{B}(Y_G)$. We thus recover, using the new method of braided doubles, the result of the first author [6] for Coxeter groups and an extension of this result to all complex reflection groups due to Kirillov and Maeno [24];

(2) An action of $\overline{H}_{0,c}(G)$ on $\mathcal{B}(Y_G)$. In Coxeter type A , the algebra $\mathcal{B}(Y_{\mathbb{S}_n})$, or its quadratic cover, coincides with the Fomin–Kirillov algebra \mathcal{E}_n from [13];

(3) An action of a rational Cherednik algebra $H_{t,c}(G)$ for $t \neq 0$ on the deformed Nichols–Woronowicz algebra $\mathcal{B}_\tau(Y_G)$; in type A , the algebra $\mathcal{B}_\tau(Y_G)$ turns out to be the universal enveloping algebra of a “*triangular Lie algebra*” introduced in [5].

We finish the paper with Appendix A which contains proofs to a number of auxiliary results on algebras with triangular decomposition.

2. Quasi-Yetter–Drinfeld modules

In this section we introduce quasi-Yetter–Drinfeld modules. Although the original motivation for these objects came from a flat deformation problem given in Section 1, we show that quasi-Yetter–Drinfeld modules arise naturally in the framework of monoidal categories. We discuss properties of quasi-Yetter–Drinfeld modules and ways to construct such modules.

2.1. Quasi-Yetter–Drinfeld modules and comodules

Recall that by a left quasicoaction of (any vector space) H on a vector space V we mean an arbitrary linear map $V \rightarrow H \otimes V$. We denote a quasicoaction by $v \mapsto v^{[-1]} \otimes v^{[0]}$.

Definition 2.1. Let H be a bialgebra over \mathbb{k} . A *quasi-Yetter–Drinfeld module* over H is a vector space V with

- (1) left H -action $\triangleright : H \otimes V \rightarrow V, h \otimes v \mapsto h \triangleright v$;
- (2) left H -quasicoaction $V \mapsto H \otimes V, v \mapsto v^{[-1]} \otimes v^{[0]}$,

which satisfy the *Yetter–Drinfeld compatibility condition*:

$$(h_{(1)} \triangleright v)^{[-1]} h_{(2)} \otimes (h_{(1)} \triangleright v)^{[0]} = h_{(1)} v^{[-1]} \otimes h_{(2)} \triangleright v^{[0]}.$$

We will often abbreviate quasi-Yetter–Drinfeld to quasi-YD.

Remark 2.2. A dual notion is that of a *quasi-Yetter–Drinfeld comodule* over H . It is a space V with an H -quasicoaction (linear map $H \otimes V \rightarrow V$) and an H -coaction, which satisfy the same Yetter–Drinfeld compatibility condition as in Definition 2.1. One can check that this compatibility condition is self-dual, and a quasi-YD module for H is a quasi-YD comodule for H^* when $\dim H < \infty$.

Yetter–Drinfeld modules, a notion widely used in modern quantum groups literature, are quasi-YD modules where the quasicoaction is in fact a coaction. Yetter–Drinfeld modules were

formally introduced by Yetter [52] under the name of “crossed bimodules” (a linearisation of crossed sets in algebraic topology, see e.g. Whitehead [50]), and were shown by Majid [29] to be the same as modules over the Drinfeld quantum double $D(H)$ of H if H is a finite-dimensional Hopf algebra.

2.2. *The map $\Psi_{V,W}$*

Let H be a bialgebra, V be an H -module with H -quasicoaction $v \mapsto v^{[-1]} \otimes v^{[0]}$, and W be an H -module. Consider the map

$$\Psi_{V,W} : V \otimes W \rightarrow W \otimes V, \quad \Psi_{V,W}(v \otimes w) = (v^{[-1]} \triangleright w) \otimes v^{[0]}.$$

The Yetter–Drinfeld compatibility condition for V can be recast in terms of the maps $\Psi_{V,W}$ where W runs over the category of H -modules:

Lemma 2.3. *An H -action and H -quasicoaction on a space V make V a quasi-Yetter–Drinfeld module, if and only if $\Psi_{V,W}$ is an H -module map for any H -module W .*

Proof. The condition that $\Psi_{V,W}$ is an H -equivariant map is

$$(h_{(1)} \triangleright v)^{[-1]} \triangleright (h_{(2)} \triangleright w) \otimes (h_{(1)} \triangleright v)^{[0]} = h_{(1)} \triangleright (v^{[-1]} \triangleright w) \otimes h_{(2)} \triangleright v^{[0]}.$$

This is just the Yetter–Drinfeld compatibility condition where the first tensor component, which is in H , is “evaluated” (via the action) on an arbitrary H -module W . \square

A categorical interpretation of the map Ψ will be given after a brief and informal reminder on

2.3. *\mathbb{k} -linear monoidal categories*

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal, or tensor, category. The monoidal product \otimes is associative, meaning that there are isomorphisms $\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ for all $X, Y, Z \in \mathcal{C}$, which are natural in X, Y, Z and satisfy Mac Lane’s pentagon condition [28, Ch. VII].

That \mathcal{C} is a \mathbb{k} -linear monoidal category ideologically means that all objects in \mathcal{C} are \mathbb{k} -vector spaces with additional structure. Formally, \mathcal{C} is a monoidal category equipped with a (forgetful) tensor functor $\mathcal{C} \rightarrow \text{Vect}_{\mathbb{k}}$ to the category of vector spaces over \mathbb{k} . For a more formal treatment, the reader is referred to [19, §8]. We may (and will) suppress the associativity isomorphisms $\Phi_{X,Y,Z}$ in all formulas, thus in fact assuming the category to be strict; see [43] for justification.

Functors between \mathbb{k} -linear monoidal categories are tensor functors which preserve the forgetful functor to $\text{Vect}_{\mathbb{k}}$. In other words, a functor does not change the underlying vector space of an object.

A *rigid* monoidal category is a category where any object V has a left dual object V^* and a right dual object *V . A left dual V^* comes with two maps, the evaluation $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V : V^* \otimes V \rightarrow \mathbb{k}$, and the coevaluation, $\text{coev} = \text{coev}_V : \mathbb{k} \rightarrow V \otimes V^*$ satisfying the axioms of the dual (as e.g. in [32, Definition 9.3.1]). Right duals are defined similarly. One may identify $({}^*V)^*$ and ${}^*(V^*)$ with V (but not V^{**} with V ; these objects may be non-isomorphic!). In a \mathbb{k} -linear rigid monoidal category, objects are finite-dimensional vector spaces. The left dual of an object V is an object with underlying vector space V^* , the linear dual of V . The coevaluation is necessarily

given by $\text{coev}_V(1) = v_a \otimes f^a$ where $\{f^a\}, \{v_a\}$ are dual bases of V^*, V with respect to the evaluation $\langle \cdot, \cdot \rangle$. (Summation over repeated indices is implied.)

Example 2.4. Let H be a bialgebra over \mathbb{k} . The category ${}_H\mathcal{M}$ of left modules over H is a \mathbb{k} -linear monoidal category. The left H -action on the tensor product $X \otimes Y$ of two modules X, Y is given by $h \triangleright (x \otimes y) = h_{(1)} \triangleright x \otimes h_{(2)} \triangleright y$. The trivial module \mathbb{k} , where H acts via $h \triangleright 1 = \epsilon(h)$, is the unit object.

If H is a Hopf algebra (with bijective antipode), the category ${}_H\mathcal{M}_{\text{f.d.}}$ of finite-dimensional H -modules is rigid. A left dual to a finite-dimensional H -module X is the dual vector space X^* with module structure given by the equation $\langle h \triangleright f, x \rangle = \langle f, Sh \triangleright x \rangle$, where $f \in X^*, x \in X$ and S is the antipode in H .

A similar example is the category ${}^H\mathcal{M}$ of left H -comodules.

Example 2.5. Let H be a bialgebra over \mathbb{k} . Define the category ${}_H\mathcal{QYD}$ as follows:

- objects of ${}_H\mathcal{QYD}$ = quasi-Yetter–Drinfeld modules over H ;
- morphisms between X and $Y = H$ -module maps $X \rightarrow Y$ (*compatibility with the quasicoaction is not required*);
- monoidal product = the structure of a quasi-YD module on $X \otimes Y$, given by
 - H -action $h \triangleright (x \otimes y) = h_{(1)} \triangleright x \otimes h_{(2)} \triangleright y$,
 - H -quasicoaction $x \otimes y \mapsto x^{[-1]}y^{[-1]} \otimes x^{[0]} \otimes y^{[0]}$.

We will refer to the latter two formulas by saying that the action and quasicoaction “respect the tensor product”.

Lemma 2.6. ${}_H\mathcal{QYD}$ is a \mathbb{k} -linear monoidal category.

Proof. Let X, Y be quasi-YD modules. We have to check that the above action and quasicoaction on $X \otimes Y$ are Yetter–Drinfeld compatible. By Lemma 2.3, it is enough to check that $\Psi_{X \otimes Y, Z} : X \otimes Y \otimes Z \rightarrow Z \otimes X \otimes Y$ is an H -module map, for an arbitrary H -module Z . Indeed,

$$\begin{aligned} \Psi_{X \otimes Y, Z}(x \otimes y \otimes z) &= x^{[-1]}y^{[-1]} \triangleright z \otimes x^{[0]} \otimes y^{[0]} \\ &= (\Psi_{X, Z} \otimes \text{id}_Y)(\text{id}_X \otimes \Psi_{Y, Z})(x \otimes y \otimes z) \end{aligned}$$

for $x \in X, y \in Y$ and $z \in Z$. Since $\Psi_{X, Z}$ and $\Psi_{Y, Z}$ are H -module morphisms, so is $\Psi_{X \otimes Y, Z}$. \square

Remark 2.7. It is convenient to think of an object in ${}_H\mathcal{QYD}$ as a pair (V, δ) , where V is an H -module and $\delta : V \rightarrow H \otimes V$ is a Yetter–Drinfeld quasicoaction on V . There is an obvious forgetful functor

$$F : {}_H\mathcal{QYD} \rightarrow {}_H\mathcal{M}, \quad F(V, \delta) = V,$$

which forgets the quasicoaction. This functor is an equivalence of monoidal categories. Indeed,

$$G : {}_H\mathcal{M} \rightarrow {}_H\mathcal{QYD}, \quad G(V) = (V, 0)$$

is a monoidal functor (the zero quasicoaction, $\delta(v) = 0$, is always Yetter–Drinfeld); $FG(V) = V$ and $GF(V, \delta) = (V, 0)$ which is naturally isomorphic to (V, δ) .

Hence the category ${}_H\mathcal{QYD}$ may be viewed as a “decorated module category”. The fibre, $F^{-1}(V)$, of the forgetful functor over an H -module V has the structure of a \mathbb{k} -vector space: if (V, δ_1) and (V, δ_2) are quasi-YD modules, then $(V, \delta_1 + \lambda\delta_2)$ is a quasi-YD module for any $\lambda \in \mathbb{k}$. This is because the Yetter–Drinfeld compatibility condition is linear in the quasicoaction.

However, we do not know the dimension of $F^{-1}(V)$ for a given H -module V , nor a sufficient condition that $\dim F^{-1}(V) > 0$.

Observe also that Yetter–Drinfeld H -coactions on V are a subset (not a subspace) in $F^{-1}(V)$; this subset does not necessarily span $F^{-1}(V)$.

Although the categories ${}_H\mathcal{QYD}$ and ${}_H\mathcal{M}$ are equivalent as monoidal categories, there is an important structure, intrinsic in ${}_H\mathcal{QYD}$, which is “forgotten” by the forgetful functor to ${}_H\mathcal{M}$. This structure is the semibraiding.

2.4. Semibraided monoidal categories

Definition 2.8. Let \mathcal{C} be a monoidal category. A *right semibraiding* on \mathcal{C} is a family $\Psi = \{\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X \mid X, Y \in \text{Ob } \mathcal{C}\}$ of morphisms, such that

1. (naturality in the right-hand argument) $\Psi_{X,Y}$ is natural in Y ;
2. (right hexagon condition) $\Psi_{X \otimes Y, Z} = (\Psi_{X,Z} \otimes \text{id}_Y)(\text{id}_X \otimes \Psi_{Y,Z})$.

Similarly, a *left semibraiding* is a family of morphisms $\Psi_{X,Y}$, natural in X and satisfying the “mirror” hexagon condition for $\Psi_{X,Y \otimes Z}$.

Remark 2.9. Naturality of $\Psi_{X,Y}$ in Y means that

$$(\phi \otimes \text{id}_X)\Psi_{X,Y} = \Psi_{X,Y'}(\text{id}_X \otimes \phi) \quad \text{for any morphism } \phi : Y \rightarrow Y'.$$

The (left and right) hexagon conditions, originally due to Mac Lane, are so named because if the associativity isomorphisms like $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ are explicitly shown, these conditions are given by hexagonal commutative diagrams. See [28, VII.7].

Remark 2.10 (Braiding). A *braiding* on a monoidal category is a collection of invertible morphisms $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ which is both right and left semibraiding. Braiding is a principal object in the theory of quantum groups, whereas semibrainings are a new notion introduced in the present paper.

We will use the term *right* (respectively *left*) *semibraided category* for a pair (\mathcal{C}, Ψ) , where \mathcal{C} is a monoidal category and Ψ is a right (respectively left) semibraiding on \mathcal{C} . Let us give an example of a semibraided category, which will turn out to be the canonical one.

For two quasi-Yetter–Drinfeld modules X, Y over a bialgebra H , define the map $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ as in 2.2. Denote by Ψ the collection of $\Psi_{X,Y}$ for all pairs $X, Y \in \text{Ob } {}_H\mathcal{QYD}$.

Lemma 2.11. $({}_H\mathcal{QYD}, \Psi)$ is a right semibraided monoidal category.

Proof. By Lemma 2.3, $\Psi_{X,Y}$ are morphisms in ${}^H\mathcal{QYD}$. Let us check the naturality in Y : for a morphism $\phi : Y \rightarrow Y'$, which is an H -module map,

$$(\phi \otimes \text{id}_X)\Psi_{X,Y}(x \otimes y) = \phi(x^{[-1]} \triangleright y) \otimes x^{[0]} = x^{[-1]} \triangleright \phi(y) \otimes x^{[0]} = \Psi_{X,Y'}(x \otimes \phi(y))$$

as required. Finally, the right hexagon condition for Ψ was explicitly checked in the proof of Lemma 2.6. \square

Remark 2.12. The same can be done for the category of quasi-Yetter–Drinfeld comodules. Let $X, Y \in \text{Ob}{}^H\mathcal{QYD}$. Denote the quasiaction of H on an object X in this category by $\blacktriangleright : H \otimes X \rightarrow X$. Let $\Psi = \{\Psi_{X,Y}\}$ where $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ is defined by the formula $\Psi_{X,Y}(x \otimes y) = x^{(-1)} \blacktriangleright y \otimes x^{(0)}$. Then $({}^H\mathcal{QYD}, \Psi)$ is a left semibraided category.

2.5. Reconstruction theorems for semibraided monoidal categories

Our next goal is to prove a “converse” of Lemma 2.11. That is, a \mathbb{k} -linear right semibraided category (\mathcal{C}, Ψ) should be realised as a semibraided subcategory of ${}^H\mathcal{QYD}$ for some bialgebra $H = H(\mathcal{C}, \Psi)$. The process of obtaining $H(\mathcal{C}, \Psi)$ from the category (\mathcal{C}, Ψ) is called reconstruction. Ideally, starting with the category ${}^H\mathcal{QYD}$, the reconstruction should yield the original bialgebra H .

Likewise, left semibraided monoidal categories are expected to be realised as subcategories of ${}^H\mathcal{QYD}$.

Known reconstruction theorems include the realisation of a \mathbb{k} -linear monoidal category \mathcal{C} , under certain finiteness assumptions, in terms of either modules or comodules over a bialgebra H (a Hopf algebra if \mathcal{C} is rigid). If \mathcal{C} is a braided category, H will be a (co)quasitriangular bialgebra. See the survey [19]; original sources include [49,52,30].

We will only state and prove a reconstruction theorem for a left semibraided category and quasi-YD comodules. “Finiteness assumptions” are easier to state for a comodule realisation. Besides that, comodules behave in a more algebraic way compared to modules. We will bear in mind that a module version also holds (an interested reader can recover it, using [32, 9.4.1] as a guide). The following finiteness assumptions are sufficient for the comodule reconstruction, and are satisfied in all our applications of the reconstruction theorem:

- monoidal categories are strict and small;
- objects in \mathbb{k} -linear monoidal categories are finite-dimensional linear spaces over \mathbb{k} .

In what follows, the symbol \blacktriangleright will denote quasiaction.

Definition 2.13. We say that a \mathbb{k} -linear left semibraided category (\mathcal{C}, Ψ) is *realised* over a bialgebra H , if there is a monoidal functor $\mathcal{C} \rightarrow {}^H\mathcal{QYD}$ which preserves the left semibraiding.

In other words, H coacts and Yetter–Drinfeld compatibly quasiacts on (the underlying vector space of) each object of \mathcal{C} , so that morphisms in \mathcal{C} commute with the coaction, the coaction and the quasiaction respect the tensor product, and

$$\Psi_{X,Y}(x \otimes y) = (x^{(-1)} \blacktriangleright y) \otimes x^{(0)}$$

for $X, Y \in \text{Ob}\mathcal{C}$.

The following lemma is easy; note that the product on H_δ arises from tensor multiplication in \mathcal{C} :

Lemma 2.14. *Suppose that a \mathbb{k} -linear left semibraided category (\mathcal{C}, Ψ) is realised over a bialgebra H . For $X \in \text{Ob } \mathcal{C}$, let $\delta_X : X \rightarrow H \otimes X$ be the coaction of H on X .*

- (a) *Denote by H_δ the minimal subspace of H such that $\delta_X(X) \subset H_\delta \otimes X$ for all $X \in \text{Ob } \mathcal{C}$. Then H_δ is a subbialgebra of H .*
- (b) *Let $I_\blacktriangleright \subset H_\delta$ be the largest biideal of H_δ which quasiacts by zero on all objects in \mathcal{C} . Then (\mathcal{C}, Ψ) is realised over the quotient bialgebra $H_\delta/I_\blacktriangleright$.*

We will call the bialgebra $H_\delta/I_\blacktriangleright$ the *minimal subquotient of H realising (\mathcal{C}, Ψ)* . Among the bialgebras which realise (\mathcal{C}, Ψ) , we will be reconstructing the bialgebra which is the smallest possible. We now state our

Theorem 2.15 (Reconstruction theorem for semibraiding). *Let (\mathcal{C}, Ψ) be a \mathbb{k} -linear left semibraided monoidal category, satisfying finiteness assumptions. There exists a bialgebra $H(\mathcal{C}, \Psi)$ such that:*

- 1. (\mathcal{C}, Ψ) is realised over $H(\mathcal{C}, \Psi)$;
- 2. (minimality) *If (\mathcal{C}, Ψ) is realised over another bialgebra H' , then the minimal subquotient of H' , realising (\mathcal{C}, Ψ) , is isomorphic to $H(\mathcal{C}, \Psi)$.*

Proof. The first step is to apply to the category \mathcal{C} (ignoring the semibraiding) the standard comodule reconstruction, see [19,49,52,30]. According to this procedure, there exists a universal bialgebra $H_{\mathcal{C}}$, which coacts on all objects of \mathcal{C} , such that the coaction respects the tensor product, and morphisms in \mathcal{C} are $H_{\mathcal{C}}$ -comodule morphisms. Universality means that for any other bialgebra H' with these properties, there is a unique map $p : H_{\mathcal{C}} \rightarrow H'$ such that the coaction of H' on all $X \in \text{Ob } \mathcal{C}$ factors through the coaction of $H_{\mathcal{C}}$: $X \rightarrow H_{\mathcal{C}} \otimes X \xrightarrow{p \otimes \text{id}} H' \otimes X$.

We will use the following description of $H_{\mathcal{C}}$, which can be found in the sources cited above.

The bialgebra $H_{\mathcal{C}}$ is spanned by comatrix elements $h_{x,\xi} = x^{(-1)}\langle x^{(0)}, \xi \rangle$, $x \in X \in \text{Ob } \mathcal{C}$, $\xi \in X^\vee$. Here X^\vee denotes the right dual vector space to X (note that X^\vee is not an object in \mathcal{C}), and $\langle x, \xi \rangle \in \mathbb{k}$ is the pairing of $x \in X$ and $\xi \in X^\vee$. The product in $H_{\mathcal{C}}$ is given by $h_{x,\xi}h_{y,\eta} = h_{x \otimes y, \eta \otimes \xi}$. Observe that the dual to the space $X \otimes Y$ is $Y^\vee \otimes X^\vee$. The unit in $H_{\mathcal{C}}$ is $h_{1,1}$ where $1 \in \mathbb{k} =$ the trivial object of \mathcal{C} . As $h_{x,\xi}$ are comatrix elements, the coproduct of $h_{x,\xi}$ is $h_{x,\xi_a} \otimes h_{x^a,\xi}$. Here $\{x^a\}, \{\xi_a\}$ is any pair of dual bases of X, X^\vee ; summation over the repeated index is implied. The counit is $\epsilon(h_{x,\xi}) = \langle x, \xi \rangle$.

The full set of relations between the comatrix elements in $H_{\mathcal{C}}$ is spanned by the obstructions for morphisms in \mathcal{C} to become comodule morphisms:

$$\text{Obstr}(\mathcal{C}) = \text{span}\{h_{\phi(x),\eta} - h_{x,\phi^\vee(\eta)} \mid \phi \in \text{Mor}(\mathcal{C}), x \in \text{source}(\phi), \eta \in \text{target}(\phi)^\vee\}.$$

Here ϕ^\vee is the linear map which is adjoint to ϕ .

This description of the bialgebra $H_{\mathcal{C}}$ is explicit enough to enable us to introduce a quasiaction of $H_{\mathcal{C}}$ on objects in \mathcal{C} , which realises the semibraiding Ψ . In fact, it is clear that there is only one choice for such quasiaction: for $x \in X, \xi \in X^\vee, y \in Y$ where X, Y are objects in \mathcal{C} , put

$$h_{x,\xi} \blacktriangleright y = (\text{id}_Y \otimes \langle \cdot, \xi \rangle) \Psi_{X,Y}(x \otimes y).$$

Let us check that the quasiaction is well defined. We need to make sure that elements of the space $\text{Obstr}(\mathcal{C})$ quasiact by zero on objects in \mathcal{C} . Indeed, let $\phi : X \rightarrow Y$ be a morphism in \mathcal{C} , $x \in X$, $\eta \in Y^\vee$. The condition that $h_{\phi(x),\eta} - h_{x,\phi^\vee(\eta)}$ quasiacts on $z \in Z$ by zero is precisely equivalent to

$$\Psi_{Y,Z}(\phi(x) \otimes z) - (\text{id}_Z \otimes \phi) \Psi_{X,Z}(x \otimes z) = 0,$$

which is the functoriality of Ψ in the left-hand argument. One easily checks that the quasiaction \blacktriangleright indeed realises Ψ : i.e., $h_{x,\xi_a} \blacktriangleright y \otimes x^a = \Psi_{X,Y}(x \otimes y)$. It follows from the left hexagon axiom for Ψ that the quasiaction respects the tensor product in \mathcal{C} .

Let us prove that the quasiaction of $H_{\mathcal{C}}$ is Yetter–Drinfeld compatible with the coaction. The condition that $\Psi_{X,Y}(x \otimes y) = x^{(-1)} \blacktriangleright y \otimes x^{(0)}$ is a morphism of comodules between $X \otimes Y$ and $Y \otimes X$ reads

$$(x^{(-2)} \blacktriangleright y)^{(-1)} x^{(-1)} \otimes (x^{(-2)} \blacktriangleright y)^{(0)} \otimes x^{(0)} = x^{(-2)} y^{(-1)} \otimes x^{(-1)} \blacktriangleright y^{(0)} \otimes x^{(0)}.$$

Evaluating the rightmost tensor factor on both sides on $\xi \in X^\vee$ and putting $h = h_{x,\xi}$, we obtain

$$(h_{(1)} \blacktriangleright y)^{(-1)} h_{(2)} \otimes (h_{(1)} \blacktriangleright y)^{(0)} = h_{(1)} y^{(-1)} \otimes h_{(2)} \blacktriangleright y^{(0)},$$

which is the required Yetter–Drinfeld compatibility condition; we reiterate that comatrix elements $h_{x,\xi}$ span $H_{\mathcal{C}}$.

