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# Lie algebras and Lie groups over noncommutative rings

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**Abstract**

The aim of this paper is to introduce and study Lie algebras and Lie groups over noncommutative rings. For any Lie algebra $\mathfrak{g}$ sitting inside an associative algebra $A$ and any associative algebra $\mathcal{F}$ we introduce and study the algebra $(\mathfrak{g}, A)(\mathcal{F})$, which is the Lie subalgebra of $\mathcal{F} \otimes A$ generated by $\mathcal{F} \otimes \mathfrak{g}$. In many examples $A$ is the universal enveloping algebra of $\mathfrak{g}$. Our description of the algebra $(\mathfrak{g}, A)(\mathcal{F})$ has a striking resemblance to the commutator expansions of $\mathcal{F}$ used by M. Kapranov in his approach to noncommutative geometry. To each algebra $(\mathfrak{g}, A)(\mathcal{F})$ we associate a "noncommutative algebraic" group which naturally acts on $(\mathfrak{g}, A)(\mathcal{F})$ by conjugations and conclude the paper with some examples of such groups.


*Keywords:* Lie algebra; Semisimple Lie algebra; Lie group; Noncommutative ring

**Contents**



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## 0. Introduction

The aim of this paper is to introduce and study algebraic groups and Lie algebras over noncommutative rings.

Our approach is motivated by the following considerations. A naive definition of a Lie algebra as a bimodule over a noncommutative associative algebra $\mathcal{F}$ (over a field $\Bbbk$) does not bring any interesting examples beyond Lie algebra $gl_n(\mathcal{F})$. Even the special Lie algebra $sl_n(\mathcal{F}) = [gl_n(\mathcal{F}), gl_n(\mathcal{F})]$ is not an $\mathcal{F}$-bimodule. Similarly, the special linear group $SL_n(\mathcal{F})$ is not defined by equations but rather by congruences given by the Dieudonné determinant (see [1]). This is why the “straightforward” approach to classical groups over rings started by J. Dieudonné in [4] and continued by O.T. O'Meara and others (see [6]) does not lead to new algebraic groups. Also, unlike in the commutative case, these methods do not employ rich structural theory of Lie algebras.

As a starting point, we observe that the Lie algebra $sl_n(\mathcal{F})$ where $\mathcal{F}$ is an associative algebra over a field $\Bbbk$ (of characteristic 0) is the Lie subalgebra of $M_n(\mathcal{F}) = \mathcal{F} \otimes M_n(\Bbbk)$ generated by $\mathcal{F} \otimes sl_n(\Bbbk)$ (all tensor products in the paper are taken over $\Bbbk$ unless specified otherwise). This motivates us to consider, for any Lie subalgebra $\mathfrak{g}$ of an associative algebra $A$, the Lie subalgebra

$$(\mathfrak{g}, A)(\mathcal{F}) \subset \mathcal{F} \otimes A \tag{0.1}$$

generated by $\mathcal{F} \otimes \mathfrak{g}$.

If $\mathcal{F}$ is commutative, then $\mathcal{F} \otimes \mathfrak{g}$ is already a Lie algebra, and $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \otimes \mathfrak{g}$. However, if $\mathcal{F}$ is noncommutative, this equality does not hold. Our first main result (Theorem 4.3) is a formula expressing $(\mathfrak{g}, A)(\mathcal{F})$ in terms of powers $\mathfrak{g}^n = Span\{g_1 g_2 \ldots g_n\colon g_1, \ldots, g_n \in \mathfrak{g}\}$ in $A$ for all *perfect pairs* $(\mathfrak{g}, A)$ in the sense of Definition 4.1. The class of perfect pairs is large enough—it includes all semisimple and Kac–Moody Lie algebras $\mathfrak{g}$.

More precisely, Theorem 4.3 states that for any perfect pair $(\mathfrak{g}, A)$ and any associative algebra $\mathcal{F}$ we have

$$(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \otimes \mathfrak{g} + \sum_{n \geqslant 1} I_n \otimes \left[\mathfrak{g}, \mathfrak{g}^{n+1}\right] + [\mathcal{F}, I_{n-1}] \otimes \mathfrak{g}^{n+1}, \tag{0.2}$$

where $I_0 \supset I_1 \subset I_2 \supset \cdots$ is the descending filtration of two-sided ideals of $\mathcal{F}$ defined inductively as $I_0 = \mathcal{F}$, $I_{n+1} = \mathcal{F}[\mathcal{F}, I_n] + [\mathcal{F}, I_n]$. It is remarkable that this filtration emerged in works by M. Kapranov in [7] and then by M. Kontsevich and A. Rosenberg in [8] as an important tool in noncommutative geometry.