The algebra $H_{\mathcal{C}}$ realises the semibraiding, but it may not be minimal. Let $I_{\blacktriangleright}(H_{\mathcal{C}})$ be the largest among (= the sum of all) biideals in $H_{\mathcal{C}}$ which quasiact on the whole category \mathcal{C} by zero. Define

$$H(\mathcal{C}, \Psi) = H_{\mathcal{C}} / I_{\blacktriangleright}(H_{\mathcal{C}}).$$

Let us show that $H(\mathcal{C}, \Psi)$ satisfies the minimality property required in the theorem. Suppose that H is another bialgebra which realises the semibraided category (\mathcal{C}, Ψ) . By the universality of $H_{\mathcal{C}}$, there is a map $p : H_{\mathcal{C}} \rightarrow H$ of bialgebras given by $p(h_{x,\xi}) = x^{(-1)} \langle x^{(0)}, \xi \rangle$. Clearly, the image of the map p is the subbialgebra of H denoted by H_δ in Lemma 2.14. Note that, since the quasiaction of H realises Ψ , the map p must commute with quasiaction:

$$p(h_{x,\xi}) \blacktriangleright y = x^{[-1]} \blacktriangleright y \langle x^{[0]}, \xi \rangle = (\text{id}_Y \otimes \langle \cdot, \xi \rangle) \Psi_{X,Y}(x \otimes y) = h_{x,\xi} \blacktriangleright y.$$

Therefore, if I_{\blacktriangleright} is the largest biideal in H_δ which quasiacts on \mathcal{C} by zero, $p^{-1}(I_{\blacktriangleright}) = I_{\blacktriangleright}(H_{\mathcal{C}})$ and the map p induces isomorphism $H(\mathcal{C}, \Psi) = H_{\mathcal{C}} / I_{\blacktriangleright}(H_{\mathcal{C}}) \xrightarrow{\sim} H_\delta / I_{\blacktriangleright}$ of bialgebras. \square

Remark 2.16. Strictly speaking, the finiteness conditions we specified do not allow us to apply reconstruction to the category ${}^H \mathcal{QYD}$. It is not clear whether we can always reconstruct H from the category $\mathcal{C} = {}^H \mathcal{QYD}_{\text{f.d.}}$ of finite-dimensional quasi-YD comodules. No doubt that $H = H_{\mathcal{C}}$; the problem is to show that there is no biideal in H which quasiacts by zero on all finite-dimensional quasi-YD comodules.

However, if $\dim H < \infty$, then $H({}^H \mathcal{QYD}_{\text{f.d.}}, \Psi) = H$.

2.6. Rigid semibraided categories and reconstruction of Hopf algebras

In the present paper, we mainly use quasi-Yetter–Drinfeld modules over Hopf algebras (not general bialgebras). In terms of reconstruction, Hopf algebras correspond to rigid monoidal categories. The following lemma (a Hopf algebra version of Lemma 2.11) is straightforward:

Lemma 2.17. *When H is a Hopf algebra with invertible antipode S , the category ${}_H\mathcal{QYD}_{\text{f.d.}}$ of finite-dimensional quasi-YD modules is a rigid right-semibraided category. The left dual V^* of an object V is the dual linear space of V , equipped with action and quasicoaction of H given by*

$$\langle h \triangleright f, v \rangle = \langle f, Sh \triangleright v \rangle, \quad f^{[-1]}(f^{[0]}, v) = S^{-1}v^{[-1]}(f, v^{[0]})$$

for all $f \in V^*$, $v \in V$, $h \in H$.

Claim 2.18. *There is a Hopf algebra version of Lemma 2.14 and Theorem 2.15, where*

- “monoidal category” is replaced with “rigid monoidal category”;
- “bialgebra” is replaced with “Hopf algebra”, “subbialgebra” with “sub-Hopf algebra” and “biideal” with “Hopf ideal”.

This follows from the fact that when \mathcal{C} is a rigid category, $H_{\mathcal{C}}$ has well-defined invertible antipode S given on comatrix elements by

$$Sh_{f,x} = h_{x,f^\vee}.$$

Here the linear isomorphism $\cdot^\vee : X^* \rightarrow X^\vee$, where X is an object in \mathcal{C} , is given by $\langle f, x \rangle = \langle x, f^\vee \rangle$. It is not a morphism in the category \mathcal{C} . One checks that S preserves the biideal of definition of $H(\mathcal{C}, \Psi)$.

Remark 2.19. If a monoidal category \mathcal{C} is a subcategory of a rigid monoidal category $\bar{\mathcal{C}}$, such that $\bar{\mathcal{C}}$ is a “rigid envelope” of \mathcal{C} in an appropriate sense, then the bialgebra $H_{\mathcal{C}}$ embeds injectively in the Hopf algebra $H_{\bar{\mathcal{C}}}$; moreover, $H_{\bar{\mathcal{C}}}$ will be the “Hopf envelope” of $H_{\mathcal{C}}$. Under some assumptions, one can construct a rigid envelope of a monoidal category. This is a categorical version of Manin’s construction of a Hopf envelope of a quadratic bialgebra in [36, Chapter 7].

2.7. Compatible braidings on a vector space

We are interested to have a supply of quasi-Yetter–Drinfeld modules over bialgebras, or, better, Hopf algebras. The idea of the reconstruction theory is that such modules naturally occur as objects of (rigid) right semibraided categories. We would thus like to have a way of constructing semibraided categories.

It is straightforward that a *braided category* is a particular case of a semibraided category. Braided category is a monoidal category with a braiding (recall Remark 2.10). Braided categories were formally introduced in [20]; see also the exposition in [32, Ch. 9]. However, braided categories are a source only of Yetter–Drinfeld modules; this class of modules is not rich enough for our purposes. We therefore need a more sophisticated example of a semibraided category. Recall the following

Definition 2.20. A braiding on a vector space V is a linear map $\Psi : V \otimes V \rightarrow V \otimes V$, satisfying the braid equation

$$(\text{id} \otimes \Psi)(\Psi \otimes \text{id})(\text{id} \otimes \Psi) = (\Psi \otimes \text{id})(\text{id} \otimes \Psi)(\Psi \otimes \text{id}).$$

Here is a new notion:

Definition 2.21. A finite set Π of braidings on a vector space V is compatible, if for all $\Psi, \Psi' \in \Pi$

$$(\Psi' \otimes \text{id})(\text{id} \otimes \Psi)(\Psi \otimes \text{id}) = (\text{id} \otimes \Psi)(\Psi \otimes \text{id})(\text{id} \otimes \Psi').$$

This is a right-handed version of compatibility; there is a left-handed version where the order of factors on both sides is opposite.

We will now show how to construct a right semibraided category from a set of compatible braidings on a vector space V . This will give a structure of a quasi-Yetter–Drinfeld module on V .

2.8. A construction of a semibraided category from a set of compatible braidings

Let Π be a finite set of compatible braidings on a space V . Let \mathcal{C}_Π be a monoidal category, whose objects are $V^{\otimes n}$, $n \geq 0$. The space $\text{Mor}(V^{\otimes m}, V^{\otimes n})$ of morphisms will be:

- \mathbb{k} , if $m = n = 1$;
- the subalgebra of $\text{End}(V^{\otimes n})$ generated by $\Psi_{i,i+1}$ (in leg notation), for all $\Psi \in \Pi$ and $1 \leq i \leq n - 1$, if $m = n > 1$;
- 0 , if $m \neq n$.

For any $n \geq 1$ and $\Psi \in \Pi$, define

$$\Psi^{1,n} : V \otimes V^{\otimes n} \rightarrow V^{\otimes n} \otimes V, \quad \Psi^{1,n} = \Psi_{n,n+1} \Psi_{n-1,n} \dots \Psi_{23} \Psi_{12}$$

(we introduce $\Psi^{1,n}$ as an endomorphism of $V^{\otimes n+1}$ and use the leg notation). In fact, $\Psi^{1,n}$ is obtained from Ψ using the “left hexagon” rule. Let

$$\Psi_\Pi^{1,n} : V \otimes V^{\otimes n} \rightarrow V^{\otimes n} \otimes V, \quad \Psi_\Pi^{1,n} = \sum_{\Psi \in \Pi} \Psi^{1,n}.$$

Now extend Ψ_Π to a map exchanging any two objects in the category:

$$\begin{aligned} \Psi_\Pi^{m,n} &: V^{\otimes m} \otimes V^{\otimes n} \rightarrow V^{\otimes n} \otimes V^{\otimes m}, \\ \Psi_\Pi^{m,n} &= (\Psi_\Pi^{1,n})_{1\dots n+1} (\Psi_\Pi^{1,n})_{2\dots n+2} \dots (\Psi_\Pi^{1,n})_{m\dots m+n}, \end{aligned}$$

that is, using the “right hexagon” rule.

Lemma 2.22. The maps $\Psi_\Pi^{m,n}$ are a right semibraiding on the category \mathcal{C}_Π .

Proof. By construction, $\Psi_{\Pi}^{m,n}$ is a morphism in \mathcal{C}_{Π} and satisfies the right hexagon rule. We need to check that $\Psi_{\Pi}^{m,n}$ is natural in its second argument. Because of the right hexagon rule, it is enough to check the naturality of $\Psi_{\Pi}^{1,n}$ in its second argument. This will follow from the naturality of $\Psi^{1,n}$ in the second argument, for any $\Psi \in \Pi$.

Let $\phi : V^{\otimes n} \rightarrow V^{\otimes n}$ be a morphism. We have to check that $\Psi^{1,n}(\text{id}_V \otimes \phi) = (\phi \otimes \text{id}_V)\Psi^{1,n}$. We may assume that $\phi = \Psi'_{i,i+1}$ for some $\Psi' \in \Pi$ and i between 1 and $n - 1$. Then the naturality equation is the same as the compatibility condition for Ψ and Ψ' . \square

Remark 2.23. Let Π be a finite compatible set of braidings on a vector space V . Denote by H_{Π} the bialgebra reconstructed, using the module version of Theorem 2.15, from the category \mathcal{C}_{Π} . Then the space V becomes a quasi-Yetter–Drinfeld module for H_{Π} .

One can show that \mathcal{C}_{Π} may be embedded in a rigid category, if and only if all braidings $\Psi \in \Pi$ are rigid (or *biinvertible*, this notion is explained in [32, 4.2]). Such a rigid category will be generated, as a monoidal category, by objects

$$\dots, \quad V^{[-2]} = **V, \quad V^{[-1]} = *V, \quad V^{[0]} = V, \quad V^{[1]} = V^*, \quad V^{[2]} = V^{**}, \quad \dots$$

such that $(V^{[m]})^* = V^{[m+1]}$ and $*(V^{[m]}) = (V^{[m-1]})$. Standard formulas show how to compute the braiding between, say, V and V^* , and this extends recursively to braidings between $V^{[m]}$ and $V^{[n]}$.

In this case, we obtain a Hopf algebra H_{Π} .

Remark 2.24. Suppose that a Hopf algebra H acts on a space V , and there are coactions, $v \mapsto v^{(-1)_i} \otimes v^{(0)_i}$, $i = 1, \dots, N$, of H on V , each satisfying the Yetter–Drinfeld condition. Then the braidings $\Psi_i(v \otimes w) = v^{(-1)_i} \triangleright w \otimes v^{(0)_i}$ are compatible. This can be checked directly. The Hopf algebra $H_{\{\Psi_1, \dots, \Psi_N\}}$ will be the minimal among subquotients of H which still act on V and coact (in N ways) on V . The quasicoaction of $H_{\{\Psi_1, \dots, \Psi_N\}}$ will be given by $v \mapsto \sum_i v^{(-1)_i} \otimes v^{(0)_i}$.

2.9. Minimal Yetter–Drinfeld realisation of a braided space

A particular case of a set of compatible braidings on V is a one-element set $\Pi = \{\Psi\}$. Assume that the braiding with such a Ψ is biinvertible. The above procedure yields a minimal Hopf algebra (denote it by H_{Ψ}) over which the braided space (V, Ψ) is realised as a Yetter–Drinfeld module.

That a braided space (V, Ψ) can be realised as a module over a coquasitriangular bialgebra (hence a Yetter–Drinfeld module), follows from the Faddeev–Reshetikhin–Takhtajan construction [40]; the latter admits a Hopf algebra version, e.g. [42,48]. The Hopf algebra H_{Ψ} which we propose to reconstruct, is not the one given by the FRT construction but rather its quotient by the left kernel of the coquasitriangular structure. Because H_{Ψ} has more relations than the FRT Hopf algebra, it looks more interesting algebraically.

Some properties of the Hopf algebra H_{Ψ} are listed as Claims 2.25–2.31 below. “Braided space” will mean a finite-dimensional space over \mathbb{k} with biinvertible braiding.

Claim 2.25. *The Hopf algebra H_{Ψ} is trivial ($H_{\Psi} \cong \mathbb{k}$) if and only if the braiding Ψ is trivial ($\Psi(x \otimes y) = y \otimes x$ for all x, y).*

Claim 2.26. *Any finite-dimensional Hopf algebra H is isomorphic to a structural Hopf algebra of some braided space (V, Ψ) . (For example, take V to be the Drinfeld double $D(H)$ of H , with*

a standard braiding. Braided spaces of dimension smaller than $\dim D(H) = (\dim H)^2$ may still give rise to the same Hopf algebra H .)

Claim 2.27. *Suppose Ψ is a braiding on a space V . Then Ψ^* is a braiding on V^* , and H_{Ψ^*} is isomorphic to H_{Ψ} . (This is because the rigid braided categories, generated by (V, Ψ) and by (V^*, Ψ^*) , are the same.)*

Claim 2.28. *If $\Psi(x \otimes y) = qy \otimes x$ for a constant $0 \neq q \in \mathbb{k}$, the Hopf algebra H_{Ψ} is isomorphic to the group algebra of $\mathbb{Z}/n\mathbb{Z}$, if q is a root of unity of order n in \mathbb{k} , or of \mathbb{Z} if q is not a root of unity. (It is easy to realise Ψ over this Hopf algebra and to show that Ψ cannot be realised over its proper subquotient.)*

Claim 2.29. *H_{Ψ} and $H_{\Psi^{-1}}$ are non-degenerately dually paired Hopf algebras. If H_{Ψ} is finite-dimensional, so is $H_{\Psi^{-1}}$, and $H_{\Psi^{-1}} = (H_{\Psi})^*$. (This follows by analysing the coquasitriangular structure on the FRT Hopf algebra of (V, Ψ) .)*

This duality pairing yields an elegant proof of the following fact:

Lemma 2.30. *Let the field \mathbb{k} be algebraically closed. The group algebra $\mathbb{k}G$ of a finitely generated Abelian group G is a self-dual Hopf algebra.*

Proof. It is enough to prove this for a group $G = \mathbb{Z}/n\mathbb{Z}$ where n is either zero or a positive integer. Let q be a root of unity of order n (or not a root of unity, if $n = 0$). Let $V = \mathbb{k}x$ and $\Psi(x \otimes x) = qx \otimes x$ be the braiding on V . Then by Claim 2.28 both H_{Ψ} and $H_{\Psi^{-1}}$ are isomorphic to the group algebra $\mathbb{k}G$. \square

Claim 2.31. *H_{Ψ} is cocommutative, if and only if the braiding Ψ is compatible with the trivial braiding τ . Dualising, H_{Ψ} is commutative if and only if Ψ^{-1} is compatible with τ .*

It seems to be a challenging problem to extract other properties of the Hopf algebra H_{Ψ} from the properties of the operator Ψ . For example, when is H_{Ψ} finite-dimensional? Semisimple? Is a group algebra? Here is a converse problem, which may also be of interest: given a finite-dimensional Hopf algebra H , find a braided space (V, Ψ) of smallest dimension, such that $H_{\Psi} \cong H$.

2.10. *Quasi-Yetter–Drinfeld modules over cocommutative or quasitriangular H*

We conclude this section with a simple but useful observation which allows us to obtain compatible braidings and to construct quasi-Yetter–Drinfeld modules from given Yetter–Drinfeld modules.

Lemma 2.32. *Let V be a Yetter–Drinfeld module over a cocommutative Hopf algebra H , with coaction $\delta(v) = v^{(-1)} \otimes v^{(0)}$. Then the induced braiding Ψ on V is compatible with the trivial braiding $\tau(v \otimes w) = w \otimes v$. For any $\lambda \in \mathbb{k}$*

$$\delta_{\Psi, \lambda\tau}(v) = v^{(-1)} \otimes v^{(0)} + \lambda \cdot 1 \otimes v$$

defines a Yetter–Drinfeld quasicoaction on V .

Proof. Any module V over a cocommutative Hopf algebra can be turned into a Yetter–Drinfeld module with the trivial coaction $v \mapsto 1 \otimes v$. Indeed, let us check that the trivial coaction is Yetter–Drinfeld compatible with any action:

$$1 \cdot h_{(2)} \otimes h_{(1)} \triangleright v = h_{(1)} \cdot 1 \otimes h_{(2)} \triangleright v,$$

which is true by cocommutativity.

Thus, the braidings Ψ and τ are realised on V via the same action and two Yetter–Drinfeld coactions of a Hopf algebra, therefore by Remark 2.24 they are compatible. The quasicoaction $\delta_{\Psi, \lambda \tau}$ is Yetter–Drinfeld as a linear combination of Yetter–Drinfeld coactions. \square

Remark 2.33. More generally, let (V, Ψ) be a Yetter–Drinfeld module over a quasitriangular Hopf algebra H . The braiding on V , induced by the quasitriangular structure $R = R^1 \otimes R^2 \in H \otimes H$, is compatible with Ψ . There is a Yetter–Drinfeld quasicoaction on V , given by

$$\delta_{\Psi, \lambda R}(v) = v^{(-1)} \otimes v^{(0)} + \lambda R^2 \otimes R^1 \triangleright x$$

for any $\lambda \in \mathbb{k}$.

3. Free braided doubles

We will now study algebras with triangular decomposition of the form $T(V) \rtimes H \ltimes T(V^*)$, where the commutator of V^* and V lies in the bialgebra H . Such algebras are called free braided doubles. The purpose of this section is to show that free braided doubles are “the same” as quasi-Yetter–Drinfeld modules over H .

3.1. Algebras \tilde{A}_β

Let H be a bialgebra, and let V be a finite-dimensional space with left H -action \triangleright . As usual, V^* denotes the linear dual of V , $\langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow \mathbb{k}$ is the canonical pairing, and V^* is viewed as a right H -module via $\langle f \triangleleft h, v \rangle = \langle f, h \triangleright v \rangle$.

Definition 3.1. To any linear map $\beta : V^* \otimes V \rightarrow H$ there corresponds an associative algebra \tilde{A}_β , generated by all $v \in V$, $h \in H$ and $f \in V^*$, subject to:

- (i) semidirect product relations $h \cdot v = (h_{(1)} \triangleright v)h_{(2)}$, $f \cdot h = h_{(1)}(f \triangleleft h_{(2)})$;
- (ii) commutator relation $f \cdot v - v \cdot f = \beta(f, v)$.

In this definition, we assume that all relations between $h \in H$ hold in \tilde{A}_β , and also that the unity in H is the unity in \tilde{A}_β : $1_{\tilde{A}_\beta} = 1_H$.

The map

$$m_\beta : T(V) \otimes H \otimes T(V^*) \rightarrow \tilde{A}_\beta,$$

of vector spaces, which is induced by the multiplication in \tilde{A}_β , is surjective. This is because any monomial in generators of \tilde{A}_β may be rewritten, using the relations, as a linear combination of monomials of the form $v_1 v_2 \dots v_m \cdot h \cdot f_1 f_2 \dots f_n$, where $v_i \in V$, $h \in H$ and $f_j \in V^*$.

If $\beta = 0$, so that the generators $f \in V^*$ commute with the generators $v \in V$, the map m_0 is an isomorphism of vector spaces:

$$m_0 : T(V) \otimes H \otimes T(V^*) \cong \tilde{A}_0.$$

Indeed, it is easy to check that multiplication on $T(V) \otimes H \otimes T(V^*)$, defined by the rule $(\eta \otimes g \otimes a)(\theta \otimes h \otimes b) = \eta(g_{(1)} \triangleright \theta) \otimes g_{(2)}h_{(1)} \otimes (a \triangleleft h_{(2)})b$, is associative, and it clearly obeys the semidirect product and commutator relations in \tilde{A}_0 . Note that the subalgebra of \tilde{A}_0 , generated by V and H , is isomorphic to the semidirect product $T(V) \rtimes H$ by the left action of H . Similarly, H and V^* generate a subalgebra isomorphic to the semidirect product $H \rtimes T(V^*)$. The algebra \tilde{A}_0 is obtained by “gluing” these two semidirect products together along H .

Definition 3.2. We say that the algebra \tilde{A}_β has *triangular decomposition* over the bialgebra H , if the map m_β is a vector space isomorphism.

Our key question in this section is, which $\beta : V^* \otimes V \rightarrow H$ have this property. A complete answer to this question is given in

Theorem 3.3. *The algebra \tilde{A}_β has triangular decomposition*

$$\tilde{A}_\beta \cong T(V) \otimes H \otimes T(V^*)$$

over the bialgebra H , if and only if the H -valued pairing $\beta : V^ \otimes V \rightarrow H$ satisfies the Yetter–Drinfeld condition:*

$$h_{(1)}\beta(f \triangleleft h_{(2)}, v) = \beta(f, h_{(1)} \triangleright v)h_{(2)}$$

for all $v \in V, h \in H, f \in V^*$.

We postpone the proof of Theorem 3.3 until the end of this section, and will now discuss the result itself.

The next lemma (which follows by easy linear algebra) clarifies why the equation for β in the theorem is termed the Yetter–Drinfeld condition:

Lemma 3.4. *Let V be a finite-dimensional module over a bialgebra H .*

1. *Linear maps $\beta : V^* \otimes V \rightarrow H$ are in one-to-one correspondence with quasicoactions (linear maps) $\delta : V \rightarrow H \otimes V$, via the formula*

$$\delta_\beta(v) = \beta(f^a, v) \otimes v_a,$$

where $\{f^a\}, \{v_a\}$ are dual bases of V^*, V .

2. *A map $\beta : V^* \otimes V \rightarrow H$ satisfies the equation in Theorem 3.3, if and only if the quasicoaction δ_β is Yetter–Drinfeld compatible with the H -action on V .*

Definition 3.5. An algebra \tilde{A}_β , satisfying the conditions of Theorem 3.3, is called a *free braided double*.

We may now restate Theorem 3.3 in the following way:

Corollary 3.6. *Free braided doubles over a bialgebra H are parametrised by finite-dimensional quasi-Yetter–Drinfeld modules over H .*

The parametrisation is as follows. Let (V, δ) be a finite-dimensional quasi-YD module over H . According to Definition 2.1, this means that V is an H -module and $\delta(v) = v^{[-1]} \otimes v^{[0]}$ is an H -quasicoaction on V satisfying the Yetter–Drinfeld compatibility condition. To (V, δ) is associated the free braided double

$$\tilde{A}(V, \delta) := T(V) \rtimes H \rtimes T(V^*) \quad \text{with defining relation } [f, v] = v^{[-1]} \{f, v^{[0]}\}.$$

Square brackets mean a commutator $fv - vf$.

Vice versa, a free braided double of the form $T(V) \rtimes H \rtimes T(V^*)$ where V is a finite-dimensional H -module, gives rise to a Yetter–Drinfeld quasicoaction on V given by $v \mapsto [f^a, v] \otimes v_a$. Here $\{f^a\}, \{v_a\}$ are dual bases of V^*, V .

3.2. Classification of one-dimensional quasi-Yetter–Drinfeld modules

We know from Section 2 that the universal source of quasi-Yetter–Drinfeld modules are right semibraided monoidal categories. This means that in general, quasi-YD modules are at least as complicated as solutions to the quantum Yang–Baxter equation.

However, one-dimensional quasi-YD modules over a Hopf algebra can be fully classified. We will do this here. A practical way to obtain some non-trivial quasi-YD modules is to take direct sums of one-dimensional modules.

One-dimensional representations of H are the same as algebra maps $H \rightarrow \mathbb{k}$. Under the convolution product of algebra maps (= tensor product of representations), these form a group $G(H^\circ)$ of grouplike elements in the finite dual H° of H [38, 9.1.4]. Quasicoactions on a 1-dimensional space V are given by $v \mapsto p \otimes v$, where $p \in H$.

The group $G(H^\circ)$ acts on H by algebra automorphisms $t_\alpha : H \rightarrow H$, defined as $t_\alpha(h) = \alpha(Sh_{(1)})h_{(2)}\alpha(h_{(3)})$ for $\alpha \in G(H^\circ)$. Let $[g, h]_{t_\alpha} = gh - t_\alpha(h)g$ be the t_α -commutator in H . One can check that the quasicoaction $v \mapsto p \otimes v$ on the representation α is Yetter–Drinfeld, if and only if

$$[p, h]_{t_\alpha} = 0 \quad \text{for all } h \in H.$$

In particular, if H is cocommutative, all t_α are the identity on H . Isomorphism classes of 1-dimensional quasi-YD modules over H then correspond to pairs

$$(\alpha, p) \in G(H^\circ) \times Z(H),$$

where $Z(H)$ is the centre of H . Under the tensor product, these isomorphism classes form a commutative monoid isomorphic to $G(H^\circ) \times Z(H)$. One-dimensional Yetter–Drinfeld modules correspond to the subgroup $G(H^\circ) \times (G(H) \cap Z(H))$.

This classification, incidentally, shows that the space of Yetter–Drinfeld quasicoactions on a given H -module V need not coincide with the linear span of Yetter–Drinfeld coactions. It is also a key ingredient in the example of braided doubles given in 1.3.

3.3. Proof of Theorem 3.3

The rest of this section will be devoted to the proof of Theorem 3.3.

It is easy to show that the Yetter–Drinfeld condition is necessary for \tilde{A}_β to have triangular decomposition over H . Indeed, let $f \in V^*$, $h \in H$ and $v \in V$. Denote $L = h_{(1)}\beta(f \triangleleft h_{(2)}, v)$ and $R = \beta(f, h_{(1)}\triangleright v)h_{(2)}$. Compute the product fhv in \tilde{A}_β in two ways: first, $fhv = h_{(1)}(f \triangleleft h_{(2)})v$, which by the commutator relation equals $L + h_{(1)}v(f \triangleleft h_{(2)}) = L + (h_{(1)}\triangleright v)h_{(2)}(f \triangleleft h_{(3)})$. Second, $fhv = f(h_{(1)}\triangleright v)h_{(2)}$, which by the commutator relation is $R + (h_{(1)}\triangleright v)fh_{(2)} = R + (h_{(1)}\triangleright v)h_{(2)}(f \triangleleft h_{(3)})$. Thus, $L = R$ in \tilde{A}_β . But H embeds in \tilde{A}_β injectively because of the triangular decomposition. Therefore, $L = R$ in H as required.

To show that the Yetter–Drinfeld condition is sufficient, it is enough to introduce on $T(V) \otimes H \otimes T(V^*)$ associative multiplication which satisfies the defining relations of \tilde{A}_β .

In order to construct such multiplication on $T(V) \otimes H \otimes T(V^*)$, we would like to use a general fact about algebra factorisations. Let X and Y be associative algebras. Denote by $m^X : X \otimes X \rightarrow X$, respectively $m^Y : Y \otimes Y \rightarrow Y$, the multiplication map for X , respectively Y . An associative product on $X \otimes Y$, which simultaneously extends m^X and m^Y (in other words, an algebra factorisation into X, Y), is defined via

$$(x \otimes y)(x' \otimes y') = (m^X \otimes m^Y)(x \otimes c(y \otimes x') \otimes y'),$$

where $c : Y \otimes X \rightarrow X \otimes Y$ is a twist map between Y and X (also called rule of exchange of tensorands). Associativity of this product is equivalent to two equations on c ; see [34, Proposition 21.4]:

Proposition 3.7. *The above product on $X \otimes Y$ is associative, if and only if*

$$(3.7a) \quad c \circ (\text{id}^Y \otimes m^X) = (m^X \otimes \text{id}^Y)(\text{id}^X \otimes c)(c \otimes \text{id}^X),$$

$$(3.7b) \quad c \circ (m^Y \otimes \text{id}^X) = (\text{id}^X \otimes m^Y)(c \otimes \text{id}^Y)(\text{id}^Y \otimes c).$$

Both sides of (3.7a) are maps from $Y \otimes X \otimes X$ to $X \otimes Y$, whereas in (3.7b) the maps are from $Y \otimes Y \otimes X$ to $X \otimes Y$.

We put $X = T(V) \rtimes H$, the semidirect product algebra arising from the left action of H on V , and $Y = T(V^*)$. In the construction of the twist map c between Y and X , one uses the fact that Y is a free tensor algebra. We use the notation $T^{\leq 1}(V^*) = \mathbb{k} \oplus V^*$.

Lemma 3.8. *Let $Y = T(V^*)$ and let $c' : V^* \otimes X \rightarrow X \otimes T^{\leq 1}(V^*)$ be a “partial twist map” that satisfies (3.7a). Then there exists a unique twist map $c : T(V^*) \otimes X \rightarrow X \otimes T(V^*)$, which extends c' and satisfies (3.7a), (3.7b) and $c(1 \otimes x) = x \otimes 1$.*

Proof. The map $c : Y \otimes X \rightarrow X \otimes Y$ is defined (in tensor leg notation) by $c(f_1 \otimes \dots \otimes f_n, x) = c'_{12}c'_{23} \dots c'_{n,n+1}(f_1 \otimes \dots \otimes f_n \otimes x)$, where f_i are elements of some basis of V^* and $n \geq 1$, and the condition $c(1 \otimes x) = x \otimes 1$. By construction, c satisfies (3.7b); this property also guarantees uniqueness of c .

Let us now check (3.7a), i.e., that $cm^X_{23}(\xi \otimes x \otimes x') = m^X_{12}c_{23}c_{12}(\xi \otimes x \otimes x')$ for all $\xi \in T(V^*)$, $x, x' \in X$. We use induction in the tensor degree n of $\xi \in V^{*\otimes n}$. When $n = 1$, the property holds

because c coincides with c' . Assume $n > 1$ and that (3.7a) holds for tensors in $T(V^*)$ of degree $< n$. Write $\xi = \eta \otimes \theta$ where η, θ are tensors in $T(V^*)$ of degree strictly less than n . We have

$$\begin{aligned} cm_{23}^X(\xi \otimes x \otimes x') &= c_{12}c_{23}m_{34}^X(\eta \otimes \theta \otimes x \otimes x') = c_{12}m_{23}^Xc_{34}c_{23}(\eta \otimes \theta \otimes x \otimes x') \\ &= m_{12}^Xc_{23}c_{12}c_{34}c_{23}(\eta \otimes \theta \otimes x \otimes x') = m_{12}^Xc_{23}c_{34}c_{12}c_{23}(\eta \otimes \theta \otimes x \otimes x'), \end{aligned}$$

where the 1st step is by property (3.7b) of c , the 2nd and the 3rd steps are by induction hypothesis, and the last step is trivial. But by property (3.7b), this expression is precisely $m_{12}^Xc_{23}c_{12}(\xi \otimes x \otimes x')$. The lemma is proved. \square

We will now construct a certain partial twist map $c' : V^* \otimes X \rightarrow X \otimes T^{\leq 1}(V^*)$, which will satisfy (3.7a). First of all, we define operators

$$\tilde{\partial}_f : T(V) \rightarrow T(V) \rtimes H, \quad \tilde{\partial}_f(V^{\otimes n}) \subset V^{\otimes n-1} \otimes H,$$

by the formula

$$\tilde{\partial}_f(v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^n (v_1 \otimes \dots \otimes v_{i-1}) \cdot \beta(f, v_i) \cdot (v_{i+1} \otimes \dots \otimes v_n),$$

where \cdot is the multiplication in the algebra $X = T(V) \rtimes H$. We put $\tilde{\partial}_f 1 = 0$.

Lemma 3.9.

1. The operators $\tilde{\partial}_f$ obey the Leibniz rule in the following form:

$$\tilde{\partial}_f(pq) = (\tilde{\partial}_f p) \cdot q + p \cdot (\tilde{\partial}_f q) \quad \text{for } p, q \in T(V),$$

where \cdot is the product in $T(V) \rtimes H$.

2. For any $b \in T(V)$,

$$h_{(1)} \cdot \tilde{\partial}_f \triangleleft_{h_{(2)}} b = \tilde{\partial}_f(h_{(1)} \triangleright b) \cdot h_{(2)}.$$

Proof. 1. The Leibniz rule is obvious from the definition of $\tilde{\partial}_f$.

2. When $b = v \in V$, this equality is the Yetter–Drinfeld condition $h_{(1)}\beta(f \triangleleft h_{(2)}, v) = \beta(f, h_{(1)} \triangleright v)h_{(2)}$. (This is the only place in the proof of Theorem 3.3 where the Yetter–Drinfeld condition is invoked.) Furthermore, it is easy to see that if the equality holds for b and for b' (where b, b' are tensors in $T(V)$), it holds for their product bb' . Indeed, $h_{(1)} \cdot \tilde{\partial}_f \triangleleft_{h_{(2)}}(bb')$ is equal, by the Leibniz rule, to $h_{(1)} \cdot \tilde{\partial}_f \triangleleft_{h_{(2)}} b \cdot b' + (h_{(1)} \triangleright b) \cdot h_{(2)} \cdot \tilde{\partial}_f \triangleleft_{h_{(3)}} b'$. Replace this with $\tilde{\partial}_f(h_{(1)} \triangleright b) \cdot h_{(2)} \cdot b' + (h_{(1)} \triangleright b) \cdot \tilde{\partial}_f(h_{(2)} \triangleright b) \cdot h_{(3)}$ which is, again by the Leibniz rule, equal to $\tilde{\partial}_f(h_{(1)} \triangleright (bb')) \cdot h_{(2)}$. The equality thus holds for any $b \in T(V)$, and the lemma is proved. \square

Now for $f \in V^*$ and $ah \in X$, where $a \in T(V)$ and $h \in H$, we put

$$c'(f, ah) = (\tilde{\partial}_f a) \cdot h \otimes 1 + ah_{(1)} \otimes f \triangleleft h_{(2)}.$$

Lemma 3.10. *The above map $c' : V^* \otimes X \rightarrow X \otimes T^{\leq 1}(V^*)$ satisfies (3.7a).*

Proof. Take $ah, bk \in X$, where $a, b \in T(V)$ and $h, k \in H$; (3.7a) is equivalent to

$$c'(f, ah \cdot bk) = (\tilde{\partial}_f a) \cdot h \cdot bk \otimes 1 + ah_{(1)} \cdot (\tilde{\partial}_{f \triangleleft h_{(2)}} b) \cdot k \otimes 1 + ah_{(1)} \cdot bk_{(1)} \otimes f \triangleleft h_{(2)} k_{(2)}. \tag{*}$$

Let us expand the left-hand side of (*). In the semidirect product algebra $T(V) \rtimes H$ the product $ah \cdot bk$ is equal to $a(h_{(1)} \triangleright b)h_{(2)}k$, hence we have $\tilde{\partial}_f(a(h_{(1)} \triangleright b)) \cdot h_{(2)}k \otimes 1 + a(h_{(1)} \triangleright b)h_{(2)}k_{(1)} \otimes f \triangleleft h_{(3)}k_{(2)}$ on the left in (*). By the Leibniz rule for $\tilde{\partial}_f$, the left-hand side of (*) is

$$(\tilde{\partial}_f a) \cdot (h_{(1)} \triangleright b)h_{(2)}k \otimes 1 + a \cdot \tilde{\partial}_f(h_{(1)} \triangleright b) \cdot h_{(2)}k \otimes 1 + a(h_{(1)} \triangleright b)h_{(2)}k_{(1)} \otimes f \triangleleft h_{(3)}k_{(2)}.$$

It is obvious that the first and the third term of this expression coincide with the respective terms on the right-hand side of (*). To see that the second terms also coincide, apply Lemma 3.9. \square

We have just constructed an algebra factorisation of the form $(T(V) \rtimes H) \otimes T(V^*)$. To show that it coincides with the algebra \tilde{A}_β , we have to check that the defining relations of \tilde{A}_β hold in this algebra factorisation. We do not need to check the relation $hv = (h_{(1)} \triangleright v)h_{(2)}$ because it is automatically fulfilled in the semidirect product algebra $T(V) \rtimes H$. Let us now compute the product fh in $(T(V) \rtimes H) \otimes T(V^*)$. We have $fh = c'(f, 1 \cdot h) = 1 \cdot h_{(1)} \otimes f \triangleleft h_{(2)} = h_{(1)}(f \triangleleft h_{(2)})$, i.e., the second defining relation of \tilde{A}_β also holds. Finally, $fv = c'(f, v \cdot 1) = \tilde{\partial}_f v \otimes 1 + v \otimes f$ where $\tilde{\partial}_f v = \beta(f, v)$. Thus, the commutator relation holds as well. Theorem 3.3 is proved.

Remark 3.11. It is clear from the proof of the theorem that the operator $\tilde{\partial}_f b \in T(V) \otimes H$ is the commutator $[f, b]$ in \tilde{A}_β , for $b \in T(V)$.

4. Braided doubles

4.1. Definition of a braided double

Recall (Corollary 3.6) that any free braided double over a bialgebra H has triangular decomposition of the form

$$\tilde{A}(V, \delta) = T(V) \otimes H \otimes T(V^*),$$

where (V, δ) is a quasi-Yetter–Drinfeld module over H .

We will now be dealing with braided doubles which are no longer “free”; that is, they have relations within $T(V)$ and within $T(V^*)$, but still have triangular decomposition over H . This is formalised as follows. Denote by $T^{>0}(V)$ the ideal $\bigoplus_{n>0} V^{\otimes n}$ of $T(V)$.

Definition 4.1. A *triangular ideal* in $\tilde{A}(V, \delta) = T(V) \rtimes H \rtimes T(V^*)$ is a two-sided ideal of the form

$$I^- \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^+,$$

where I^-, I^+ are subspaces (and automatically two-sided H -invariant ideals) in $T^{>0}(V)$ and $T^{>0}(V^*)$, respectively.

Definition 4.2. A *braided double* is a quotient of a free braided double modulo a triangular ideal.

Where (V, δ) is a quasi-YD module over a bialgebra H , we will refer to a quotient of $\tilde{A}(V, \delta)$ modulo a triangular ideal as a (V, δ) -braided double.

4.2. Hierarchy of braided doubles

In what follows, we will use the facts about triangular decomposition over a bialgebra and triangular ideals, collected and proved in Appendix A.

Denote by $\mathcal{D}(V, \delta)$ the set of (V, δ) -braided doubles. This set is partially ordered by the reverse inclusion of triangular ideals $I \subset \tilde{A}(V, \delta)$. Note that if $I_1 \subseteq I_2$ are triangular ideals, the double $\tilde{A}(V, \delta)/I_2$ is a *triangular quotient* of $\tilde{A}(V, \delta)/I_1$. This notion is defined in A.1.

All (V, δ) -braided doubles are triangular quotients of the free double $\tilde{A}(V, \delta)$, the greatest element of $\mathcal{D}(V, \delta)$. By Corollary A.3, a sum of triangular ideals in $\tilde{A}(V, \delta)$ is a triangular ideal; therefore, all (V, δ) -braided doubles have a common triangular quotient:

Definition 4.3. Let (V, δ) be a quasi-Yetter–Drinfeld module over a bialgebra H . Denote by

$$I(V, \delta) \subset T(V), \quad I(V^*, \delta) \subset T(V^*)$$

the pair of H -invariant two-sided ideals such that $I_{\tilde{A}(V, \delta)} = I(V, \delta) \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I(V^*, \delta)$ is the largest among (= the sum of all) triangular ideals in $\tilde{A}(V, \delta)$. Define two algebras:

$$U(V, \delta) = T(V)/I(V, \delta), \quad U(V^*, \delta) = T(V^*)/I(V^*, \delta).$$

The braided double

$$\bar{A}(V, \delta) = \tilde{A}(V, \delta)/I_{\tilde{A}(V, \delta)} = U(V, \delta) \rtimes H \rtimes U(V^*, \delta)$$

is called the *minimal double* associated to (V, δ) .

There are other distinguished (V, δ) -braided doubles which lie between $\tilde{A}(V, \delta)$ and $\bar{A}(V, \delta)$ in the above partial order. *Quadratic doubles* are braided doubles of the form $T(V)/I^- \otimes H \otimes T(V^*)/I^+$ where I^- and I^+ are quadratic ideals in $T(V)$, $T(V^*)$ (i.e., are generated by subsets of $V^{\otimes 2}$ and $V^{*\otimes 2}$, respectively). The lowest element in this class is the *minimal quadratic double*

$$\bar{A}_{quad}(V, \delta) \cong T(V)/I_{quad}(V, \delta) \rtimes H \rtimes T(V^*)/I_{quad}^*(V, \delta).$$

One can deduce from Theorem 4.11 below that

$$I_{quad}(V, \delta) = \langle I(V, \delta) \cap V^{\otimes 2} \rangle, \quad I_{quad}^*(V, \delta) = \langle I(V^*, \delta) \cap V^{*\otimes 2} \rangle,$$

where $\langle \dots \rangle$ denotes the two-sided ideal with given generators. There is a canonical surjection $\bar{A}_{quad}(V, \delta) \rightarrow \bar{A}(V, \delta)$, and we may regard the double $\bar{A}_{quad}(V, \delta)$ as a ‘first approximation’ to $\bar{A}(V, \delta)$.

4.3. Relations in the minimal double

Our goal in this section is to describe the largest triangular ideal in the free braided double $\tilde{A}(V, \delta)$ —or, the same, the relations in the algebras $U(V, \delta)$ and $U(V^*, \delta)$ —in terms of the quasi-Yetter–Drinfeld structure on V . The first step is the following

Lemma 4.4. *Triangular ideals in $\tilde{A}(V, \delta)$ are subspaces $J \subset \tilde{A}(V, \delta)$ of the form $J = J^-HT(V^*) + T(V)HJ^+$, where $J^- \subset T^{>0}(V)$ and $J^+ \subset T^{>0}(V^*)$ are H -invariant two-sided ideals, such that*

$$[f, J^-] \subset J^- \otimes H, \quad [J^+, v] \subset H \otimes J^+$$

for all $f \in V^*, v \in V$.

Proof. Triangular ideals are described by Proposition A.2, and we only have to adapt that description to the case of braided doubles. Denote $U^- = T(V)$, $U^+ = T(V^*)$, and let $\epsilon^\pm : U^\pm \rightarrow \mathbb{k}$ be the projections to degree zero component. By Proposition A.2, triangular ideals are of the form $J = J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$, where $J^\pm \subset \ker \epsilon^\pm$ are H -invariant two-sided ideals in the algebras U^\pm , such that $U^+ \cdot J^-, J^+ \cdot U^-$ lie in J . Since V (respectively V^*) generates U^- (respectively U^+) as an algebra, this is equivalent to

$$f \cdot J^-, J^+ \cdot v \subset J^-HU^+ + U^-HJ^+$$

for all $v \in V$ and $f \in V^*$. Now, since $J^- \cdot f$ obviously lies in J^-HU^+ and $v \cdot J^+$ lies in U^-HJ^+ , we may replace products by commutators and rewrite this condition as

$$[f, J^-], [J^+, v] \subset J^-HU^+ + U^-HJ^+.$$

Finally, we observe that by Remark 3.11, $[f, J^-]$ lies in U^-H , and, similarly, $[J^+, v]$ lies in HU^+ . Therefore, the condition splits into two separate inclusions, $[f, J^-] \subset J^-H$ and $[J^+, v] \subset HJ^+$. \square

Remark 4.5. Note that the lemma implies that any triangular ideal in $\tilde{A}(V, \delta)$ is a sum of two triangular ideals of special form: one $J^-HT(V^*)$ and the other $T(V)HJ^+$.

Our next step is to show that the defining ideals $I(V, \delta), I(V^*, \delta)$ of the minimal double are graded ideals in $T(V), T(V^*)$, respectively. We call a triangular ideal $I^- \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^+$ *graded*, if I^- (respectively I^+) is a graded ideal in $T^{>0}(V)$ (respectively $T^{>0}(V^*)$). A *graded braided double* is a quotient of a free braided double by a graded triangular ideal.

Lemma 4.6. *Any triangular ideal in a free braided double is contained in a graded triangular ideal.*

Proof. Let $\tilde{A}(V, \delta) \cong T(V) \rtimes H \rtimes T(V^*)$ be a free braided double, and J be a triangular ideal in $\tilde{A}(V, \delta)$. Denote $U^- = T(V)$ and $U^+ = T(V^*)$. By Remark 4.5, it is enough to consider the cases $J = J^-HU^+$ and $J = U^-HJ^+$. We will assume $J = J^-HU^+$, the other case being analogous. By Lemma 4.4, J^- is a two-sided ideal in $U^-_{>0}$ is an $[f, J^-] \subset J^- \otimes H$ for any $f \in V^*$.

Denote by p_n the projection map from U^- onto its n th homogeneous component U_n^- . Put $J_n^- = p_n(J^-)$ and let $J_{\text{gr}}^- = \bigoplus_{n>0} J_n^-$. Let us check that the space J_{gr}^- satisfies the conditions of Lemma 4.4. Indeed, J_{gr}^- is a two-sided ideal in U^- , because $vJ_n^- = p_{n+1}(vJ^-) \subset p_{n+1}(J^-) = J_{n+1}^-$ for any $v \in V$, and similarly $J_n^-v \subset J_{n+1}^-$. Subspaces U_n^- are H -submodules of U^- , and p_n are H -equivariant maps; thus, J_{gr}^- is H -invariant. By construction, J_{gr}^- lies in $U_{>0}^- = \ker \epsilon^-$ and contains J^- . Finally, it is clear (e.g. from the definition of the operator $\tilde{\partial}_f = [f, \cdot]$ in the proof of Theorem 3.3) that the commutator $[f, \cdot]$ is lowering the degree in $U^- \otimes H$ by one:

$$[f, U_n^-] \subset U_{n-1}^- \otimes H,$$

therefore $[f, J_n^-] \subset J_{n-1}^- \otimes H$. Thus, J^-HU^+ is contained in a graded triangular ideal $J_{\text{gr}}^-HU^+$ of $\tilde{A}(V, \delta)$. \square

Corollary 4.7. *$I(V, \delta), I(V^*, \delta)$ are graded ideals in $T(V), T(V^*)$, respectively.*

Proof. Observe that by the lemma, the largest triangular ideal in a free braided double is graded. \square

4.4. *Computation of the ideals $I(V, \delta), I(V^*, \delta)$*

To proceed with the computation of the maximal triangular ideal $I(V, \delta)HT(V^*) + T(V)HI(V^*, \delta)$ of $\tilde{A}(V, \delta)$, we assume that H is a Hopf algebra. To make the exposition concise, let us focus on the ideal $I(V, \delta)$; we will state the final result for $I(V^*, \delta)$ later in Remark 4.12. We say that a subspace $W \subset T^{>0}(V)$ is “preserved by commutators”, if $[f, W] \subset W \otimes H$ for any $f \in V^*$.

Lemma 4.8. *$I(V, \delta)$ is the maximal subspace in $T^{>0}(V)$, preserved by commutators.*

Proof. Let W be a subspace of $T^{>0}(V)$, preserved by commutators. Then $W' = H \triangleright W$ is a subspace of $T^{>0}(V)$. Let us show that W' is also preserved by commutators. By Lemma 3.9,

$$[f, h \triangleright b] = \tilde{\partial}_f(h \triangleright b) = h_{(1)} \cdot (\tilde{\partial}_f \lrcorner_{h_{(2)}} b) \cdot Sh_{(3)} \quad \text{for } b \in T(V);$$

applying this to $b \in W$ shows that $[f, W']$ lies in $H \cdot W \cdot H \subset W'H$.

It follows that the maximal subspace preserved by commutators is H -invariant. Assume now that W is an H -invariant subspace of $T^{>0}(V)$, preserved by commutators. Let us show that W is contained in an H -invariant two-sided ideal in $T^{>0}(V)$, preserved by commutators. Indeed, apply $\tilde{\partial}_f$ to the ideal $T(V) \cdot W \cdot T(V)$. By the Leibniz rule, the result lies in $(T(V)H) \cdot W \cdot T(V) + T(V) \cdot (WH) \cdot T(V) + T(V) \cdot W \cdot (T(V)H)$. Since W and $T(V)$ are H -invariant subspaces of $T(V)$, this coincides with $(T(V) \cdot W \cdot T(V)) \otimes H$.

Thus, the maximal subspace of $T^{>0}(V)$, preserved by commutators, is an H -invariant two-sided ideal with this property. But the maximal among such ideals is $I(V, \delta)$. \square

4.5. *Quasibraided integers and quasibraided factorials*

We are ready to describe the graded components of the ideal $I(V, \delta) \subset T(V)$ as kernels of *quasibraided factorials*, which we now introduce.