We also get a compact formula (Theorem 4.9) for $(sl_2(\Bbbk), A)(\mathcal{F})$ which, hopefully, has physical implications.

For unital $\mathcal{F}$ we define the *“noncommutative” current group* or, in short, the $\mathcal{N}$-current group $G_{\mathfrak{g},A}(\mathcal{F})$ to be the set of all invertible elements $X \in \mathcal{F} \otimes A$ such that $X \cdot (\mathfrak{g}, A)(\mathcal{F}) \cdot X^{-1} = (\mathfrak{g}, A)(\mathcal{F})$. This is our generalization of $GL_n(\mathcal{F})$. In fact, if $\mathfrak{g} = sl_n(\Bbbk) \subset M_n(\Bbbk) = A$ then $G_{\mathfrak{g},A}(\mathcal{F}) = GL_n(\Bbbk)$.

However, for other compatible pairs the structure of $G_{\mathfrak{g},A}(\mathcal{F})$ is rather nontrivial even for classical Lie algebras $\mathfrak{g} = o_n(\Bbbk)$ and $g = sp_n(\Bbbk)$ and $A = M_n(\Bbbk)$. To demonstrate this, we explicitly compute the "Cartan subgroup" of $G_{\mathfrak{g},A}(\mathcal{F})$ (Proposition 5.11) as follows. For the above classical compatible pair $(\mathfrak{g}, A)$ an invertible diagonal matrix $D = diag(f_1, \dots, f_n) \in M_n(\mathcal{F})$ belongs to the $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if

$$f_i f_{n-i+1} - f_1 f_n \in I_1 = \mathcal{F}[\mathcal{F}, \mathcal{F}]$$

for $i = 1, \dots, n$.

Our computation of the "Cartan subgroup" for $\mathfrak{g} = sl_2(\Bbbk)$ and $A = M_n(\Bbbk)$ is dramatically harder and constitutes the second main result of this paper (Theorem 5.12). More precisely, let $\mathfrak{g} = sl_2(\Bbbk) \subset End_{\Bbbk}(V) = M_n(\Bbbk) = A$, where $V = S^{n-1}(\Bbbk^2)$ is the simple $n$-dimensional $sl_2(\Bbbk)$-module. Then an invertible diagonal matrix $diag(f_1, f_2, \dots, f_n)$ belongs to $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} f_{i+1} f_{i+2}^{-1} \in I_m \tag{0.3}$$

for $m = 1, 2, \dots, n-2$.

We can also apply our functorial generalization of $GL_n(\mathcal{F})$ to $K$-theory (however, we postpone all computations for concrete rings $\mathcal{F}$ until a separate paper). We propose to generalize the fundamental inclusion which plays a pivotal role in the algebraic $K$-theory

$$E_n(\mathcal{F}) \subset GL_n(\mathcal{F}) \tag{0.4}$$

where $E_n(\mathcal{F})$ is the subgroup generated by matrices $1 + tE_{ij}$, $t \in \mathcal{F}$, $i \neq j$. Here the $E_{ij}$ are elementary matrices. It is well known and widely used that $E_n(\mathcal{F})$ is normal in $GL_n(\mathcal{F})$ for $n \geqslant 3$ (and for certain algebras $\mathcal{F}$ when $n = 2$).

It turns out that both groups $E_n(\mathcal{F})$ and $GL_n(\mathcal{F})$ are completely determined by the compatible pair $(\mathfrak{g}, A)$, where $\mathfrak{g} := sl_n(\Bbbk) \subset M_n(\Bbbk) = A$. Then $sl_n(\mathcal{F}) = (\mathfrak{g}, A)(\mathcal{F})$, $GL_n(\mathcal{F}) = G_{\mathfrak{g},A}(\mathcal{F})$ and the group $E_n(\mathcal{F})$ is generated by all elements $g \in G_{\mathfrak{g},A}(\mathcal{F})$ of the form $g = 1 + t \otimes E$ where $E \in \mathfrak{g}$ is a nilpotent in $A$, $t \in \mathcal{F}$.