Definition 4.9. Let (V, δ) be a quasi-Yetter–Drinfeld module over a bialgebra H , with action \triangleright and quasicoaction $\delta(v) = v^{[-1]} \otimes v^{[0]}$. The *quasibraided integers* are maps

$$\begin{aligned} \widetilde{[n]}_\delta &: V^{\otimes n} \rightarrow V^{\otimes n-1} \otimes H \otimes V, \\ \widetilde{[n]}_\delta(v_1 \otimes \cdots \otimes v_n) &= \sum_{i=1}^n v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i^{[-1]} \underset{(1)}{\triangleright} (v_{i+1} \otimes \cdots \otimes v_n) \otimes v_i^{[-1]} \underset{(2)}{\otimes} v_i^{[0]}. \end{aligned}$$

The *quasibraided factorials* are maps

$$\widetilde{[n]}!_\delta : V^{\otimes n} \rightarrow (H \otimes V)^{\otimes n}, \quad \widetilde{[n]}!_\delta = (\widetilde{[1]}_\delta \otimes \text{id}_{H \otimes V}^{\otimes n-1}) \circ (\widetilde{[2]}_\delta \otimes \text{id}_{H \otimes V}^{\otimes n-2}) \circ \cdots \circ \widetilde{[n]}_\delta.$$

We also put $\widetilde{[0]}!_\delta = 1$.

Lemma 4.10. The commutator of $f \in V^*$ and $b \in V^{\otimes n}$ in the free braided double $\widetilde{A}(V, \delta)$ is given by

$$[f, b] = (\text{id}_V^{\otimes n-1} \otimes \text{id}_H \otimes \langle f, - \rangle) \widetilde{[n]}_\delta b.$$

Proof. By Remark 3.11, $[f, b] = \widetilde{\partial}_f b$ where the operator $\widetilde{\partial}_f$ was introduced in the proof of Theorem 3.3. Recall that $\beta(f, v) = v^{[-1]} \langle f, v^{[0]} \rangle$, and rewrite the formula for $\widetilde{\partial}_f$ in terms of the quasicoaction:

$$\widetilde{\partial}_f(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n (v_1 \otimes \cdots \otimes v_{i-1}) \cdot v_i^{[-1]} \langle f, v_i^{[0]} \rangle \cdot (v_{i+1} \otimes \cdots \otimes v_n).$$

The lemma now follows from the relations in the semidirect product $T(V) \rtimes H$. \square

Theorem 4.11. Let (V, δ) be a quasi-YD module over a Hopf algebra H . The ideal $I(V, \delta)$ in $T(V)$ is given by

$$I(V, \delta) = \bigoplus_{n=1}^{\infty} \ker \widetilde{[n]}!_\delta.$$

Proof. The ideal $I(V, \delta) \subset T^{>0}(V)$ is graded by Corollary 4.7. Write $I(V, \delta) = I_0 \oplus I_1 \oplus \cdots$, where $I_n = I(V, \delta) \cap V^{\otimes n}$. By Lemma 4.8, $I(V, \delta)$ is the maximal subspace of $T^{>0}(V)$ preserved by commutators, which in terms of the graded components rewrites as

$$I_n = \{b \in V^{\otimes n} : [f, b] \in I_{n-1} \otimes H \text{ for all } f \in V^*\}, \quad n \geq 1.$$

Let us show that $I_n = \ker \widetilde{[n]}!_\delta$. This is true for $n = 0$ (because $I_0 = 0$); assume this to be true for $n - 1$. Substitute the commutator $[f, b]$ with its expression via the quasibraided integer from Lemma 4.10:

$$I_n = \{b \in V^{\otimes n} : (\text{id}_V^{\otimes n-1} \otimes \text{id}_H \otimes \langle f, - \rangle) \widetilde{[n]}_\delta b \in (\ker \widetilde{[n-1]}!_\delta \otimes H)\}.$$

Since this holds for arbitrary $f \in V^*$, the space I_n consists of $b \in V^{\otimes n}$ such that $[\widetilde{n}]_\delta b$ is in $(\ker [\widetilde{n-1}]_\delta) \otimes H \otimes V$. That is, $I_n = \ker [\widetilde{n}]_\delta$. The theorem follows by induction. \square

Remark 4.12. The ideal $I(V^*, \delta)$ of $T(V^*)$ has a description of the same nature. Consider the right quasicoaction $\delta_r : V^* \rightarrow V^* \otimes H$, $f \mapsto f^{[0]} \otimes f^{[1]}$, which is given by $\beta(f, v) = \langle f^{[0]}, v \rangle f^{[1]}$. This, together with the right action of H on V^* , gives rise to right-handed quasibraided integers $[\widetilde{n}]_{\delta_r} : V^{*\otimes n} \rightarrow V^* \otimes H \otimes V^{*\otimes n-1}$ and to right-handed quasibraided factorials $[\widetilde{n}]_{\delta_r}! : V^{*\otimes n} \rightarrow (V^* \otimes H)^{\otimes n}$. One has $I(V^*, \delta) = \bigoplus_n \ker [\widetilde{n}]_{\delta_r}!$.

The following corollary gives a useful criterion of minimality of a graded braided double.

Corollary 4.13. (Minimality criterion) *Let $A = U^- \rtimes H \rtimes U^+$ be a (V, δ) -braided double which is graded: $U^- = \bigoplus_{n=0}^\infty U_n^-$, $U^+ = \bigoplus_{n=0}^\infty U_n^+$. Then A is a minimal double, when and only when*

- (a) if $b \in U^-$, $[f, b] = 0$ for all $f \in V^*$, then $b \in U_0^-$;
- (b) if $\phi \in U^+$, $[\phi, v] = 0$ for all $v \in V$, then $\phi \in U_0^+$.

Proof. $U^- \neq U(V, \delta)$, if and only if $U_{>0}^-$ contains a graded subspace preserved by commutators $[f, \cdot]$ and not contained in degree 0 of grading. Such a subspace exists if and only if there is a homogeneous element b of positive degree (in the lowest degree component of the subspace) which commutes with all $f \in V^*$. Similarly, $U^+ \neq U(V^*, \delta)$, if and only if there is $\phi \in U_{>0}^+$ which commutes with all $v \in V$. \square

The next corollary will be useful in constructing graded braided doubles which are not minimal.

Corollary 4.14. *Let (V, δ) be a quasi-YD module over a Hopf algebra H . Let $I^- \subset T^{>0}(V)$ be a graded two-sided ideal. Then $I^- \otimes H \otimes T(V^*)$ is a triangular ideal in $\widetilde{A}(V, \delta)$, if and only if I^- has an H -invariant generating space $R = \bigoplus_{n>0} R_n$, $R_n \subset V^{\otimes n}$, such that*

$$[\widetilde{n}]_\delta R_n \subseteq R_{n-1} \otimes H \otimes V$$

(assuming $R_0 = 0$). A similar statement holds for ideals $I^+ \subset T^{>0}(V)$ and right-handed quasibraided integers $[\widetilde{n}]_{\delta_r}$.

Proof. By Lemma 4.10, the equation on R is equivalent to saying that $[f, R_n] \subseteq R_{n-1} \otimes H$ for all $f \in V^*$. This implies that the ideal $I^- = \langle R \rangle$ is an H -equivariant ideal in $T^{>0}(V)$, such that $[f, I^-] \subseteq I^- \otimes H$. By Lemma 4.4, $I^- HT(V^*)$ is a triangular ideal in $\widetilde{A}(V, \delta)$. This is the ‘if’ part; the ‘only if’ part follows by putting $R = I^-$. \square

4.6. Standard modules for braided doubles

Clearly, minimality of a braided double should influence its representation theory. Let us mention a construction which yields a family of standard modules for an algebra with triangular decomposition (known as Verma modules in Lie theory and used also for rational Cherednik algebras).

Let $A = U^- \rtimes H \rtimes U^+$ be a graded (V, δ) -braided double over H . To any irreducible left representation $\rho : H \rightarrow \text{End}(L_\rho)$ of H is associated a left A -module:

$$M_\rho = \text{Ind}_{H \otimes U^+}^A(\rho \otimes \epsilon^+).$$

As a vector space, M_ρ is the tensor product $U^- \otimes L_\rho$. The standard modules M_ρ are crucial in the Bernstein–Gelfand–Gelfand theory of category \mathcal{O} for $U(\mathfrak{g})$ [8] and its more recent version for rational Cherednik algebras as in [15].

Observe, however, that all M_ρ are reducible A -modules unless $U^- = U(V, \delta)$. Indeed, let $\overline{M}_\rho \cong U(V, \delta) \otimes L_\rho$ be the induced module for the minimal double $\overline{A}(V, \delta)$. Then \overline{M}_ρ , which is an A -module via the quotient map $A \twoheadrightarrow \overline{A}(V, \delta)$, is a quotient of M_ρ . We therefore suggest $\{\overline{M}_\rho \mid \rho \in \text{Irr}(H)\}$ as a family of standard modules for any (V, δ) -braided double.

4.7. *The Harish-Chandra pairing and minimality*

We will now suggest a useful method for proving minimality of a given braided double. Let us introduce the Harish-Chandra pairing in braided doubles; in fact, this can be done for any algebra with triangular decomposition over a bialgebra, see Appendix A, Section A.2.

Definition 4.15. Let $A = U^- \rtimes H \rtimes U^+$ be a graded braided double over a bialgebra H . Let $\epsilon^\pm : U^\pm \rightarrow \mathbb{k}$ be projections onto degree 0 components in U^\pm . Denote

$$p_H = \epsilon^- \otimes \text{id}_H \otimes \epsilon^+ : A \twoheadrightarrow H.$$

The *Harish-Chandra pairing* in A is

$$(\cdot, \cdot)_H : U^+ \times U^- \rightarrow H, \quad (\phi, b)_H = p_H(\phi b).$$

The terminology is inspired by the example of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra \mathfrak{g} . The next theorem follows from a general result on algebras with triangular decomposition (Proposition A.4):

Theorem 4.16. *If the Harish-Chandra pairing in a braided double A is non-degenerate, then A is a minimal double.*

We will now give a formula for the Harish-Chandra pairing in a free braided double $\widetilde{A}(V, \delta)$ in terms of quasibraided factorials. (It works as well for any graded (V, δ) -braided double.) We will use the notation

$$m_H : (H \otimes V)^{\otimes n} \rightarrow H \otimes V^{\otimes n},$$

$$m_H(h_1 \otimes v_1 \otimes h_2 \otimes v_2 \otimes \cdots \otimes h_n \otimes v_n) = h_1 h_2 \dots h_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n.$$

Proposition 4.17. *Let (V, δ) be a quasi-Yetter–Drinfeld double over a bialgebra H . The Harish-Chandra pairing in $\widetilde{A}(V, \delta)$ is given by*

$$\phi \in V^{*\otimes n}, b \in V^{\otimes n} \mapsto (\phi, b)_H = (\text{id}_H \otimes \langle \phi, - \rangle_{V^{\otimes n}}) m_H[\widetilde{n}]!_\delta b$$

and $(V^{*\otimes n}, V^{\otimes m})_H = 0$ if $n \neq m$.

Proof. Recall the operator $\tilde{\partial}_f : T(V) \rightarrow T(V) \rtimes H$, $\tilde{\partial}_f b = [f, b]$ (commutator in the double $\tilde{A}(V, \delta)$), and extend it to $T(V) \rtimes H$ by

$$\tilde{\partial}_f(b \otimes h) = (\tilde{\partial}_f b) \cdot h, \quad b \in T(V), \quad h \in H, \quad f \in V^*.$$

Consider the subspace $A^+ = T(V) \otimes H \otimes T^{>0}(V^*)$ of $\tilde{A}(V, \delta)$. It has the property that $V^*A^+ \subset A^+$ and $A^+H \subset A^+$. Therefore, for any $b \in T(V)$, $h \in H$ and $f \in V^*$ we have

$$fbh \simeq \tilde{\partial}_f(bh) \quad \text{modulo } A^+.$$

Let $\phi = f_1 \otimes f_2 \otimes \dots \otimes f_n \in V^{*\otimes n}$ and $b \in V^{\otimes m}$. The subspace A^+ lies in the kernel of the map p_H , therefore

$$(\phi, b)_H = p_H(\phi b) = p_H(\tilde{\partial}_{f_1} \tilde{\partial}_{f_2} \dots \tilde{\partial}_{f_n} b).$$

If $m = n$, it is easy to deduce from Lemma 4.10 that the right-hand side is equal to $(\text{id}_H \otimes \langle f_1, - \rangle \otimes \dots \otimes \langle f_n, - \rangle) m_H \tilde{[n]}_\delta$. If $m > n$, then $\tilde{\partial}_{f_1} \dots \tilde{\partial}_{f_n} b$ lies in the space $V^{\otimes m-n} \otimes H \subset \ker p_H$. Finally, if $m < n$, then $\tilde{\partial}_{f_1} \dots \tilde{\partial}_{f_n} b = 0$. \square

4.8. Non-degeneracy of the Harish-Chandra pairing and ideals

We would like to make a simple observation concerning ideals in braided doubles, which will not be used in the sequel. It is here to highlight a possible direction of further research.

The study of ideals in universal enveloping algebras $U(\mathfrak{g})$ was a significant topic in representation theory in the second half of the 20th century. It has been observed, however, that some important results on ideals may be deduced from the fact that the algebra has a triangular structure of a certain kind, cf. [14]. This allows one to extend such results to objects of more recent vintage such as rational Cherednik algebras.

Let us extend the Harish-Chandra pairing in a braided double $A = U^- \rtimes H \rtimes U^+$ to obtain pairings

$$(\cdot, \cdot)_H : U^+ \times U^- H \rightarrow H, \quad (\cdot, \cdot)_H : H U^+ \times U^- \rightarrow H,$$

both defined by the same formula $(y, x)_H = p_H(yx)$ and denoted by the same symbol. We say that the Harish-Chandra pairing in A is *strongly non-degenerate*, if these two extensions are non-degenerate pairings.

Remark 4.18. One can show that if a scalar pairing $\lambda((\cdot, \cdot)_H)$ is non-degenerate for an algebra homomorphism $\lambda : H \rightarrow \mathbb{k}$ (e.g. for the counit $\lambda = \epsilon$), then the Harish-Chandra pairing is strongly non-degenerate.

Proposition 4.19. *Let $A = U^- \rtimes H \rtimes U^+$ be a braided double with strongly non-degenerate Harish-Chandra pairing. Then $p_H(I) \neq 0$ for any non-zero two-sided ideal I of A .*

Note that the proposition links ideals in the algebra A (in general with no Hopf algebra structure) and ideals in the Hopf algebra H . In particular, if H is commutative or super-commutative, this may allow one to define associated (super)varieties in $\text{Spec}(H)$ for two-sided ideals in A (see, e.g., [14]).

Proof of Proposition 4.19. Denote the subalgebra HU^+ of A by B^+ , and denote by N^\pm the respective kernels of $\epsilon^\pm: U^\pm \rightarrow \mathbb{k}$. Let $p_{B^+} = \epsilon^- \otimes \text{id}_{B^+}$ be the projection onto B^+ . Let us show that if $p_{B^+}(I) \neq 0$ for an ideal I of A , then $p_H(I) \neq 0$. Indeed, $p_{B^+}(I) \neq 0$ means that I contains an element $\phi \in \phi' + N^-HU^+ = \phi' + N^-A$ for some non-zero $\phi' \in B^+$. By strong non-degeneracy, there is $b \in U^-$ such that $p_H(\phi'b) = (\phi', b)_H \neq 0$. Since $\phi'b$ differs from ϕb by an element from N^-A , which is in the kernel of the Harish-Chandra projection p_H , one has $p_H(\phi b) \neq 0$; it remains to note that $\phi b \in I$.

We now have to check that if I is a non-zero two-sided ideal in A , then $\text{pr}_{B^+}(I) \neq 0$. By Theorem 4.16, A is a minimal double, hence U^+ is graded by Corollary 4.7. Choose a graded basis \mathcal{B} of U^+ . Take a non-zero element in I and write it in the form $a_1 \otimes u_1 + \dots + a_n \otimes u_n$, where a_i are nonzero elements of U^-H and u_i are in \mathcal{B} . Without loss of generality, assume that u_1, \dots, u_k ($1 \leq k \leq n$) are of lowest degree, say m , among all u_i . Using non-degeneracy, find an element $v \in U^+$ such that $(v, a_1)_H = h_1 \neq 0$. This means that va_1 lies in $1 \otimes h_1 \otimes 1 + N^- \otimes H \otimes U^+ + U^- \otimes H \otimes N^+$, therefore va_1u_1 is in $1 \otimes h_1 \otimes u_1 + N^-A + A(N^+)^{m+1}$. If $(v, a_i)_H$ is denoted by h_i (h_i may be zero for $i > 1$), then

$$\sum_{1 \leq i \leq n} va_iu_i \equiv \sum_{1 \leq i \leq k} 1 \otimes h_i \otimes u_i \pmod{N^-A + A(N^+)^{m+1}}.$$

Projecting the element $\sum_i va_iu_i$ of I onto B^+ , then projecting further onto the quotient $B^+/B^+(N^+)^{m+1} = H \otimes (U^+/(N^+)^{m+1})$ gives $\sum_{1 \leq i \leq k} h_i \otimes (u_i \text{ mod } (N^+)^{m+1})$. This is not zero, since $h_1 \neq 0$ and u_1, \dots, u_k are linearly independent modulo $(N^+)^{m+1}$. Thus, $\text{pr}_{B^+}(I) \neq 0$ as required. \square

4.9. Two examples of braided doubles

We would like to finish this section with two (counter)examples. The first example shows that a minimal double may have degenerate Harish-Chandra pairing.

Example 4.20. Let $\mathbb{k}x$ be a one-dimensional module over a Hopf algebra H , with trivial action $h \triangleright x = \epsilon(h)x$. By 3.2, any quasi-Yetter–Drinfeld structure on $\mathbb{k}x$ is given by $\delta(x) = a \otimes x$, where

- (1) a is a central element in H .

We will write a instead of δ . The free braided double $\tilde{A}(\mathbb{k}x, a)$ has triangular decomposition $\mathbb{k}[x] \otimes H \otimes \mathbb{k}[y]$ where y is the spanning vector of the module dual to $\mathbb{k}x$. The quasibraided factorial is given by

$$\tilde{[n]}!_a(x^{\otimes n}) = n! (a \otimes x)^{\otimes n}.$$

It follows that $\tilde{A}(\mathbb{k}x, a)$ is a minimal double, if and only if

- (2) $a \neq 0$ and \mathbb{k} is of characteristic zero.

Assume $\text{char } \mathbb{k} = 0$. By Proposition 4.17, the Harish-Chandra pairing in $\tilde{A}(\mathbb{k}x, a)$ is given by

$$(y^{\otimes n}, x^{\otimes n})_H = n!(y, x)^n a^n.$$

The Harish-Chandra pairing is degenerate, if and only if

(3) a is a nilpotent element in H .

Thus, any Hopf algebra in characteristic zero with a nonzero central nilpotent element gives rise to a braided double which is minimal but has degenerate Harish-Chandra pairing.

Remark 4.21 (*Kaplansky’s third conjecture*). A conjecture that a Hopf algebra with the above properties *does not exist*, was number 3 in a list of ten conjectures on Hopf algebras published by I. Kaplansky in 1975. For some time, this third conjecture has been known to be false. The survey [46] contains historical remarks and a comprehensive account of progress made in relation to Kaplansky’s conjectures, including counterexamples to the third conjecture.

In order to complete Example 4.20, we give a new explicit counterexample to the third Kaplansky’s conjecture (a Hopf algebra of dimension 8) below in Example 5.8.

The second construction of a braided double shows that the Hilbert series of the two graded “halves”, $U(V, \delta)$ and $U(V^*, \delta)$, of a minimal double may be different (even in degree 1). In particular, V which is a “space of generators” for the algebra $U(V, \delta)$, may not embed injectively in $U(V, \delta)$.

Example 4.22. Let $\mathbb{Z}_2 = \{1, s\}$ be the two-element group. We take V to be a two-dimensional $\mathbb{k}\mathbb{Z}_2$ -module where s acts as a multiplication by -1 . Let v_1, v_2 be a basis of V and f_1, f_2 be the dual basis of V^* . Consider a $\mathbb{k}\mathbb{Z}_2$ -valued pairing between V^* and V , defined on the bases as follows:

$$\beta(f_1, v_1) = 1, \quad \beta(f_1, v_2) = s, \quad \beta(f_2, v_1) = \beta(f_2, v_2) = 0.$$

It is easy to see that the pairing β satisfies the Yetter–Drinfeld condition as in Theorem 3.3. Hence V becomes a quasi-Yetter–Drinfeld module over $\mathbb{k}\mathbb{Z}_2$. The quasicoactions $\delta : V \rightarrow \mathbb{k}\mathbb{Z}_2 \otimes V$ and $\delta_r : V^* \rightarrow V^* \otimes \mathbb{k}\mathbb{Z}_2$ are given by

$$\delta(v_1) = 1 \otimes v_1, \quad \delta(v_2) = s \otimes v_1; \quad \delta_r(f_1) = f_1 \otimes 1 + f_2 \otimes s, \quad \delta_r(f_2) = 0.$$

It follows that

$$I(V, \delta) \cap V^{\otimes 1} = \ker \delta = 0, \quad I(V^*, \delta) \cap V^{*\otimes 1} = \ker \delta_r = \mathbb{k}f_2.$$

The degree 1 component in the graded algebra $U(V, \delta)$ has dimension 2, whereas the degree 1 component in $U(V^*, \delta)$ is one-dimensional. It is not difficult to check that $f_1^{\otimes n}$ does not vanish under the quasibraided factorial $[\widetilde{n}]!_{\delta_r}$ in characteristic 0; therefore, $U(V^*, \delta)$ is isomorphic to the polynomial algebra $\mathbb{k}[f_1]$.

The algebra $U(V, \delta)$ is, however, not commutative. One may compute

$$\begin{aligned} [\widetilde{2}]!_{\delta}(v_1 \otimes v_2) &= 1 \otimes v_1 \otimes s \otimes v_1 + s \otimes v_1 \otimes 1 \otimes v_1, \\ [\widetilde{2}]!_{\delta}(v_2 \otimes v_1) &= s \otimes v_1 \otimes 1 \otimes v_1 - 1 \otimes v_1 \otimes s \otimes v_1, \end{aligned}$$

hence (recall Theorem 4.11) the commutator $v_1 \otimes v_2 - v_2 \otimes v_1$ is not in $I(V, \delta)$.

5. Braided Heisenberg doubles

In the previous section, we associated to every quasi-Yetter–Drinfeld module (V, δ) , $\dim V < \infty$, over a Hopf algebra H a pair of two-sided graded H -invariant ideals

$$I(V, \delta) \subset T(V), \quad I(V^*, \delta) \subset T(V^*)$$

which are the defining ideals in the minimal double $\bar{A}(V, \delta) \cong U(V, \delta) \otimes H \otimes U(V^*, \delta)$. However, the description of $I(V, \delta)$, $I(V^*, \delta)$ as kernels of quasibraided factorials may be far from satisfactory: the factorials are operators from $V^{\otimes n}$ to $(H \otimes V)^{\otimes n}$, which possibly is an infinite-dimensional space. We have also seen that the algebras $U(V, \delta)$ and $U(V^*, \delta)$ may not look similar at all (Example 4.22).

The goal of this section is to analyse $I(V, \delta)$, $I(V^*, \delta)$ and the minimal double $\bar{A}(V, \delta)$ in the case when (V, δ) is a Yetter–Drinfeld module over a Hopf algebra H (without the “quasi” prefix).

Write the H -coaction on V as $\delta: v \mapsto v^{(-1)} \otimes v^{(0)}$. Let $\Psi: V \otimes V \rightarrow V \otimes V$ be the braiding on the space V induced by the Yetter–Drinfeld structure:

$$\Psi(v \otimes w) = v^{(-1)} \triangleright w \otimes v^{(0)}.$$

We will show that $I(V, \delta)$ is the kernel of the Woronowicz symmetriser $\text{Wor}(\Psi) \in \text{End } T(V)$. This is because our factorial $[\tilde{n}]_\delta$ specialises, in the case of Yetter–Drinfeld module, to the braided factorial of Majid, which is an endomorphism of $V^{\otimes n}$. Braided doubles associated to Yetter–Drinfeld modules will never have pathological properties such as those demonstrated in Examples 4.20 and 4.22.

5.1. Free doubles $\tilde{A}(V, \delta)$ and $\tilde{A}(V, \Psi)$

Let V be a finite-dimensional Yetter–Drinfeld module over H . We denote by \triangleright the H -action on V , and by $v \mapsto v^{(-1)} \otimes v^{(0)} \in H \otimes V$ the H -coaction on V . Recall that the free braided double associated to V has triangular decomposition $T(V^*) \rtimes H \ltimes T(V)$, and the multiplication is defined by the relations

$$h \cdot v = (h_{(1)} \triangleright v) \cdot h_{(2)}, \quad f \cdot h = h_{(1)} \cdot (f \triangleleft h_{(2)}), \quad [f, v] = v^{(-1)} \langle f, v^{(0)} \rangle$$

for $v \in V$, $f \in V^*$, $h \in H$.

Note that these free braided doubles can be associated to any braided space with biinvertible braiding; that is, the Hopf algebra H does not have to be a part of the input. If Ψ is a biinvertible braiding on a finite-dimensional space V , then by 2.9 (V, Ψ) is a Yetter–Drinfeld module over the Hopf algebra H_Ψ , which is a canonical minimal realisation of the braiding Ψ . This yields a free braided double $\bar{A}(V, \Psi) \cong T(V) \otimes H_\Psi \otimes T(V^*)$ canonically associated to a braided space (V, Ψ) .

5.2. Braided integers and braided derivatives

The coaction $\delta: v \mapsto v^{(-1)} \otimes v^{(0)}$ on V satisfies the usual coaction property

$$v^{(-1)}_{(1)} \otimes v^{(-1)}_{(2)} \otimes v^{(0)} = v^{(-1)} \otimes v^{(0)(-1)} \otimes v^{(0)(0)}.$$

Hence the formula for the quasibraided integer $\widetilde{[n]}_\delta : V^{\otimes n} \rightarrow (H \otimes V)^{\otimes n}$ from Definition 4.9 can be rewritten as

$$\widetilde{[n]}_\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i^{(-1)} \triangleright (v_{i+1} \otimes \cdots \otimes v_n) \otimes v_i^{(0)(-1)} \otimes v_i^{(0)(0)}.$$

These operators can be expressed in terms of the braiding Ψ on V . We need the following

Definition 5.1. Let (V, Ψ) be a braided space. *Braided integers* are operators

$$[n]_\Psi = \text{id}_{V^{\otimes n}} + \Psi_{n-1,n} + \Psi_{n-1,n} \Psi_{n-2,n-1} + \cdots + \Psi_{n-1,n} \Psi_{n-2,n-1} \cdots \Psi_{1,2} \in \text{End}(V^{\otimes n}).$$

We are using the leg notation, thus $\Psi_{i,i+1}$ stands for the operator Ψ applied at positions $i, i + 1$ in the tensor product. In particular, $[1]_\Psi = \text{id}_V$ and $[2]_\Psi = \text{id}_{V^{\otimes 2}} + \Psi$.

Braided factorials are operators

$$[n]!_\Psi = ([1]_\Psi \otimes \text{id}^{\otimes n-1}) \circ ([2]_\Psi \otimes \text{id}^{\otimes n-2}) \circ \cdots \circ [n]_\Psi \in \text{End}(V^{\otimes n}).$$

Here is a new formula for quasibraided integers and factorials on a Yetter–Drinfeld module:

Lemma 5.2. Let V be a Yetter–Drinfeld module over H , with coaction δ inducing a braiding Ψ . Then

$$\widetilde{[n]}_\delta = (\text{id}_V^{\otimes n-1} \otimes \delta) \circ [n]_\Psi, \quad \widetilde{[n]}!_\delta = \delta^{\otimes n} \circ [n]!_\Psi.$$

Let $\pi : H \otimes V \rightarrow V$ be the projection $\pi = \epsilon \otimes \text{id}_V$, where ϵ is the counit of H . Then

$$[n]_\Psi = (\text{id}_V^{\otimes n-1} \otimes \pi) \circ \widetilde{[n]}_\delta, \quad [n]!_\Psi = \pi^{\otimes n} \circ \widetilde{[n]}!_\delta.$$

Proof. The formula for $\widetilde{[n]}_\delta$ follows from $\Psi(v \otimes w) = v^{(-1)} \triangleright w \otimes v^{(0)}$. For example, $v_1^{(-1)} \triangleright (v_2 \otimes \cdots \otimes v_n) \otimes v_1^{(0)(-1)} \otimes v_1^{(0)(0)}$ rewrites as $\delta_n \Psi_{n-1,n} \cdots \Psi_{23} \Psi_{12}(v_1 \otimes \cdots \otimes v_n)$. The formula $\widetilde{[n]}!_\delta$ is then immediate from the definition of the quasibraided factorial. The rest is an immediate consequence of counitality of the coaction, $\pi \circ \delta = \text{id}_V$. \square

Braided integers and braided factorials, which we have just obtained as a particular case of their quasibraided analogues, were introduced by Majid [31] (a book reference is [32, 10.4]). When the braided space is 1-dimensional, the braiding Ψ is multiplication by constant $q \in \mathbb{k}$, and braided integers are the well-known q -integers $[n]_q = \frac{1-q^n}{1-q}$. Another important form of the braided factorial is

5.3. The Woronowicz symmetriser

For a permutation σ in the symmetric group \mathbb{S}_n , let $\sigma = (i_1 i_1 + 1) \cdots (i_l i_l + 1)$ be a reduced (i.e., shortest) decomposition of σ into elementary transpositions. For a braiding Ψ on V , define Ψ_σ to be equal to the operator $\Psi_{i_1, i_1+1} \cdots \Psi_{i_l, i_l+1}$ on $V^{\otimes n}$ (this does not depend on the choice of

a reduced decomposition of σ because Ψ satisfies the braid equation). The above expression for the braided factorial expands into

$$[n]!_{\Psi} = \sum_{\sigma \in \mathbb{S}_n} \Psi_{\sigma}.$$

This endomorphism of $V^{\otimes n}$, associated to a braiding Ψ , is called the *Woronowicz symmetriser of degree n* . It was introduced by Woronowicz in [51] (as an “antisymmetriser” with $-\Psi$ instead of Ψ), and its factorial expression was given by Majid [32, 10.4].

We will consider an endomorphism of the whole tensor algebra $T(V)$, given on tensor powers by the braided factorials:

$$\text{Wor}(\Psi) : T(V) \rightarrow T(V), \quad \text{Wor}(\Psi)|_{V^{\otimes n}} := [n]!_{\Psi}.$$

Note that $[0]!_{\Psi} = 1$ and $[1]!_{\Psi} = \text{id}_V$. We refer to $\text{Wor}(\Psi)$ simply as the *Woronowicz symmetriser*.

In the following definition, we identify $V^* \otimes V^*$ with the dual space to $V \otimes V$ in a standard way via $\langle f \otimes g, v \otimes w \rangle = \langle f, v \rangle \langle g, w \rangle$. This allows us to view Ψ^* as a braiding on V^* .

Definition 5.3. Let (V, Ψ) be a braided space. The graded algebras

$$\mathcal{B}(V, \Psi) = T(V) / \ker \text{Wor}(\Psi), \quad \mathcal{B}(V^*, \Psi^*) = T(V^*) / \ker \text{Wor}(\Psi^*),$$

are called the *Nichols–Woronowicz algebras* associated to (V, Ψ) .

We are now ready to give a description of minimal doubles specific to the case Yetter–Drinfeld modules.

Theorem 5.4 (*Doubles of Nichols–Woronowicz algebras*). *Let (V, δ) be a Yetter–Drinfeld module for a Hopf algebra H , and let Ψ be the induced braiding on V . Then*

$$U(V, \delta) = \mathcal{B}(V, \Psi), \quad U(V^*, \delta) = \mathcal{B}(V^*, \Psi^*)$$

are Nichols–Woronowicz algebras. The minimal double $\mathcal{H}_V := \bar{A}(V, \delta)$ has triangular decomposition

$$\mathcal{H}_V = \mathcal{B}(V, \Psi) \rtimes H \ltimes \mathcal{B}(V^*, \Psi^*).$$

Proof. By Theorem 4.11, $I(V, \delta) = \bigoplus_n \ker \widetilde{[n]}!_{\delta}$ where δ is the coaction on V . But by Lemma 5.2, $\ker \widetilde{[n]}!_{\delta} = \ker [n]_{\Psi}$. Hence $I(V, \delta) = \ker \text{Wor}(\Psi)$ as required. The formula $I(V^*, \delta) = \ker \text{Wor}(\Psi^*)$ can be obtained in a similar way, using Remark 4.12. \square

We will call \mathcal{H}_V the *braided Heisenberg double* associated to the Yetter–Drinfeld module V . If H is the trivial Hopf algebra, $H = \mathbb{k}$, then \mathcal{H}_V is the Heisenberg–Weyl algebra $S(V) \otimes S(V^*)$.

Similarly to the free doubles, the Hopf algebra H does not need to be in the picture: to any braided space (V, Ψ) , of finite dimension and with biinvertible braiding, is associated the minimal braided Heisenberg double $\mathcal{H}_{(V, \Psi)}$, defined as in the Proposition with $H = H_{\Psi}$. Observe that the ideals $I(V, \delta)$, $I(V^*, \delta)$ depend only on the braiding Ψ on V , not on the Hopf algebra H ; and that it automatically follows that $\ker \text{Wor}(\Psi)$ is a two-sided ideal in $T(V)$.

An important feature of the braided Heisenberg double \mathcal{H}_V is that its Harish-Chandra pairing is non-degenerate. In fact, a stronger property holds, and \mathcal{H}_V satisfies the conditions in Proposition 4.19:

Lemma 5.5. *The scalar-valued pairing $\epsilon((\cdot, \cdot)_H)$ between $\mathcal{B}(V^*, \Psi^*)$ and $\mathcal{B}(V, \Psi)$ in \mathcal{H}_V is non-degenerate.*

Proof. Let $\phi \in V^{*\otimes n}$ and $b \in V^{\otimes n}$. One deduces from Proposition 4.17 and Lemma 5.2 that

$$\epsilon((\phi, b)_H) = \langle \phi, [n]!_{\Psi} b \rangle.$$

This is a non-degenerate pairing between $\mathcal{B}(V^*, \Psi^*)$ and $\mathcal{B}(V, \Psi)$, because the kernels of the braided factorials are quotiented out. \square

The non-degenerate pairing between $\mathcal{B}(V^*, \Psi^*)$ and $\mathcal{B}(V, \Psi)$ induces on each of these algebras a coassociative coproduct. Via the associativity of multiplication in \mathcal{H}_V , or otherwise, it can be shown that $\mathcal{B}(V^*, \Psi^*)$ and $\mathcal{B}(V, \Psi)$ become dually paired *braided Hopf algebras*. Braided Hopf algebras are a relatively recent branch of the Hopf algebra theory and quantum algebra. We will not give details here and refer the reader to [34]. The braided coproduct on $v \in V \subset \mathcal{B}(V)$ is $\underline{\Delta}v = v \otimes 1 + 1 \otimes v$; the braided coproduct is not multiplicative, but rather braided-multiplicative, and this allows one to extend $\underline{\Delta}$ to the whole of $\mathcal{B}(V, \Psi)$.

Remark 5.6. In fact, one can construct a braided Heisenberg double of any pair of dually paired braided Hopf algebras. Such a construction should be viewed as a quantum analogue of the algebra $\mathcal{H}(A)$ referred to as the *Heisenberg double* of A [44,26], where A is a finite-dimensional Hopf algebra. One has $\mathcal{H}(A) \cong A \otimes A^*$ with defining relation $\phi a = \langle \phi_{(1)}, a_{(2)} \rangle a_{(1)} \phi_{(2)}$ between $\phi \in A^*$ and $a \in A$. The Heisenberg double produces canonical solutions to the *pentagon equation* in the same way as the Drinfeld double $D(A)$ works for the quantum Yang–Baxter equation, see [4] (based on earlier work by Woronowicz) and a more algebraic exposition in [37]. However, $\mathcal{H}(A)$ is not a Hopf algebra; it is a simple (matrix) algebra [38, 9.4.3].

5.4. *The Hopf algebra structure on $\mathcal{B}(V, \Psi) \rtimes H$ and on $H \ltimes \mathcal{B}(V^*, \Psi^*)$*

It follows from the theory of braided Hopf algebras that, while the Nichols–Woronowicz algebras $\mathcal{B}(V, \Psi)$ and $\mathcal{B}(V^*, \Psi^*)$ are braided Hopf algebras, the algebras $\mathcal{B}(V) \rtimes H$ and $H \ltimes \mathcal{B}(V^*)$ have the structure of ordinary Hopf algebras. This structure is called *biproduct bosonisation*, and is due to Majid. We give the following proposition without proof; it can be deduced from [33].

Proposition 5.7. *Let V be a Yetter–Drinfeld module over a Hopf algebra H , with braiding Ψ . The algebra $\mathcal{B}(V, \Psi) \rtimes H$ has the structure of an ordinary Hopf algebra, which contains $\mathcal{B}(V, \Psi)$ as a subalgebra and H as a sub-Hopf algebra. Write a typical element of $\mathcal{B}(V, \Psi) \rtimes H$ as $\phi \cdot h$ where $\phi \in \mathcal{B}(V, \Psi)$ and $h \in H$. The coproduct on $v \cdot 1$, where $v \in V$, is defined by*

$$\Delta(v \cdot 1) = (v \cdot 1) \otimes (1 \cdot 1) + (1 \cdot v^{(-1)}) \otimes (v^{(0)} \cdot 1),$$

and extends to $\mathcal{B}(V, \Psi) \rtimes H$ by multiplicativity.

Biproduct bosonisations, like the one given by the proposition, are a powerful tool in the structural theory of Hopf algebras. We will now use a biproduct bosonisation to obtain the following

Example 5.8 (*Counterexample to the third Kaplansky’s conjecture*). As we mentioned in Remark 4.21, a counterexample to the third Kaplansky’s conjecture is a Hopf algebra over a field of characteristic 0, which has a nonzero central nilpotent element.

It is easy to obtain a counterexample which is a braided Hopf algebra (namely, a Nichols–Woronowicz algebra). Let V be a finite-dimensional vector space with braiding $\Psi(v \otimes w) = -\tau(v \otimes w) = -w \otimes v$. The Woronowicz symmetriser associated to $-\tau$ is the standard antisymmetriser:

$$[n]!_{-\tau} = \sum_{\sigma \in \mathbb{S}_n} (-1)^{\ell(\sigma)} \sigma \quad \text{acting on } V^{\otimes n}.$$

It follows that the Nichols–Woronowicz algebra of $(V, -\tau)$ is the exterior algebra $\wedge V$.

Let $n = \dim V > 0$. Consider the element ω spanning the top degree, $\wedge^n V$, of the exterior algebra. Clearly, ω is a nonzero nilpotent in $\wedge V$ as $\omega \wedge \omega = 0$. If $\dim V$ is even, ω is central in $\wedge V$.

To obtain an ordinary Hopf algebra, let us consider the biproduct bosonisation of $\wedge V$ over the Hopf algebra $H_{-\tau}$, which is the minimal Hopf algebra realising the braiding $-\tau$. According to 2.9, $H_{-\tau} = \mathbb{k}\mathbb{Z}_2$. One can check that the action of $\mathbb{Z}_2 = \{1, s\}$ on V is given by $s(v) = -v$, and the coaction is $\delta(v) = s \otimes v$ for any $v \in V$. If $n = \dim V$ is even, $s(\omega) = \omega$, hence ω commutes with s and is central nilpotent in the biproduct bosonisation $\wedge V \rtimes \mathbb{k}\mathbb{Z}_2$ given by Proposition 5.7.

The dimension of the Hopf algebra $\wedge V \rtimes \mathbb{k}\mathbb{Z}_2$ is 2^{n+1} . The minimum is 8 for $n = 2$.

Remark 5.9. Note that in general, there is no canonical Hopf algebra structure on $B \otimes H \otimes B'$ where B, B' are dually paired braided Hopf algebras. The difficulty in some of existing approaches to “doubling” a braided Hopf algebra is that the double is expected to be a Hopf algebra. It is in fact possible to “double” a braided Hopf algebra in ${}^H_H\mathcal{YD}$ if H is a self-dual Hopf algebra, such as $\mathbb{k}G$ where G is a finitely generated Abelian group. A principal example of such a double which is a Hopf algebra is the construction of the quantised universal enveloping algebra $U_q(\mathfrak{g})$ as a braided double of two Nichols–Woronowicz algebras [27,33].

5.5. *Mixed Yetter–Drinfeld structures and compatible braidings*

In the rest of this section, we consider minimal doubles corresponding to quasi-Yetter–Drinfeld modules V of the following special structure.

Let V be a finite-dimensional module over a Hopf algebra H , with several coactions $\delta_1, \dots, \delta_N$ of H on V , each of them Yetter–Drinfeld compatible with the action. Let t_1, t_2, \dots, t_N be scalar parameters. Put

$$\delta = t_1\delta_1 + \dots + t_N\delta_N : V^* \otimes V \rightarrow H,$$

so that δ is a Yetter–Drinfeld quasicoaction on V (in general, not a coaction). We will refer to this as a *mixed Yetter–Drinfeld structure*. If Ψ_k is the braiding on V induced by the Yetter–Drinfeld coaction δ_k , the mixed Yetter–Drinfeld structure realises the endomorphism

$$t_1\Psi_1 + \dots + t_N\Psi_N \in \text{End}(V \otimes V).$$

Note that the braidings Ψ_k satisfy the compatibility equation in Definition 2.21.

We will now give a description of the minimal braided double $\tilde{A}(V, \delta)$, associated to a mixed Yetter–Drinfeld structure on V , in the case when the coefficients t_1, \dots, t_N are *generic*. This applies, for example, if t_1, \dots, t_N are independent formal parameters. If the t_k are elements of \mathbb{k} , ‘generic’ will mean that (t_1, \dots, t_N) are outside of a union of countably many hyperplanes in \mathbb{k}^N ; obviously, generic tuples are guaranteed to exist only if \mathbb{k} is uncountable. When we regard a braided integer $[m]_\Psi$ as an operator on $V^{\otimes n}$, $n > m$, we imply that it acts on the first m tensor components in $V^{\otimes n}$.

Proposition 5.10 (Minimal doubles for compatible braidings). *For generic coefficients t_1, \dots, t_N , the graded components $I_n = I_n(V, \delta)$ of the defining ideal $I(V, \delta)$ in the minimal double $\tilde{A}(V, \delta)$ are given by $I_0 = I_1 = 0$,*

$$I_n = \bigcap \ker([2]_{\Psi_{k_2}} [3]_{\Psi_{k_3}} \dots [n]_{\Psi_{k_n}}),$$

for $n \geq 2$, where the intersection on the right is over all sequences $\mathbf{k} = (k_2, \dots, k_n)$ in $\{1, \dots, N\}^{n-1}$.

Remark 5.11. If the parameters t_k are not generic, the ideals $I(V, \delta)$, $I(V^*, \delta)$ may only be bigger than in the generic case.

Proof of Proposition 5.10. It is easy to see that the quasibraided integer $\tilde{[n]}_\delta$ is given by $\sum_k t_k (\text{id}_V^{\otimes n-1} \otimes \delta_k) [n]_{\Psi_k}$. By Theorem 4.11 and its proof, I_n consists of all $\in V^{\otimes n}$ such that $\tilde{[n]}_\delta b$ lies in $I_{n-1} \otimes H \otimes V$; for generic t_k , this means $[n]_{\Psi_k} b \in I_{n-1} \otimes V$ for every k . Trivially, $I_0 = 0$ and $[1]_{\Psi_k} = \text{id}_V$. The proposition now follows by induction. \square

5.6. Deformation of the Nichols–Woronowicz algebra

We will now consider the example of a pair of compatible braidings, provided by Lemma 2.32, and obtain an interesting (non-flat) deformation the Nichols–Woronowicz algebra.

Let (V, Ψ) be a finite-dimensional braided space with biinvertible braiding. Assume that Ψ is compatible with the trivial braiding τ (this is a homogeneous quadratic constraint on Ψ). Then (V, Ψ) can be realised as a Yetter–Drinfeld module over a cocommutative Hopf algebra H_Ψ , see Claim 2.31. Let $v \mapsto v^{(-1)} \otimes v^{(0)}$ be the coaction of H_Ψ on V .

By Lemma 2.32, there is a Yetter–Drinfeld quasicoaction on V given by $\delta_{\Psi, t\tau}(v) = v^{(-1)} \otimes v^{(0)} + t \cdot 1 \otimes v$. The free double $\tilde{A}(V, \delta_{\Psi, t\tau})$ has commutation relation $[f, v] = v^{(-1)} \langle f, v^{(0)} \rangle + t \langle f, v \rangle \cdot 1$ between $f \in V^*$ and $v \in V$. The introduction of the extra term $t \langle f, v \rangle \cdot 1$ makes the maximal triangular ideal smaller:

Proposition 5.12 (Deformed braided Heisenberg double). *For t generic,*

$$\tilde{A}(V, \delta_{\Psi, t\tau}) \cong \mathcal{B}_\tau(V, \Psi) \rtimes H_\Psi \rtimes \mathcal{B}_\tau(V^*, \Psi^*),$$

where the graded algebra $\mathcal{B}_\tau(V, \Psi)$ does not depend on t , and its homogeneous components are $\mathcal{B}_\tau(V, \Psi)_n = V^{\otimes n} / \ker[n]!_{\Psi, \tau}$. The ‘deformed braided factorial’ $[n]!_{\Psi, \tau}$ can be written as

$$[n]!_{\Psi, \tau} = ([2]_\Psi + u_2[2]_\tau) \dots ([n-1]_\Psi + u_{n-1}[n-1]_\tau) ([n]_\Psi + u_n[n]_\tau)$$

with independent formal parameters u_k .

We will call $\mathcal{B}_\tau(V, \Psi)$ the *deformed Nichols–Woronowicz algebra*. Note that this construction works only for Ψ satisfying the quadratic equation of compatibility with the trivial braiding. Clearly there is a surjective map $\mathcal{B}_\tau(V, \Psi) \twoheadrightarrow \mathcal{B}(V, \Psi)$ onto the Nichols–Woronowicz algebra of (V, Ψ) .

Proof of Proposition 5.12. The result follows immediately from Proposition 5.10, because obviously

$$\ker[n]!_{\Psi, \tau} = \bigcap \ker[2]_{\Psi_{k_2}} \dots [n-1]_{\Psi_{k_{n-1}}} [n]_{\Psi_{k_n}},$$

where the intersection on the right is over all $k_i \in \{1, 2\}$, $\Psi_1 := \Psi$ and $\Psi_2 := \tau$. \square

5.7. *Minimal quadratic doubles associated to compatible braidings*

We will finish the Section with “quadratic versions” of all braided double constructions presented here. Recall the definition of the minimal quadratic double $\tilde{A}_{quad}(V, \delta)$ from Section 4.

Lemma 5.13. *Let V be a finite-dimensional vector space.*

- (1) *If Ψ is a biinvertible braiding on V , the minimal quadratic double associated to (V, Ψ) is $\mathcal{B}_{quad}(V, \Psi) \otimes H_\Psi \otimes \mathcal{B}_{quad}(V^*, \Psi^*)$. Here*

$$\mathcal{B}_{quad}(V, \Psi) = T(V) / \langle \ker(\text{id}_{V^{\otimes 2}} + \Psi) \rangle$$

is the quadratic cover of the Nichols–Woronowicz algebra $\mathcal{B}(V, \Psi)$.

- (2) *If Ψ_1, \dots, Ψ_N are braidings on V arising from Yetter–Drinfeld coactions over the same Hopf algebra H , the corresponding minimal quadratic double*

$$T(V) / I_{quad}(V, \delta) \otimes H \otimes T(V^*) / I_{quad}^*(V, \delta)$$

has

$$I_{quad}(V, \delta) = \langle \bigcap_{k=1}^N \ker(\text{id}_{V^{\otimes 2}} + \Psi_k) \rangle.$$

- (3) *If Ψ is a braiding on V compatible with τ , the above construction with $\Psi_1 = \Psi$, $\Psi_2 = \tau$ leads to the deformation*

$$\mathcal{B}_{quad_\tau}(V, \Psi) = T(V) / \langle \ker(\text{id}_{V^{\otimes 2}} + \Psi) \cap \wedge^2 V \rangle$$

of the algebra $\mathcal{B}_{quad}(V, \Psi)$. Here $\wedge^2 V$ is the space of skew-symmetric tensors in $V^{\otimes 2}$ which is the kernel of $\text{id}_{V^{\otimes 2}} + \tau$.

Proof. All statements are obtained by leaving only quadratic relations in $T(V)$, $T(V^*)$ in minimal doubles constructed in this section. \square

6. The category of braided doubles over H . Perfect subquotients

6.1. Morphisms between braided doubles over H

We will now describe the category \mathcal{D}_H , whose objects are braided doubles over a Hopf algebra H . As in 4.2, for a finite-dimensional quasi-Yetter–Drinfeld module (V, δ) over H denote the set of (V, δ) -braided doubles by $\mathcal{D}(V, \delta)$. We have

$$\text{Ob } \mathcal{D}_H = \bigcup_{(V, \delta) \in \text{Ob}_H \mathcal{QYD}} \mathcal{D}(V, \delta).$$

Before we define morphisms in \mathcal{D}_H , let us introduce a small bit of notation. If $\mu : V \rightarrow W$ is a map of vector spaces, denote by $T(\mu) : T(V) \rightarrow T(W)$ the linear map which coincides with $\mu^{\otimes n}$ on $V^{\otimes n}$. This is an algebra homomorphism. If $I \subset T(V)$ and $J \subset T(W)$ are two-sided ideals such that $\mu(I) \subseteq J$, there is an algebra homomorphism $\overline{T(\mu)} : T(V)/I \rightarrow T(W)/J$. We say that this map is *induced by μ* .

We define morphisms in \mathcal{D}_H in a natural way: they must be triangular maps between algebras with triangular decomposition over H (see A.1), and should come from the maps between the generating quasi-YD modules.

Definition 6.1. Let $(V, \delta_V), (W, \delta_W)$ be two finite-dimensional quasi-Yetter–Drinfeld modules over a Hopf algebra H , and let

$$A = T(V)/I^- \rtimes H \rtimes T(V^*)/I^+, \quad B = T(W)/J^- \rtimes H \rtimes T(W^*)/J^+$$

be a (V, δ_V) - and a (W, δ_W) -braided double, respectively. Morphisms between A and B are algebra maps

$$\overline{T(\mu)} \otimes \text{id}_H \otimes \overline{T(\nu)} : A \rightarrow B,$$

induced by a pair of H -module maps $\mu : V \rightarrow W, \nu : W \rightarrow V$.

Lemma 6.2. Let $(V, \delta_V), (W, \delta_W)$ be finite-dimensional quasi-YD modules over H .

1. If a pair $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ of H -module maps induces a morphism between a (V, δ_V) -braided double and a (W, δ_W) -braided double, then $\delta_V = (\text{id}_H \otimes \nu) \circ \delta_W \circ \mu$.
2. Any H -module maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ satisfying 1. induce a morphism $\tilde{A}(V, \delta_V) \rightarrow \tilde{A}(W, \delta_W)$ between free braided doubles.

Proof. 1. Take $v \in V$ and $f \in V^*$. The commutator $[f, v]$ in any (V, δ_V) -braided double is equal to $v^{[-1]} \langle f, v^{[0]} \rangle$. As μ, ν induce a morphism of braided doubles, the same commutator (an element of H) must be equal to

$$\mu(v)^{[-1]} \langle \nu^*(f), \mu(v)^{[0]} \rangle_W = \mu(v)^{[-1]} \langle f, \nu(\mu(v)^{[0]}) \rangle_V.$$

Here, $\mu(v)^{[-1]} \otimes \mu(v)^{[0]}$ is $\delta_W(\mu(v))$; the pairing between W^* and W is denoted by $\langle \cdot, \cdot \rangle_W$, the pairing between V^* and V is denoted by $\langle \cdot, \cdot \rangle_V$. Since $f \in V^*$ is arbitrary, it follows that $\delta_V = (\text{id}_H \otimes \nu) \circ \delta_W \circ \mu$.

2. Let us show that

$$j = T(\mu) \otimes \text{id}_H \otimes T(v^*) : \tilde{A}(V, \delta_V) \rightarrow \tilde{A}(W, \delta_W)$$

is a map of algebras. The semidirect product relations in $\tilde{A}(V, \delta_V)$ are preserved by j because $T(\mu), T(v^*)$ are H -equivariant maps. The commutator relation between $v^*(f)$ and $\mu(v)$ is equivalent to $\delta_V = (\text{id}_H \otimes v) \circ \delta_W \circ \mu$, as was shown in the proof of part 1; and there are no other relations in $\tilde{A}(V, \delta_V)$. \square

Thus, morphisms between free braided doubles are the same as pairs of maps between quasi-YD modules, satisfying

Definition 6.3 (Subquotients). Let V, W be quasi-Yetter–Drinfeld modules over a Hopf algebra H . We say that V is a *subquotient* of W via the maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$, if μ, ν are H -module maps such that the quasicoaction δ_V is induced from δ_W via $\delta_V = (\text{id}_H \otimes \nu) \circ \delta_W \circ \mu$.

We stress that none of the maps μ, ν in this definition is required to be injective or surjective.

Remark 6.4. Let $\mathcal{D}_H^{\text{free}}$ be the full subcategory of \mathcal{D}_H consisting of all free braided doubles over H . It follows from Lemma 6.2 that $\mathcal{D}_H^{\text{free}}$ is equivalent to the following category:

- objects: finite-dimensional quasi-YD modules V over H ;
- morphisms between V and W : diagrams $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ which make V a subquotient of W ;
- composition: the composition of $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ and $W \xrightarrow{\mu'} X \xrightarrow{\nu'} W$ is the diagram $V \xrightarrow{\mu' \circ \mu} X \xrightarrow{\nu \circ \nu'} V$.

6.2. *Perfect subquotients*

Let $(V, \delta_V), (W, \delta_W)$ be finite-dimensional quasi-YD modules over H . If V is a subquotient of W , we have a map $j : \tilde{A}(V, \delta_V) \rightarrow \tilde{A}(W, \delta_W)$ between free braided doubles. Of course, we may consider a composite map

$$\bar{j} : \tilde{A}(V, \delta_V) \xrightarrow{j} \tilde{A}(W, \delta_W) \twoheadrightarrow \bar{A}(W, \delta_W)$$

into the minimal double associated to W . Hence there is some (V, δ) -braided double $\tilde{A}(V, \delta) / \ker \bar{j}$ which embeds injectively in $\bar{A}(W, \delta_W)$; but this may not be the minimal double $\bar{A}(V, \delta)$. This is the content of the following

Proposition 6.5. *If V is a subquotient of W via the maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$, then $I(V, \delta_V)$ contains the preimage $T(\mu)^{-1}I(W, \delta_W)$.*

Proof. For finite-dimensional V and W this is an immediate consequence of the above: the kernel $T(\mu)^{-1}I(W, \delta_W) \cdot HT(V^*) + T(V)H \cdot T(v^*)^{-1}I(W^*)$ of \bar{j} is a triangular ideal in $\tilde{A}(V, \delta)$, hence is contained in $I(V, \delta_V)HT(V^*) + T(V)HI(V^*, \delta_V)$.

Here is an alternative proof which works for infinite-dimensional V, W . Direct computation shows

$$(\text{id}_H \otimes \nu)^{\otimes n} \circ [\widetilde{n}]!_{\delta_W} \circ \mu^{\otimes n} = [\widetilde{n}]!_{\delta_V}.$$

Therefore, $I(V, \delta_V) \cap V^{\otimes n} = \ker [\widetilde{n}]!_{\delta_V}$ is equal to $T(\mu)^{-1} \ker((\text{id}_H \otimes \mu)^{\otimes n} \circ [\widetilde{n}]!_{\delta_W})$. The latter contains $T(\mu)^{-1} \ker [\widetilde{n}]!_{\delta_W} = (T(\mu)^{-1} I(W, \delta_W)) \cap V^{\otimes n}$. \square

The proposition immediately leads to the following

Definition 6.6. Let $(V, \delta_V), (W, \delta_W)$ be quasi-YD modules such that V is a subquotient of W via the maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$. We say that V is a *perfect subquotient* of W , if $I(V, \delta_V) = T(\mu)^{-1} I(W, \delta_W)$.

Remark 6.7. Observe that if V is a perfect subquotient of W via the maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$, and W is a perfect subquotient of X via the maps $W \xrightarrow{\mu'} X \xrightarrow{\nu'} W$, then V is a perfect subquotient of X via the composition of these two diagrams.

6.3. Every quasi-YD module is a perfect subquotient of a Yetter–Drinfeld module

In the previous section, we identified doubles of Nichols–Woronowicz algebras as a distinguished class of minimal doubles. Now, given a quasi-Yetter–Drinfeld module (V, δ) over a Hopf algebra H , we would like to embed the algebra $U(V, \delta)$ (the “lower part” of the minimal double $\bar{A}(V, \delta)$) in a Nichols–Woronowicz algebra of some Yetter–Drinfeld module Y . This is achieved if V is a perfect subquotient of Y .

We show in the next theorem that for any quasi-YD module V , there exists a Yetter–Drinfeld module Y such that V is a perfect subquotient of Y . However, only in some cases can we guarantee that Y can be chosen to be finite-dimensional.

The theorem will use the following lemma:

Lemma 6.8. Let (V, δ) be a quasi-Yetter–Drinfeld module over a Hopf algebra H . Let $Y(V)$ be the space $H \otimes V$.

(a) $Y(V)$ is a module over H with respect to the following action:

$$h \triangleright (x \otimes v) = h_{(1)} x S h_{(3)} \otimes h_{(2)} \triangleright v, \quad x \in H, v \in V.$$

(b) The Yetter–Drinfeld condition on δ is equivalent to the map $\delta : V \rightarrow H \otimes V = Y(V)$ being a morphism of H -modules.

(c) The map

$$Y(V) \rightarrow H \otimes Y(V), \quad x \otimes v \mapsto x_{(1)} \otimes x_{(2)} \otimes v$$

is an H -coaction on $Y(V)$, which makes $Y(V)$ a Yetter–Drinfeld module over H .

Proof. It is straightforward to check that the formula given in (a) indeed defines an action of H on $Y(V)$. Part (b) follows by rewriting the definition of a Yetter–Drinfeld module over H in an equivalent form suitable for Hopf algebras:

$$\delta(h \triangleright v) = h_{(1)}v^{(-1)}Sh_{(3)} \otimes h_{(2)} \triangleright v^{(0)}.$$

Part (c) is also easy, and is left as an exercise to the reader. \square

Theorem 6.9.

1. Let V be a quasi-Yetter–Drinfeld module over a Hopf algebra H , with quasicoaction $\delta : V \rightarrow H \otimes V$. Then V is a perfect subquotient of the Yetter–Drinfeld module $Y(V)$, via the maps $V \xrightarrow{\delta} Y(V) \xrightarrow{\epsilon \otimes \text{id}_V} V$.
2. If V is finite-dimensional and
 - $\dim H < \infty$, or
 - H is commutative and cocommutative,
 then there exists a finite-dimensional Yetter–Drinfeld module Y of which V is a perfect subquotient.

Proof. The map $\mu = \delta : V \rightarrow Y(V)$ is an H -module map by Lemma 6.8. It is easy to see that $\nu = \epsilon \otimes \text{id}_V$ is also an H -module map. Let us check that $V \xrightarrow{\mu} Y(V) \xrightarrow{\nu} V$ is indeed a subquotient: $\mu(v)^{(-1)} \otimes \nu(\mu(v)^{(0)})$ is equal to $v^{[-1]}_{(1)} \otimes \epsilon(v^{[-1]}_{(2)})v^{[0]} = \delta(v)$ as required.

To show that V is a perfect subquotient of $Y(V)$, denote the braiding on $Y(V)$ by Ψ and observe that

$$[\widetilde{n}]!_{\delta} = [n]_{\Psi} \circ \delta^{\otimes n},$$

where both sides are maps $V^{\otimes n} \rightarrow Y(V)^{\otimes n}$. This formula is straightforward to verify, and immediately implies that $\ker [\widetilde{n}]!_{\delta} = (\delta^{\otimes n})^{-1} \ker [n]_{\Psi}$, precisely as required by the definition of a perfect subquotient.

Now assume that V is finite-dimensional. If $\dim H < \infty$, we may take Y to be the Yetter–Drinfeld module $Y(V)$, because $\dim Y(V) < \infty$.

If H is commutative and cocommutative, but not necessarily of finite dimension, it is enough to choose a finite-dimensional Yetter–Drinfeld submodule in $Y(V)$ containing $\delta(V)$. Let $H' \subset H$ be a subspace, $\dim H' < \infty$, such that $\delta(V) \subset H' \otimes V$. By the fundamental theorem on coalgebras [47, Corollary 2.2.2], $H' \subset C \subset H$ where C is a finite-dimensional subcoalgebra of H . Put $Y = C \otimes V \subset Y(V)$. Then Y is a submodule of $Y(V)$, because, by commutativity and cocommutativity of H ,

$$h \triangleright (c \otimes v) = c \otimes (h \triangleright v)$$

for any $c \otimes v \in C \otimes V$; clearly, Y is a subcomodule of $Y(V)$ because C is a subcoalgebra of H . The quasi-YD module V will be a perfect subquotient of Y via the maps μ and $\nu|_{C \otimes V}$. \square

6.4. Subquotients in right quasi-Yetter–Drinfeld modules

It is also useful to consider right quasi-YD modules over H . If (V, δ) is a (left) quasi-YD module, then V^* is naturally a right quasi-YD module over H , with right quasicoaction $\delta_r : V^* \rightarrow$

$V^* \otimes H$ as defined in 4.12. There is a notion of *subquotient* for right quasi-YD modules. A left quasi-YD module V is a subquotient of W via the maps $V \xrightarrow{\mu} W \xrightarrow{\nu} V$, if and only if the right quasi-YD module V^* is a subquotient of W^* via the maps $V^* \xrightarrow{\nu^*} W^* \xrightarrow{\mu^*} V^*$.

The notion of *perfect subquotient* is also defined for right modules. Theorem 6.9 clearly admits a version for right quasi-Yetter–Drinfeld modules. A word of warning: if $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ is a perfect subquotient, $V^* \xrightarrow{\nu^*} W^* \xrightarrow{\mu^*} V^*$ is not necessarily a perfect subquotient.

If μ, ν is a pair of morphisms such that both $V \xrightarrow{\mu} W \xrightarrow{\nu} V$ and $V^* \xrightarrow{\nu^*} W^* \xrightarrow{\mu^*} V^*$ are perfect subquotients, the minimal double $\tilde{A}(V, \delta)$ embeds as a subdouble in the minimal double $\tilde{A}(W, \delta_W)$.

Ideally, we would like to look for such an embedding of any minimal double $\tilde{A}(V, \delta)$ into a braided Heisenberg double, corresponding to some Yetter–Drinfeld module Y . But in general, we have no tools to achieve this: although the pair of morphisms $V \xrightarrow{\delta} Y(V) \xrightarrow{\epsilon \otimes \text{id}_V} V$, produced by Theorem 6.9 for finite-dimensional H , is a perfect subquotient, the adjoint map $(\epsilon \otimes \text{id}_V)^*: V^* \rightarrow V^* \otimes H^*$ does not typically give a perfect subquotient. It is instructive to check this when H is a group algebra of a finite group.

However, we manage to embed those rational Cherednik algebras, which are minimal doubles of a special kind over a group algebra, in braided Heisenberg doubles. This will be done in the next section.

7. Rational Cherednik algebras

Up to now we have been dealing with common properties of braided doubles attached to any finite-dimensional module over some Hopf algebra H . In this section, we will soon fix a \mathbb{k} -vector space V , $\dim V < \infty$, and take the group algebra $\mathbb{k}G$ of an *irreducible linear group* $G \leq GL(V)$ as H . Our task is to study braided doubles $U^- \rtimes \mathbb{k}G \rtimes U^+$ where U^- and U^+ are commutative algebras.

7.1. Quasi-Yetter–Drinfeld modules over a group algebra

Initially, let G be an arbitrary group. The coproduct, counit and antipode in $H = \mathbb{k}G$ are defined on $g \in G$ by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1},$$

respectively. Unlike for general Hopf algebras, we use traditional notation $(g, v) \mapsto g(v)$ for an action of G on a space V . The definition of a quasi-Yetter–Drinfeld module is restated in the group algebra case as follows:

Lemma 7.1. *A quasi-Yetter–Drinfeld module over a group G is a representation V of G , equipped with a map*

$$\delta: V \rightarrow \mathbb{k}G \otimes V, \quad \delta(v) = \sum_{h \in G} h \otimes L_h(v),$$

where the linear maps $L_h \in \text{End}(V)$ satisfy

$$g(L_h(v)) = L_{ghg^{-1}}(g(v)), \quad g, h \in G, \quad v \in V.$$

Proof. Note that $g(L_h(v)) = L_{ghg^{-1}}(g(v))$ is just the G -equivariance condition for the map $\delta : V \rightarrow Y(V)$, where the G -action on $Y(V) = \mathbb{k}G \otimes V$ is given, as in Lemma 6.8, by $g(x \otimes v) = gxg^{-1} \otimes g(v)$. \square

If V is a G -module, we will, as usual, write (V, δ) to denote a particular quasi-YD module structure on V given by a quasicoaction δ .

Remark 7.2. We allow the group G to be infinite; however, in the present paper we do not explore continuous versions of our constructions and treat the fields and groups as discrete objects. Accordingly, whenever a summation over group elements is present, the sum should be well-defined; e.g., although all maps $\{L_h \in \text{End}(V) : h \in G\}$ may be non-zero, for any fixed $v \in V$ all but a finite number of $L_h(v)$ must be zero.

Let us now give a definition of a Yetter–Drinfeld module in a form more suitable for group algebras.

7.2. *Yetter–Drinfeld modules over groups*

First, observe that a coaction of the group algebra $\mathbb{k}G$ on a vector space Y is the same as a G -grading on Y .

Indeed, write the coaction as $\delta(y) = \sum_{h \in G} h \otimes L_h(y)$ as in Lemma 7.1. The comultiplicativity axiom, $(\Delta \otimes \text{id}_Y)\delta = (\text{id}_{\mathbb{k}G} \otimes \delta)\delta$, means that $L_g L_h$ equals L_h if $h = g$, or 0 otherwise. The counitality axiom, $(\epsilon \otimes \text{id}_Y)\delta = \text{id}_Y$, is equivalent to $\sum_{h \in G} L_h(y) = y$. Thus, $\{L_h : h \in G\}$ is a complete set of pairwise orthogonal idempotents on Y , and $Y = \bigoplus_{h \in G} Y_h$ where $Y_h = L_h Y$. Therefore, the definition of a Yetter–Drinfeld module (cf. 2.1) looks in the group algebra case as follows:

Lemma 7.3. *A Yetter–Drinfeld module over a group G is a vector space Y such that*

1. G acts on Y : $(g, y) \in G \times Y \mapsto g(y) \in Y$;
2. Y is a G -graded space: $Y = \bigoplus_{h \in G} Y_h$;
3. the grading is compatible with the action: $g(Y_h) \subseteq Y_{ghg^{-1}}$, $g, h \in G$.

The Yetter–Drinfeld structure induces a braiding on Y by the formula

$$\Psi(y \otimes z) = h(z) \otimes y, \quad y \in Y_h, z \in Y.$$

7.3. *Quasi-Yetter–Drinfeld modules (V, δ) with commutative $U(V, \delta)$*

Let us establish an equation on the quasicoaction δ , which is equivalent to the commutativity of the algebra $U(V, \delta)$.

Lemma 7.4. *Let V be a finite-dimensional quasi-YD module over $\mathbb{k}G$, with quasicoaction $\delta(v) = \sum_{h \in G} h \otimes L_h(v)$. The algebra $U(V, \delta)$ is commutative, if and only if*

$$\delta(v - h(v)) \otimes L_h(w) = \delta(w - h(w)) \otimes L_h(v) \quad \text{for any } h \in G, v, w \in V.$$

Proof. By Theorem 6.9, (V, δ) is a perfect subquotient of the Yetter–Drinfeld module $Y(V)$ with underlying vector space $\mathbb{k}G \otimes V$. The map $\delta : V \rightarrow Y(V)$ induces an embedding

$$U(V, \delta) \hookrightarrow \mathcal{B}(Y(V))$$

of algebras. Because the algebra $U(V, \delta)$ is generated by elements of V , it is commutative, if and only if for any $v, w \in V$ the elements $\delta(v)$ and $\delta(w)$ commute in the Nichols–Woronowicz algebra $\mathcal{B}(Y(V))$. By Definition 5.3 of the Nichols–Woronowicz algebra, this is equivalent to

$$(\text{id} + \Psi)(\delta(v) \otimes \delta(w) - \delta(w) \otimes \delta(v)) = 0 \quad \text{for any } v, w \in V,$$

where Ψ is the braiding on $Y(V)$. This is because the quadratic relations in $\mathcal{B}(Y(V))$ are the kernel of $[2]!_\Psi = \text{id} + \Psi$. In other words, $\delta(V)$ must be an *Abelian subspace* of the braided space $(Y(V), \Psi)$ —the term is from [1, I.C]. The condition rewrites as

$$\delta(v) \otimes \delta(w) - \Psi(\delta(w) \otimes \delta(v)) = \delta(w) \otimes \delta(v) - \Psi(\delta(v) \otimes \delta(w)).$$

Note that the G -coaction on $Y(V) \cong H \otimes V$ is given by $h \otimes v \mapsto h \otimes h \otimes v$. Substituting $\delta(\cdot) = \sum_{h \in G} h \otimes L_h(\cdot)$ and using the formula for Ψ from Lemma 7.3, rewrite the commutativity equation as

$$\delta(v) \otimes \delta(w) - \sum_h h(\delta(v)) \otimes h \otimes L_h(w) = \delta(w) \otimes \delta(v) - \sum_h h(\delta(w)) \otimes h \otimes L_h(v).$$

The left-hand side is $\sum_h (1 - h)\delta(v) \otimes h \otimes L_h(w)$, and this expression must be symmetric in v and w . Equivalently, $(1 - h)\delta(v) \otimes L_h(w) = (1 - h)\delta(w) \otimes L_h(v)$ for any $h \in G$. We may interchange the action of $1 - h$ and δ because δ is G -equivariant. \square

7.4. The commutativity equation for an irreducible linear group G

From now on we take G to be an irreducible linear group, that is, $G \leq GL(V)$ such that V is an irreducible G -module. Elements of the set

$$\mathcal{S} = \{s \in G : \dim(1 - s)V = 1\}$$

are called *complex reflections* in G . Note that we do not restrict the characteristic of the ground field \mathbb{k} ; an alternative term, more commonly used for linear groups in positive characteristic, is *pseudoreflection*.

We will use the following easy observation about complex reflections. By $\langle \cdot, \cdot \rangle$ is denoted the pairing between V^* and V .

Lemma 7.5. *Let $s \in \mathcal{S}$.*

1. *There are non-zero vectors $\alpha_s \in V^*$, $\check{\alpha}_s \in V$ such that*

$$s(v) = v - \langle \alpha_s, v \rangle \check{\alpha}_s \quad \text{for } v \in V.$$

The vectors $\alpha_s, \check{\alpha}_s$ are defined up to a simultaneous rescaling which leaves $\check{\alpha}_s \otimes \alpha_s$ fixed.

2. For any $g \in G$, the element $gs g^{-1}$ is also a complex reflection, and

$$\check{\alpha}_{gsg^{-1}} \otimes \alpha_{gsg^{-1}} = g(\check{\alpha}_s) \otimes g(\alpha_s).$$

We will refer to α_s (respectively $\check{\alpha}_s$) as the *root* (respectively the *coroot*) of a complex reflection s . Note that if $V \otimes V^*$ is identified with the algebra $\text{End}(V)$, the tensor $\check{\alpha}_s \otimes \alpha_s$ is equal to the endomorphism $1 - s$ of V .

Proposition 7.6. *Let $\delta : V \rightarrow \mathbb{k}G \otimes V$ be a Yetter–Drinfeld quasicoaction on V . The algebra $U(V, \delta)$ is commutative, if and only if δ is of the form*

$$\delta(v) = t \cdot 1 \otimes v + \sum_{s \in \mathcal{S}} s \otimes \langle \alpha_s, v \rangle b_s,$$

for some constant $t \in \mathbb{k}$ and vectors $b_s \in V$, or $\mathcal{S} = \emptyset$ and $\delta(v) = 1 \otimes L(v)$ for some G -automorphism L of V .

Proof. As $\delta : V \rightarrow Y(V)$ is a map of G -modules (Lemma 6.8), $\ker \delta$ is a G -submodule of V . By irreducibility of V , the quasicoaction δ is either zero or injective. In the trivial case $\delta = 0$ one has $U(V, 0) = \mathbb{k}$, and we may put $t = 0$, $b_s = 0$ for all $s \in \mathcal{S}$.

We now assume that δ is injective. Then $\delta(\cdot)$ may be dropped from the commutativity equation in Lemma 7.4. If $\delta(v) = \sum_{h \in G} h \otimes L_h(v)$ where $L_h \in \text{End}(V)$, the commutativity equation for fixed $s \in G$ is now as follows:

$$(v - s(v)) \otimes L_s(w) = (w - s(w)) \otimes L_s(v).$$

It is easy to see that this tensor equation can hold for arbitrary $v, w \in V$ only if one of the following holds:

- (a) $s = 1$; or
- (b) $L_s = 0$; or
- (c) $\dim(1 - s)V = \dim L_s(V) = 1$.

Conditions (a) and (b) are sufficient for the commutativity equation to hold. With regard to (a), it follows from Lemma 7.1 that the map $L := L_1 \in \text{End}(V)$ satisfies $L(g(v)) = g(L(v))$, i.e., is a G -endomorphism of V . If L is not zero, L is invertible by irreducibility of V .

Suppose that \mathcal{S} is not empty. The fact that L commutes with $1 - s$, $s \in \mathcal{S}$, implies that $L(\check{\alpha}_s) = t_s \check{\alpha}_s$ for some $t_s \in \mathbb{k}$. The G -equivariance of L means that $t_{gsg^{-1}} = t_s$ for all $g \in G$; irreducibility of V implies that $\{\check{\alpha}_{gsg^{-1}} = g(\check{\alpha}_s) \mid g \in G\}$ span V for any fixed $s \in \mathcal{S}$. It follows that $L = t \cdot \text{id}_V$ for a constant $t = t_s$.

In (c), the element $s \in G$ must be a complex reflection, and the commutativity equation rewrites as

$$\langle \alpha_s, v \rangle \check{\alpha}_s \otimes L_s(w) = \langle \alpha_s, w \rangle \check{\alpha}_s \otimes L_s(v).$$

This holds, if and only if $L_s(v) = \langle \alpha_s, v \rangle b_s$ for some vector $b_s \in V$. □

If $\langle \alpha_s, \check{\alpha}_s \rangle \neq 0$, one can show, using the G -equivariance of δ , that the vectors b_s must be proportional to $\check{\alpha}_s$. But if $\text{char } \mathbb{k} > 0$, it may happen that $\langle \alpha_s, \check{\alpha}_s \rangle = 0$ for a pseudoreflection s . Nevertheless, the next theorem will show that any possible pathological solutions are eliminated if the algebra $U(V^*, \delta)$ is also assumed to be commutative.

7.5. Rational Cherednik algebras

We denote by $\mathbb{k}(S)^G$ the space of \mathbb{k} -valued functions c on the set S of complex reflections, $s \mapsto c_s$, such that $c_{gs g^{-1}} = c_s$ for any $g \in G$.

Theorem 7.7. *Let $G \leq GL(V)$ be an irreducible linear group and S be the set of all complex reflections in G .*

1. *There exists a (V, δ) -braided double $U^- \rtimes \mathbb{k}G \rtimes U^+$ with commutative algebras U^- and U^+ , if and only if the quasicoaction δ is*

$$\delta(v) = \delta_{t,c}(v) := 1 \otimes tv + \sum_{s \in S} c_s s \otimes (v - s(v)),$$

where $t \in \mathbb{k}$ and $c \in \mathbb{k}(S)^G$, or $S = \emptyset$ and $\delta(v) = 1 \otimes L(v)$ for some G -automorphism L of V .

2. *For any t and c as above, there exists a $(V, \delta_{t,c})$ -braided double $H_{t,c}(G)$ of the form $S(V) \rtimes \mathbb{k}G \rtimes S(V^*)$.*

The theorem leads to the following

Definition 7.8. For $(t, c) \in \mathbb{k} \times \mathbb{k}(S)^G$, the braided double $H_{t,c}(G)$, given by the theorem, is called a *rational Cherednik algebra* of G .

The defining relations in $H_{t,c}(G)$ thus are:

$$\begin{aligned} [v, v'] &= 0, & [f, f'] &= 0, & gv &= g(v)g, & gf &= g(f)g, \\ [f, v] &= t \langle f, v \rangle \cdot 1 + \sum_{s \in S} c_s \langle f, (1-s)v \rangle \cdot s & & \in \mathbb{k}G \end{aligned}$$

for $v, v' \in V, g \in G, f, f' \in V^*$.

Remark 7.9.

- (a) Rational Cherednik algebras were introduced by Etingof and Ginzburg in [12] as symplectic reflection algebras for the symplectic space $V \oplus V^*$. It is proved in [12] (for finite G and in characteristic 0) that $H_{t,c}(G)$ are the only braided doubles of the form $S(V) \rtimes \mathbb{k}G \rtimes S(V^*)$. We obtain the same result (for any $|G|$ and $\text{char } \mathbb{k}$) using a different approach via braided doubles and Nichols–Woronowicz algebras. With our construction, we get “for free” an embedding of $H_{t,c}(G)$ in a braided Heisenberg double, see 7.9.
- (b) It is clear that essentially, rational Cherednik algebras are defined over groups generated by complex reflections (pseudoreflections). There is classification of such groups, both in characteristic zero and in characteristic p case.
- (c) The “exceptional” braided doubles with $\delta(v) = 1 \otimes L(v)$, where $L \neq t \cdot \text{id}_V$ is a G -module automorphism of V , may exist only when the field \mathbb{k} is not algebraically closed (otherwise such L are prohibited by Schur’s lemma). Even then, such a double is nothing but semidirect

product of $\mathbb{k}G$ and a Weyl algebra of V built out of a G -invariant scalar pairing $\langle \cdot, L(\cdot) \rangle$ between V^* and V , and is isomorphic to $H_{1,0}(G)$.

Before proving the theorem, let us give an example of rational Cherednik algebras over an infinite group.

Example 7.10 (*$U_q(sl_2)$ and quantum Smith algebras*). Let $V = \mathbb{k}x$ be a one-dimensional space. Let $q \in \mathbb{k}$ be not a root of unity, and denote by G_q the infinite cyclic subgroup of $GL(\mathbb{k}x)$ generated by q . Write the group algebra $\mathbb{k}G_q$ as $\mathbb{k}\{z^n : n \in \mathbb{Z}\}$; it acts on $\mathbb{k}x$ via $z(x) = qx$. Let $\mathbb{k}y$ be the dual space to $\mathbb{k}x$.

All $z^n, n \neq 0$, trivially are complex reflections. To any sequence $(c_n)_{n \in \mathbb{Z}}$, where all but a finite number of entries are zero, there is associated a rational Cherednik algebra $H_c(G_q)$ with relations

$$zx = qxz, \quad zy = q^{-1}yz, \quad [y, x] = \sum_{n \in \mathbb{Z}} c_n z^n$$

(note that the role of the parameter t is played here by c_0). These may be viewed as “quantum Smith algebras” (recall Smith algebras from 1.3). A particular case when $[y, x] = z - z^{-1}$ gives the quantised universal enveloping algebra $U_q(sl_2)$.

A version of $H_c(G_q)$ over $\mathbb{k} = \mathbb{C}$ where the commutator $[y, x]$ is an infinite power series in z might be interesting from the viewpoint of physical applications.

Proof of Theorem 7.7. We are going to ignore the case where $\delta(v) = 1 \otimes L(v)$ with L not proportional to id_V , due to the reason outlined in Remark 7.9(c).

1. Call a Yetter–Drinfeld quasicoaction δ on V “left-good” (respectively “right-good”) if there exists a (V, δ) -braided double $U^- \rtimes \mathbb{k}G \rtimes U^+$ with commutative U^- (respectively with commutative U^+). By Proposition 7.6, any left-good δ has the form $\delta(v) = 1 \otimes tv + \sum_{s \in \mathcal{S}} s \otimes \langle \alpha_s, v \rangle b_s$ for some vectors $b_s \in V$. The corresponding right-hand quasicoaction $\delta_r : V^* \rightarrow V^* \otimes \mathbb{k}G$ on the dual G -module V^* is given by $\delta_r(f) = tf \otimes 1 + \sum_s \langle f, b_s \rangle \alpha_s \otimes s$. Note that an element $s \in G$ is a complex reflection on V , if and only if s is a complex reflection on V^* ; furthermore, the complex reflection $s|_{V^*}$ has $\check{\alpha}_s$ as the root and α_s as the coroot.

We may now apply a straightforward analogue of Proposition 7.6 for the dual module V^* and the quasicoaction δ_r . It follows that a left-good δ is also right-good, if and only if for any complex reflection $s \in G$ the vector b_s is proportional to the root $\check{\alpha}_s$ of $s|_{V^*}$. That is, $b_s = c_s \check{\alpha}_s$ where $c_s \in \mathbb{k}$ are some constants, so that $\langle \alpha_s, v \rangle b_s = c_s(v - s(v))$.

We have shown that a quasicoaction δ is left-good and right-good, if and only if $\delta = \delta_{t,c}$ where t is a constant and c is a scalar function on \mathcal{S} . It remains to check what maps $\delta_{t,c}$ are G -equivariant, which by Lemma 7.1 means $g \circ \delta_{t,c} \circ g^{-1} = \delta_{t,c}$. The left-hand side equals $1 \otimes tv + \sum_s c_s \cdot gsg^{-1} \otimes (v - gsg^{-1}(v))$. Hence $\delta_{t,c}$ is a Yetter–Drinfeld quasicoaction on V , if and only if $c_{gsg^{-1}} = c_s$ for all $g \in G, s \in \mathcal{S}$.

2. Denote by $\wedge^2 V$ the subspace of $V^{\otimes 2}$ spanned by $v \otimes w - w \otimes v$ for all $v, w \in V$. Let

$$I^- = \langle \wedge^2 V \rangle \subset T(V), \quad I^+ = \langle \wedge^2 V^* \rangle \subset T(V^*)$$

be the ideals of definition of the symmetric algebras $S(V)$ and $S(V^*)$, respectively. We have to show that $I^- \otimes \mathbb{k}G \otimes T(V^*)$ and $T(V) \otimes \mathbb{k}G \otimes I^+$ are triangular ideals in $\hat{A}((V, \delta_{t,c}), \mathbb{k}G)$.

By part 1, the algebra $T(V)/I(V, \delta_{t,c})$ is commutative, therefore $\wedge^2 V \subset I(V, \delta_{t,c})$. By Theorem 4.11, this is equivalent to $[\widetilde{2}]!_{\delta_{t,c}}(\wedge^2 V) = 0$. By definition of the quasibraided factorial, $[\widetilde{2}]!_{\delta_{t,c}} = (\delta_{t,c} \otimes \text{id}_{\mathbb{k}G \otimes V})[\widetilde{2}]_{\delta_{t,c}}$. By irreducibility of V , the quasicoaction $\delta_{t,c}$ is injective (unless we are in the trivial case $t = 0, c_s = 0$ for all s). Hence $[\widetilde{2}]_{\delta_{t,c}}(\wedge^2 V) = 0$, so by Corollary 4.14 the ideal $I^- \otimes \mathbb{k}G \otimes T(V^*)$ is triangular. The argument for I^+ is analogous. \square

7.6. *Minimality of $H_{t,c}(G)$ for $t \neq 0$*

We would like to know if the rational Cherednik algebra $H_{t,c}(G) \cong S(V) \rtimes \mathbb{k}G \rtimes S(V^*)$ is a minimal double (i.e., has no proper quotient doubles). To investigate this, we are going to use the minimality criterion from Corollary 4.13. It is not difficult to deduce the following expression for the commutator of $\phi \in S(V^*)$ and $v \in V$ in $H_{t,c}(G)$:

$$[\phi, v] = t \cdot 1 \otimes \frac{\partial \phi}{\partial v} + \sum_{s \in \mathcal{S}} c_s \langle \alpha_s, v \rangle \cdot s \otimes \frac{\phi - s(\phi)}{\alpha_s}.$$

This looks very similar to the celebrated *Dunkl operator* acting on ϕ (see e.g. [11] for the complex reflection group case). However, note that the right-hand side lies in $\mathbb{k}G \otimes S(V^*)$. Here, $\frac{\partial \phi}{\partial v}$ stands for the derivative of the polynomial ϕ along the vector v , and $\frac{\phi - s(\phi)}{\alpha_s}$ is the divided difference operator, sometimes called the BGG-Demazure operator.

We first determine minimality of $H_{t,c}(G)$ for $t \neq 0$.

Proposition 7.11. *Let $t \neq 0$. If the characteristic of \mathbb{k} is zero, $H_{t,c}(G)$ is a minimal double. If $\text{char } \mathbb{k} > 0$, $H_{t,c}(G)$ is not a minimal double.*

Proof. Let $\text{char } \mathbb{k} = 0$. As there is no non-constant polynomial $\phi \in S(V^*)$ such that $\frac{\partial \phi}{\partial v} = 0$ for any $v \in V$, the commutator $[\phi, v]$ cannot be zero for all $v \in V$. The same reasoning applies to commutators $[f, b]$ where $f \in V^*, b \in S(V)$. Thus, by Corollary 4.13, $H_{t,c}(G)$ is a minimal double.

If $\text{char } \mathbb{k} = p > 0$, take ϕ to be a G -invariant in $V^{*\otimes pr}$ for some r . Then both the differential and the difference parts of $[\phi, v]$ vanish for all v , so by Corollary 4.13 $H_{t,c}(G)$ is not a minimal double. \square

Remark 7.12. In positive characteristic, one may consider standard modules $\{\overline{M}_\rho : \rho \in \text{Irr}(G)\}$ for $H_{t,c}(G)$ as was suggested in 4.6. One has $\overline{M}_\rho \cong U(V, \delta_{t,c}) \otimes \rho$, where $U(V, \delta_{t,c})$ is a proper quotient of the polynomial algebra $S(V)$. One should study the dimension of these standard $H_{t,c}(G)$ -modules (it may be finite) and their reducibility. In a particular rank 1 case this was done by Latour [25].

7.7. *Restricted Cherednik algebras*

The algebra $H_{0,c}(G)$ is never a minimal double. In fact, a more appropriate object from the point of view of minimality is a finite-dimensional quotient double of $H_{0,c}(G)$ called *restricted Cherednik algebra*, the definition of which we now recall. We consider only the most familiar case, where $\mathbb{k} = \mathbb{C}$ and G is a complex reflection group.

Let $S(V)_+^G \subset S(V)$ be the set of G -invariant polynomials with zero constant term. The algebra $S(V)_G = S(V)/\langle S(V)_+^G \rangle$ is termed *the coinvariant algebra* of G . The algebra

$$\overline{H}_{0,c}(G) = H_{0,c}(G)/\langle S(V)_+^G, S(V^*)_+^G \rangle \cong S(V)_G \rtimes \mathbb{C}G \rtimes S(V^*)_G$$

is the restricted Cherednik algebra of G . “Baby Verma modules” for $\overline{H}_{0,c}(G)$ are important for the representation theory of the rational Cherednik algebra $H_{0,c}(G)$ at $t = 0$; they were introduced and studied by Gordon [16].

Proposition 7.13. *Let $\mathbb{k} = \mathbb{C}$ and $G \leq GL(V)$ be a complex reflection group. If $c_s \neq 0$ for all $s \in \mathcal{S}$, the algebra $\overline{H}_{0,c}(G)$ is a minimal double.*

Proof. For a non-constant $\phi \in S(V^*)_G$, expressions $\phi - s(\phi)$ cannot vanish simultaneously for all $s \in \mathcal{S}$ (otherwise, as \mathcal{S} generates G , ϕ would be a nontrivial G -invariant in $S(V^*)_G$). It follows that $[\phi, v]$ cannot be zero for all $v \in V$. Likewise for $S(V)_G$. The minimality follows by Corollary 4.13. \square

Remark 7.14. Note that the assumption $c_s \neq 0 \forall s \in \mathcal{S}$, made in the proposition, can be slightly relaxed and replaced with the following: complex reflections s , for which $c_s \neq 0$, generate G .

7.8. *The Yetter–Drinfeld module $Y_{\mathcal{S}}(G)$*

For an irreducible linear group $G \leq GL(V)$ we obtained a complete classification of braided doubles of the form $S(V) \rtimes \mathbb{k}G \rtimes S(V)$ (Theorem 7.7). This was achieved by observing that V must identify, via the quasicoaction $\delta: V \rightarrow Y(V)$, with an Abelian subspace in the Yetter–Drinfeld module $Y(V)$. The quasi-Yetter–Drinfeld module (V, δ) then becomes a perfect subquotient of $Y(V)$.

But in fact, the Abelian subspace $\delta(V) \subset Y(V)$ will always lie in a proper submodule $Y_{\mathcal{S}}(G)$ of $Y(V)$. We will now give an abstract description of the Yetter–Drinfeld module $Y_{\mathcal{S}}(G)$. Let us start with its “essential part”, denoted by $\overline{Y}_{\mathcal{S}}(G)$.

Definition 7.15. Define $\overline{Y}_{\mathcal{S}}(G)$ to be the following Yetter–Drinfeld module over G :

- $\overline{Y}_{\mathcal{S}}(G)$ is a vector space spanned by symbols $[s]$, indexed by $s \in \mathcal{S}$;
- G -action on $\overline{Y}_{\mathcal{S}}(G)$: for $g \in G$,

$$g([s]) = \lambda(g, s)[gs g^{-1}],$$

where $\lambda(g, s) \in \mathbb{k}$ is such that $g(\check{\alpha}_s) = \lambda(g, s)\check{\alpha}_{gs g^{-1}}$;

- G -grading: $\overline{Y}_{\mathcal{S}}(G) = \bigoplus_{s \in \mathcal{S}} \overline{Y}_{\mathcal{S}}(G)_s$, where $\overline{Y}_{\mathcal{S}}(G)_s = \mathbb{k} \cdot [s]$.

Remark 7.16. The function $\lambda: G \times \mathcal{S} \rightarrow \mathbb{k}^\times$ is a 1-cocycle, in the sense that $\lambda(gh, s) = \lambda(g, hsh^{-1})\lambda(h, s)$. Each of the coroots $\check{\alpha}_s \in V$ is defined only up to a scalar factor; the G -module structure on $\overline{Y}_{\mathcal{S}}(G)$ does not depend on a choice of such factors, which changes $\lambda(g, s)$ by a coboundary.

Definition 7.17. The Yetter–Drinfeld module $Y_{\mathcal{S}}(G)$ is defined as $Y_{\mathcal{S}}(G) = V_1 \oplus \bar{Y}_{\mathcal{S}}(G)$. Here V_1 is a copy of the module V with the trivial Yetter–Drinfeld structure, which is given by the coaction $v \mapsto 1 \otimes v$.

In the next lemma, a typical element of $Y_{\mathcal{S}}(G)$ is written as $v \oplus \bigoplus_s a_s [s]$ for some $v \in V$ and $a_s \in \mathbb{k}$.

Lemma 7.18. Let $\delta_{t,c}$ be the quasicoaction on V introduced in Theorem 7.7. The quasi-Yetter–Drinfeld module $(V, \delta_{t,c})$ is a perfect subquotient of $Y_{\mathcal{S}}(G)$ via the maps

$$\begin{aligned} \mu : V &\rightarrow Y_{\mathcal{S}}(G), & \mu(v) &= tv \oplus \bigoplus_{s \in \mathcal{S}} c_s \langle \alpha_s, v \rangle [s] \quad \text{and} \\ \nu : Y_{\mathcal{S}}(G) &\rightarrow V, & \nu|_{V_1} &= \text{id}_V, & \nu([s]) &= \check{\alpha}_s. \end{aligned}$$

Proof. By Theorem 6.9, (V, δ) is a perfect subquotient of the Yetter–Drinfeld module $Y(V) \cong \mathbb{k}G \otimes V$ via the maps $V \xrightarrow{\delta} Y(V) \xrightarrow{\epsilon \otimes \text{id}_V} V$. Denote by Y' the subspace of $Y(V)$ spanned by $1 \otimes v$ for $v \in V$ and by $s \otimes \check{\alpha}_s$ for $s \in \mathcal{S}$. It follows from Lemma 7.5 and the definition of the G -action on $Y(V)$ that Y' is a submodule of $Y(V)$; obviously, Y' is a subcomodule of V , hence a Yetter–Drinfeld submodule. Observe that $\delta_{t,c}(V) \subset Y'$, hence $(V, \delta_{t,c})$ is a perfect subquotient of Y' .

We identify $Y_{\mathcal{S}}(G)$ with Y' via a linear isomorphism $i : Y_{\mathcal{S}}(G) \rightarrow Y'$, defined by $i(v \oplus \bigoplus_s a_s [s]) = 1 \otimes v + \sum_s a_s s \otimes \check{\alpha}_s$. The G -action on $Y_{\mathcal{S}}(G)$ is so defined that i is a G -module map; i preserves the G -grading, hence i is a Yetter–Drinfeld module isomorphism. It remains to note that $\mu = i^{-1} \circ \delta$ and $\nu = (\epsilon \otimes \text{id}_V) \circ i$, thus by Remark 6.7 $(V, \delta_{t,c})$ is a perfect subquotient of $Y_{\mathcal{S}}(G)$ via the maps μ, ν . \square

7.9. *Embedding of $H_{t,c}(G)$ in a braided Heisenberg double*

We assume that the set \mathcal{S} is finite, so that the Yetter–Drinfeld module $Y_{\mathcal{S}}(G)$ is finite-dimensional.

Observe that a complex reflection $s \in G$ acting on V^* has $\check{\alpha}_s$ as the root and α_s as the coroot. The right Yetter–Drinfeld module $\bar{Y}_{\mathcal{S}}(G)^*$ will be spanned by symbols $[s]^*$ which are a basis dual to $[s] \in \bar{Y}_{\mathcal{S}}(G)$. We have $Y_{\mathcal{S}}(G)^* = V_1^* \oplus \bar{Y}_{\mathcal{S}}(G)^*$ where V_1^* coincides with V^* as a right G -module and has trivial right G -coaction.

We are ready to construct a triangular map from the rational Cherednik algebra $H_{t,c}(G)$ into the braided Heisenberg double $\mathcal{H}_{Y_{\mathcal{S}}(G)}$.

Proposition 7.19. Let $G \leq GL(V)$ be a finite linear group, and let (t, c) be a parameter from $\mathbb{k} \times \mathbb{k}(\mathcal{S})^G$. The maps

$$\begin{aligned} \mu : V &\rightarrow Y_{\mathcal{S}}(G), & \mu(v) &= tv \oplus \bigoplus_{s \in \mathcal{S}} c_s \langle \alpha_s, v \rangle [s], \\ \nu^* : V^* &\rightarrow Y_{\mathcal{S}}(G)^*, & \nu^*(f) &= f \oplus \bigoplus_{s \in \mathcal{S}} \langle f, \check{\alpha}_s \rangle [s]^* \end{aligned}$$

induce a morphism of braided doubles from the rational Cherednik algebra $H_{t,c}(G)$ to the braided Heisenberg double $\mathcal{B}(Y_{\mathcal{S}}(G)) \rtimes \mathbb{k}G \rtimes \mathcal{B}(Y_{\mathcal{S}}(G)^*)$. If $t \neq 0$ and $\text{char } \mathbb{k} = 0$, this morphism is injective.

Proof. One should note that the map denoted by v^* in the proposition is indeed adjoint to the map $v : Y_{\mathcal{S}}(G) \rightarrow V$ defined in Lemma 7.18. By Lemma 7.18, the maps $V \xrightarrow{\mu} Y_{\mathcal{S}}(G) \xrightarrow{v} V$ make the quasi-Yetter–Drinfeld module $(V, \delta_{t,c})$ a perfect subquotient of $Y_{\mathcal{S}}(G)$.

In a fashion similar to Lemma 7.18 we can show that the right quasi-YD module V^* is a perfect subquotient of $Y_{\mathcal{S}}(G)^*$ via the maps $V^* \xrightarrow{v^*} Y_{\mathcal{S}}(G)^* \xrightarrow{\mu^*} V^*$.

Since both pairs of maps μ, v and v^*, μ^* are perfect subquotients, the situation is now precisely as described in 6.4. That is, the maps μ and v^* induce an injective embedding of the minimal double associated to $(V, \delta_{t,c})$ (which itself is a triangular quotient of $H_{t,c}(G)$, and coincides with $H_{t,c}(G)$ when $t \neq 0$ and $\text{char } \mathbb{k} = 0$) in the braided Heisenberg double $\mathcal{H}_{Y_{\mathcal{S}}(G)}$. \square

7.10. Braided doubles containing rational Cherednik algebras of complex reflection groups

From now on we assume that

- the field \mathbb{k} is \mathbb{C} ;
- G is an irreducible finite complex reflection group;
- V is the reflection representation of G .

We will now use general results up to and including Proposition 7.19 to construct two interesting realisations of rational Cherednik algebras of complex reflection groups: one for $t = 0$, the other for $t \neq 0$. Both realisations are in braided doubles associated to the Yetter–Drinfeld module $Y_G := \bar{Y}_{\mathcal{S}}(G)$ (see Definition 7.15), which looks “nicer” than $Y_{\mathcal{S}}(G)$ because it does not contain an extra copy of the space V . Thus, Y_G is a vector space spanned by symbols $\{[s] : s \in \mathcal{S}\}$, with G -action $g([s]) = \lambda(g, s)[gs g^{-1}]$ and braiding given by $\Psi([r] \otimes [s]) = r([s]) \otimes [r]$ for $r, s \in \mathcal{S}$. We will illustrate the constructions by examples for $G = \mathbb{S}_n$.

7.11. The case $t = 0$

Let \mathcal{H}_{Y_G} be the braided Heisenberg double associated to the Yetter–Drinfeld module Y_G . This is an algebra with generators $[s], [s]^*$ ($s \in \mathcal{S}$) and $g \in G$.

To understand what are the relations between these generators in \mathcal{H}_{Y_G} , denote by $I(Y_G)$ the ideal of relations between the generators $[s]$ in the Nichols–Woronowicz algebra $\mathcal{B}(Y_G)$. It is the kernel of the Woronowicz symmetriser $\text{Wor}(\Psi)$. Similarly, let $I(Y_G^*)$ be the ideal of relations between the generators $[s]^*$ in $\mathcal{B}(Y_G^*)$. The defining relations in \mathcal{H}_{Y_G} are

$$\begin{aligned}
 I(Y_G), \quad I(Y_G^*), \quad g \cdot [s] &= g([s]) \cdot g, \quad g \cdot [s]^* = g([s]^*) \cdot g, \\
 [r]^* \cdot [s] - [s] \cdot [r]^* &= \begin{cases} s, & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases}
 \end{aligned}$$

We will now embed the restricted Cherednik algebra $\bar{H}_{0,c}(G)$ in the braided Heisenberg double \mathcal{H}_{Y_G} .

Theorem 7.20. *Let G be an irreducible complex reflection group, $S \subset G$ be the set of complex reflections, and c be a conjugation-invariant function on S . Define a linear map M_c from $V \oplus \mathbb{C}G \oplus V^*$ to $Y_G \oplus \mathbb{C}G \oplus Y_G^*$ by*

$$M_c(v) = \sum_{s \in S} c_s \langle \alpha_s, v \rangle [s], \quad M_c(f) = \sum_{s \in S} \langle f, \check{\alpha}_s \rangle [s]^*, \quad M_c(g) = g,$$

where $v \in V$, $f \in V^*$, $g \in G$. Then the map M_c extends to an algebra homomorphism $M_c : \overline{H}_{0,c}(G) \rightarrow \mathcal{H}_{Y_G}$. If c is generic, for example $c_s \neq 0$ for all s , the homomorphism M_c is injective.

Proof. Proposition 7.19 gives a triangular map $H_{0,c}(G) \rightarrow \mathcal{H}_{Y_S(G)}$ where $Y_S(G) = V_1 \oplus Y_G$ (direct sum of Yetter–Drinfeld modules), and that triangular map is defined by the same formulas as M_c . Moreover, since $t = 0$, the image of that triangular map lies in the subdouble of $\mathcal{H}_{Y_S(G)}$ generated by Y_G and Y_G^* . This subdouble is precisely \mathcal{H}_{Y_G} . The image of $H_{0,c}(G)$ in the braided Heisenberg double $\mathcal{H}_{Y_G} \cong \mathcal{B}(Y_G) \rtimes \mathbb{C}G \rtimes \mathcal{B}(Y_G^*)$ is the minimal double which is the quotient of $H_{0,c}(G)$; for generic c , this is $\overline{H}_{0,c}(G)$ by Proposition 7.13. \square

Remark 7.21. The theorem implies (and provides a new proof of) the realisation of the coinvariant algebra $S(V)_G$ in the Nichols–Woronowicz algebra $\mathcal{B}(Y_G)$, which was the main result of [6] when G is a Coxeter group, and of [24] when G is an arbitrary complex reflection group.

Now observe that for any algebra $A = U^- \rtimes H \rtimes U^+$ with triangular decomposition over a bialgebra H , the subalgebra U^- is a left A -module (it is the induced module M_ϵ in the notation of 4.6, where ϵ is the counit viewed as a 1-dimensional representation of H). In particular, the Nichols–Woronowicz algebra $\mathcal{B}(Y_G)$ is a left module for the braided Heisenberg double \mathcal{H}_{Y_G} . Since $\overline{H}_{0,c}$ embeds or maps into \mathcal{H}_{Y_G} , we have

Corollary 7.22. *The Nichols–Woronowicz algebra $\mathcal{B}(Y_G)$ is a module over the restricted Cherednik algebra $\overline{H}_{0,c}$, for any c .*

Example 7.23. Let us consider the case when $G = \mathbb{S}_n$ is a symmetric group acting on \mathbb{C}^n . The action restricts onto the subspace $V = \{(x_1, \dots, x_n) : \sum x_i = 0\}$ which is the irreducible reflection representation of \mathbb{S}_n . We have the restricted Cherednik algebra $\overline{H}_{0,c}(\mathbb{S}_n)$ for any $c \in \mathbb{C}$; all such algebras are isomorphic for $c \neq 0$.

The reflections in \mathbb{S}_n are transpositions (ij) , $1 \leq i < j \leq n$. The Nichols–Woronowicz algebra $\mathcal{B}(Y_{\mathbb{S}_n})$ has generators $[ij]$. The \mathbb{S}_n -action on generators is via $g([ij]) = [g(i)g(j)]$ for $g \in \mathbb{S}_n$, if we agree that the symbol $[ij]$ stands for $-[ji]$ whenever $i > j$. The quadratic relations in $\mathcal{B}(Y_{\mathbb{S}_n})$ are

$$[ij]^2 = 0, \quad [ij][kl] = [kl][ij], \quad [ij][jk] + [jk][ki] + [ki][ij] = 0 \quad \text{for distinct } i, j, k, l,$$

so that the quadratic cover $\mathcal{B}_{quad}(Y_{\mathbb{S}_n})$ of $\mathcal{B}(Y_{\mathbb{S}_n})$ is the Fomin–Kirillov quadratic algebra \mathcal{E}_n , introduced in [13]. Using our method and results from [13], one can check that the restricted Cherednik algebra $\overline{H}_{0,c}(\mathbb{S}_n)$ acts both on $\mathcal{B}(Y_{\mathbb{S}_n})$ and on \mathcal{E}_n .

It is a conjecture (which is at least ten years old at the time when the present paper is being written, and is still open) that the Nichols–Woronowicz algebra $\mathcal{B}(Y_{\mathbb{S}_n})$ is quadratic and coincides with \mathcal{E}_n . The algebras \mathcal{E}_n are finite-dimensional for $n \leq 5$. Open questions about \mathcal{E}_n include infinite-dimensionality for $n > 5$ and the Hilbert series. Overall, very little is known about the structure of \mathcal{E}_n .

It therefore may prove to be helpful to study the $\overline{H}_{0,c}(\mathbb{S}_n)$ -action on \mathcal{E}_n and to use the representation theory of the restricted Cherednik algebra to obtain information about the Fomin–Kirillov algebra. For example, any simple $\overline{H}_{0,c}(\mathbb{S}_n)$ -module, viewed as an \mathbb{S}_n -module, is a regular representation of \mathbb{S}_n [16]. Thus the very fact that $\overline{H}_{0,c}(\mathbb{S}_n)$ acts on \mathcal{E}_n implies a new result:

Proposition 7.24. *As an \mathbb{S}_n -module, the Fomin–Kirillov algebra \mathcal{E}_n is a direct sum of copies of the regular representation of \mathbb{S}_n .*

7.12. *The case $t \neq 0$*

When $t \neq 0$, the algebra $H_{t,c}(G)$ embeds, as a subalgebra, in a “deformed quadratic double” associated to Y_G .

Note that Y_G is a Yetter–Drinfeld module over $\mathbb{C}G$ which is a cocommutative Hopf algebra. Hence there exists a deformed Nichols–Woronowicz algebra $\mathcal{B}_\tau(Y_G)$. We will only need its quadratic cover

$$\mathcal{B}_{quad_\tau}(Y_G) = T(Y_G)/I_{quad_\tau}(Y_G)$$

given in Lemma 5.13, where the ideal $I_{quad_\tau}(Y_G)$ is generated by $\ker(\text{id} + \Psi) \cap \wedge^2 Y_G$. The corresponding braided double is $\mathcal{H}_t(Y_G)$ with relations

$$\begin{aligned} I_{quad_\tau}(Y_G), \quad I_{quad_\tau}(Y_G^*), \quad g \cdot [s] &= g([s]) \cdot g, \quad g \cdot [s]^* = g([s]^*) \cdot g, \\ [r]^* \cdot [s] - [s] \cdot [r]^* &= \begin{cases} s + t \cdot 1, & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases} \end{aligned}$$

Theorem 7.25. *Let $t \neq 0$ and c be generic. In the notation of Theorem 7.20, the map M_c extends to an injective algebra homomorphism $H_{t,c}(G) \rightarrow \mathcal{H}_{t'}(Y_G)$ for some t' depending on t and c .*

Proof. Computing the commutator of $M_c(f)$ and $M_c(v)$ in a braided double $\mathcal{H}_{t'}(Y_G)$, we obtain

$$[M_c(f), M_c(v)] = \sum_{s \in \mathcal{S}} c_s \langle f, (1 - s)v \rangle (s + t' \cdot 1).$$

Note that $\sum_s c_s \langle f, (1 - s)v \rangle$ is a G -invariant pairing between V^* and V , and because V is irreducible, this pairing is proportional to $\langle \cdot, \cdot \rangle$ (and is non-zero for c generic).

Choose t' in such a way that $\sum_s c_s \langle f, (1 - s)v \rangle \cdot t' = t$. Then M_c extends to a morphism of braided doubles between $T(V) \rtimes \mathbb{C} \rtimes T(V^*)$ (with commutator given by the quasisoaction $\delta_{t,c}$) and $\mathcal{H}_{t'}(Y_G)$. It follows from Lemma 5.13 that $M_c(V)$ (respectively $M_c(V^*)$) generates a commutative subalgebra in $\mathcal{B}_{quad_\tau}(Y_G)$ (respectively $\mathcal{B}_{quad_\tau}(Y_G^*)$). Therefore, M_c factors through the rational Cherednik algebra $H_{t,c}(G)$, and is injective on this algebra because $H_{t,c}(G)$ is a minimal double. \square

Remark 7.26. In fact, by modifying the map M_c on the space V^* , one can relax the condition that c is generic and only assume that c is not identically zero on \mathcal{S} .

Example 7.27. In the case $G = \mathbb{S}_n$, the quadratic algebra $\mathcal{B}_{quad_\tau}(Y_G)$ coincides with the universal enveloping algebra $U(\text{tr}_n)$ associated to the classical Yang–Baxter equation and introduced in [5]. It is a Koszul algebra. We thus obtain an action of $H_{r,c}(\mathbb{S}_n)$ on $U(\text{tr}_n)$ and a generalisation of $U(\text{tr}_n)$ for an arbitrary irreducible complex reflection group.

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Appendix A. Triangular decomposition over a bialgebra

This appendix contains proofs to a number of facts about triangular ideals in braided doubles. We do it in somewhat more general situation, for algebras with triangular decomposition over a bialgebra. Triangular decomposition of universal enveloping algebras of Kac–Moody algebras has been generalised in several useful ways [14,22]. Most relevant for the present paper is the following definition (all algebras are assumed to be associative and unital):

Definition A.1. An algebra A has *triangular decomposition over a bialgebra H* if A has distinguished subalgebras $H \hookrightarrow A$, $U^- \hookrightarrow A$, $U^+ \hookrightarrow A$ such that:

- H acts covariantly from the left on the algebra U^- , and from the right on U^+ ;
- the multiplication in A induces a linear isomorphism $U^- \otimes H \otimes U^+ \rightarrow A$; it makes $U^- \otimes H$ a subalgebra of A isomorphic to the semidirect product $U^- \rtimes H$ by the left H -action, and similarly $H \otimes U^+$ a subalgebra isomorphic to the semidirect product $H \rtimes U^+$;
- the algebras U^- , U^+ are equipped with H -equivariant characters (algebra homomorphisms) $\epsilon^- : U^- \rightarrow \mathbb{k}$, $\epsilon^+ : U^+ \rightarrow \mathbb{k}$.

An H -equivariant character $\epsilon^\pm : U^\pm \rightarrow \mathbb{k}$ is a homomorphism of H -modules, where the action of H on \mathbb{k} is via the counit ϵ of H . (These characters are required to ensure that A has a maximal triangular ideal—see below.) Semidirect products are also known as smash products [38, Definition 4.1.3].

A.1. Triangular subalgebras, ideals and quotients. Triangular-simple algebras

Let H be a bialgebra. Algebras $A \cong U^- \otimes H \otimes U^+$ with triangular decomposition over H form a category. Morphisms in this category are algebra maps of the form

$$\mu = \mu^- \otimes \text{id}_H \otimes \mu^+ : U_1^- \otimes H \otimes U_1^+ \rightarrow U_2^- \otimes H \otimes U_2^+,$$

where $\mu^- : U_1^- \rightarrow U_2^-$ (respectively $\mu^+ : U_1^+ \rightarrow U_2^+$) is a left (respectively right) H -module algebra homomorphism, which intertwines the characters ϵ^- (resp. ϵ^+). Among morphisms are embeddings of triangular subalgebras

$$A' = U'^- \otimes H \otimes U'^+ \hookrightarrow A = U^- \otimes H \otimes U^+,$$

where U'^{\pm} embed in U^{\pm} as subalgebras, and triangular quotient maps

$$A = U^- \otimes H \otimes U^+ \twoheadrightarrow A'' = U''^- \otimes H \otimes U''^+,$$

induced by pairs $U^{\pm} \twoheadrightarrow U''^{\pm}$ of surjective algebra maps.

A kernel of a triangular quotient map will be called a *triangular ideal*. We say that $A = U^- \otimes H \otimes U^+$ is a *triangular-simple algebra* over H , if A has no non-trivial triangular quotients.

The next proposition describes triangular ideals.

Proposition A.2. *Let $A = U^- \otimes H \otimes U^+$ be an algebra with triangular decomposition. A linear subspace $J \subset A$ is a triangular ideal in A , if and only if*

$$J = J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+,$$

where J^- (respectively J^+) is a two-sided ideal in U^- (respectively U^+), satisfying the following:

- (A.2.1) J^-, J^+ are invariant with respect to the H -action on U^-, U^+ ;
- (A.2.2) $J^- \subset \ker \epsilon^-, J^+ \subset \ker \epsilon^+$;
- (A.2.3) $U^+ \cdot J^-$ and $J^+ \cdot U^-$ (the products with respect to the multiplication in the algebra A) lie in $J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$.

Proof. We start with the ‘only if’ part. Let J be a triangular ideal, i.e., the kernel of a map

$$\mu^- \otimes \text{id}_H \otimes \mu^+ : U^- \otimes H \otimes U^+ \rightarrow U'^- \otimes H \otimes U'^+,$$

which is a morphism of algebras with triangular decomposition over H . Then $J = J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$ where $J^{\pm} = \ker \mu^{\pm}$. Since μ^- is an algebra morphism, J^- is a two-sided ideal in U^- . Furthermore, J^- is H -invariant because μ^- is an H -morphism, and $J^- \subset \ker \epsilon^-$ because $\epsilon^-|_{U^-} = \epsilon^-|_{U'^-} \circ \mu^-$. This verifies properties (A.2.1), (A.2.2) for the ideal J^- , and they are verified for J^+ in the same way. Property (A.2.3) follows from the fact that $J^- = J^- \otimes 1 \otimes 1$ and $J^+ = 1 \otimes 1 \otimes J^+$ lie in J which is a two-sided ideal in A .

To prove the ‘if’ part of the proposition, we must show that if two-sided ideals $J^{\pm} \subset U^{\pm}$ satisfy (A.2.1)–(A.2.3), then $J = J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$ is a triangular ideal in A . We begin by checking that J is a two-sided ideal in A . First,

$$U^- \cdot J \subset U^- \cdot J^- H U^+ + U^- \cdot U^- H J^+ = J^- H U^+ + U^- H J^+ = J$$

because J^- is a left ideal in U^- . Similarly, $J \cdot U^+ \subset J$. Next,

$$H \cdot J \subset H \cdot J^- H U^+ + H \cdot U^- H J^+ \subset J^- H U^+ + U^- H J^+ = J,$$

as $h \cdot J^- = (h_{(1)} \triangleright J^-) h_{(2)} \subset J^- \cdot H$ by (A.2.1) for any $h \in H$. Similarly, $J \cdot H \subset J$. Finally, using (A.2.3),

$$U^+ \cdot J \subset (U^+ \cdot J^-) H U^+ + (U^+ \cdot U^- H) J^+ \subset J H U^+ + (U^- H U^+) J^+;$$

we have already established that $JHU^+ \subset J$, and $(U^-HU^+)J^+ = U^-HJ^+$ because J^+ is a two-sided ideal in U^+ . Thus, $U^+ \cdot J \subset J$. Quite similar argument shows that $J \cdot U^- \subset J$. The subalgebras U^- , H and U^+ generate A as an algebra, hence $A \cdot J, J \cdot A \subset J$ as required.

Let $p: A \rightarrow A/J$ be the quotient map of algebras. Our goal is to show that A/J is an algebra with triangular decomposition over H , and p is a triangular morphism (then $J = \ker p$ is, by definition, a triangular ideal).

Since $J = J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$, we have a vector space tensor product decomposition

$$A/J = (U^-/J^-) \otimes H \otimes (U^+/J^+),$$

and $p = p^- \otimes \text{id}_H \otimes p^+$ where $p^\pm: U^\pm \rightarrow U^\pm/J^\pm$ are quotient maps. We observe that U^\pm/J^\pm and H are subalgebras in A/J (these are p -images of U^\pm and H , respectively). Moreover, by (A.2.1), one has the induced H -action on U^\pm/J^\pm , and p^\pm are H -algebra homomorphisms. The relation between $\bar{b} = p^-(b) \in U^-/J^-$ and $h \in H$,

$$h \cdot \bar{b} = p(h \cdot b) = p((h_{(1)} \triangleright b)h_{(2)}) = (h_{(1)} \triangleright \bar{b})h_{(2)},$$

is the cross-product relation between U^-/J^- and H . Similarly for H and U^+/J^+ . And finally, by (A.2.2), there are induced characters $\epsilon^\pm: U^\pm/J^\pm \rightarrow \mathbb{k}$, such that p^- intertwines the characters ϵ^- on U^- and U^-/J^- (similarly for p^+). Thus, A/J is indeed an algebra with triangular decomposition over H , and $p: A \rightarrow A/J$ is a triangular morphism. \square

The proposition and its proof have the following important corollary:

Corollary A.3. *Let A be an algebra with triangular decomposition over a bialgebra H .*

1. *If J is a triangular ideal in A , the quotient algebra A/J has triangular decomposition over H .*
2. *A surjective morphism of algebras with triangular decomposition over H maps triangular ideals to triangular ideals.*
3. *A sum of triangular ideals in A is a triangular ideal in A .*
4. *The algebra A has a greatest triangular ideal I_A .*
5. *The algebra A has a unique triangular-simple quotient over H , which is A/I_A .*

Proof. 1. In the proof of the ‘if’ part of the proposition, the quotient A/J was explicitly constructed and shown to have triangular decomposition.

2. Let $\mu = \mu^- \otimes \text{id}_H \otimes \mu^+$ be a triangular morphism from $A = U^- \otimes H \otimes U^+$ onto $A'' = U'^- \otimes H \otimes U''^+$. It is easy to see that if $J^\pm \subset U^\pm$ are ideals satisfying (A.2.1)–(A.2.3), then $\mu^\pm(J^\pm)$ are ideals in $U''^\pm = \mu^\pm(U^\pm)$ which, too, satisfy (A.2.1)–(A.2.3).

3. Let $\{J_\alpha\}$ be a family of triangular ideals in A . Then for each index α , $J_\alpha = J_\alpha^- \otimes H \otimes U^+ + U^- \otimes H \otimes J_\alpha^+$, where $J_\alpha^\pm \subset U^\pm$ are a pair of two-sided ideals satisfying (A.2.1)–(A.2.3). It is clear that $J^- = \sum_\alpha J_\alpha^-$, $J^+ = \sum_\alpha J_\alpha^+$ is a pair of two-sided ideals in U^- , U^+ satisfying (A.2.1)–(A.2.3) (in particular, $J^\pm \neq U^\pm$ because $J^\pm \subset \ker \epsilon^\pm$). Thus, $J = \sum_\alpha J_\alpha = J^-HU^+ + U^-HJ^+$ is a triangular ideal in A .

4. The ideal I_A is the sum of all triangular ideals in A .

5. It follows from 2. that if a triangular ideal J is not greatest in A , then A/J has a non-trivial triangular ideal I_A/J , hence is not triangular-simple. It remains to observe that A/I_A is triangular-simple, because a non-trivial triangular quotient map $A/I_A \rightarrow A''$ would give rise to composite triangular quotient map $A \rightarrow A/I_A \rightarrow A''$ whose kernel is a triangular ideal strictly larger than I_A . \square

A key question about an algebra A with triangular decomposition over H is to find its unique maximal triangular ideal I_A . We will now show that there is a natural “upper bound” for I_A , given by kernels of the *Harish-Chandra pairing* in A .

A.2. *The Harish-Chandra pairing*

Let $A = U^- \otimes H \otimes U^+$ be an algebra with triangular decomposition over a bialgebra H . The *Harish-Chandra projection map* is a linear map from A onto H defined as

$$\text{pr}_H = \epsilon^- \otimes \text{id}_H \otimes \epsilon^+ : U^- \otimes H \otimes U^+ \rightarrow H.$$

The *Harish-Chandra pairing* is an H -valued bilinear pairing between U^+ and U^- :

$$(\cdot, \cdot)_H : U^+ \times U^- \rightarrow H, \quad (\phi, b)_H = \text{pr}_H(\phi b),$$

where the product of $\phi \in U^+$ and $b \in U^-$ is taken in A .

The Harish-Chandra projection map pr_H will in general not be an algebra homomorphism. However, it is an H - H bimodule map.

A.3. *The kernels of the Harish-Chandra pairing*

Let $A = U^- \otimes H \otimes U^+$ be any algebra with triangular decomposition over H . Let

$$K_A^- = \{b \in U^- \mid (\phi, b)_H = 0 \forall \phi \in U^+\}, \quad K_A^+ = \{\phi \in U^+ \mid (\phi, b)_H = 0 \forall b \in U^-\}$$

be the kernels of the Harish-Chandra pairing in U^- and U^+ . We have the following

Proposition A.4. *All triangular ideals in A lie in $K_A^- \otimes H \otimes U^+ + U^- \otimes H \otimes K_A^+$.*

Proof. A triangular ideal in A is of the form $J^- \otimes H \otimes U^+ + U^- \otimes H \otimes J^+$, where J^\pm are ideals in U^\pm satisfying (A.2.1)–(A.2.3). In particular, (A.2.2) says that $J^\pm \subset \ker \epsilon^\pm$, therefore J lies in the kernel of the Harish-Chandra projection $\text{pr}_H = \epsilon^- \otimes \text{id}_H \otimes \epsilon^+$; and (A.2.3) says that $U^+ \cdot J^- \subset J$, hence $(U^+, J^-)_H = \text{pr}_H(U^+ \cdot J^-) = 0$ and $J^- \subset K_A^-$. Similarly, $J^+ \subset K_A^+$. \square

Thus, if the Harish-Chandra pairing in A is non-degenerate, the algebra A is automatically triangular-simple. The converse is not true (and has explicit counterexamples). Note that $K_A^- H U^+ + U^- H K_A^+$ is not even guaranteed to be an ideal in A .

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