Motivated by this observation, we propose to generalize the inclusion (0.4) to all of our $\mathcal{N}$-current groups as follows. We define (Section 5.1) the group $E_{\mathfrak{g},A}(\mathcal{F})$ to be the sub-group of $G_{\mathfrak{g},A}(\mathcal{F})$ generated by all elements of the form $1 + t \otimes s$ where $t \in \mathcal{F}$ and $s \in \mathfrak{g}$ is a nilpotent in $A$. Clearly, $E_{\mathfrak{g},A}(\mathcal{F})$ is a normal subgroup of $G_{\mathfrak{g},A}(\mathcal{F})$ which plays the role of $E_n(\mathcal{F})$ in $GL_n(\mathcal{F})$. This defines a generalized $K_1$-functor

$$\mathcal{F} \mapsto K_1^{\mathfrak{g},A}(\mathcal{F}) = G_{\mathfrak{g},A}(\mathcal{F}) / E_{\mathfrak{g},A}(\mathcal{F}).$$

In Section 5.1 we also construct other generalized $K_1$-functors in which we replace $E_{\mathfrak{g},A}(\mathcal{F})$ by $E_{S,\mathfrak{g},A}(\mathcal{F})$, the subgroup generated by $1 + S$, where $S$ is a $G_{\mathfrak{g},A}(\mathcal{F})$-invariant subset of nilpotents in $\mathcal{F} \otimes A$. However, computation of the generalized $K_1$-functors is beyond the scope of the present paper and will be performed in a separate publication.

The paper is organized as follows.

• Section 1 contains some preliminary results on ideals in associative algebras $\mathcal{F}$ generated by $k$th commutator spaces of $\mathcal{F}$. Several key results are based on the Jacobi–Leibniz type identity (1.5).

• In Section 2 we introduce $\mathcal{N}$-Lie algebras and their important subclass: $\mathcal{N}$-current Lie algebras $(\mathfrak{g}, A)(\mathcal{F})$ over Lie algebras $\mathfrak{g}$. As our first examples, we describe algebras $(\mathfrak{g}, A)(\mathcal{F})$ for all classical Lie algebras.

• Section 3 contains upper bounds for $\mathcal{N}$-current Lie algebras.

• Section 4 contains our main result for Lie algebras $(\mathfrak{g}, A)(\mathcal{F})$: upper bounds for algebras $(\mathfrak{g}, A)(\mathcal{F})$ coincide with them for a large class of compatible pairs $(\mathfrak{g}, A)$ including all such pairs for semisimple Lie algebras $\mathfrak{g}$.

• In Section 5 we introduce affine $\mathcal{N}$-groups and $\mathcal{N}$-current groups, there relation with $\mathcal{N}$-Lie algebras and $\mathcal{N}$-current Lie algebras, important classes of their normal subgroups similar to subgroups $E_n(\mathcal{F})$ and the corresponding $K_1$-functors. We also consider useful examples of $\mathcal{N}$-subgroups and their "Cartan subgroups" attached to the standard representations of classical Lie algebras $\mathfrak{g}$ and to various representations of $\mathfrak{g} = sl_2(\Bbbk)$. Our description of these subgroups is based on a new class of algebraic identities for noncommutative difference derivatives (Lemma 5.25) which are of interest by themselves.

The present paper continues our study of algebraic groups over noncommutative rings and their representations started in [2]. Part of our results was published in [3]. In the next paper we will focus on "reductive groups over noncommutative rings," their geometric structure and representations.

Throughout the paper, **Alg** will denote the category which objects are associative algebras (not necessarily with 1) over a field $\Bbbk$ of characteristic zero and morphisms are algebra homomorphisms; and $\mathcal{N}$ will stand for a sub-category of **Alg**. Also $\mathbf{Alg}_1$ will denote that sub-category of **Alg** which objects are unital $\Bbbk$-algebras over $\Bbbk$ and arrows are homomorphisms of unital algebras.

## 1. Commutator expansions and identities

Given an object $\mathcal{F} \in \mathbf{Alg}$, for $k \geqslant 0$ define the $k$th *commutator space* $\mathcal{F}^{(k)}$ of $\mathcal{F}$ recursively as $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(1)} = \mathcal{F}' = [\mathcal{F}, \mathcal{F}]$, $\mathcal{F}^{(2)} = \mathcal{F}'' = [\mathcal{F}, \mathcal{F}']$, $\ldots$, $\mathcal{F}^{(k)} = [\mathcal{F}, \mathcal{F}^{(k-1)}]$, $\ldots$, where for any subsets $S_1, S_2$ of $\mathcal{F}$ the notation $[S_1, S_2]$ stands for the linear span of all commutators $[a, b] = ab - ba$, $a \in S_1$, $b \in S_2$. In a similar fashion, for each subset $S \subset \mathcal{F}$ define the subspaces $S^{(k)} \subset \mathcal{F}^{(k)}$ by $S^{(0)} = span(S)$, $S^{(k)} = [S, S^{(k-1)}]$ for $k > 0$; and the subspace $S^{(\bullet)}$ in $\mathcal{F}$ by

$$S^{(\bullet)} = \sum_{k \geqslant 0} S^{(k)}. \tag{1.1}$$

The following result is obvious.

**Lemma 1.1.** *For any $S \subset \mathcal{F}$ the subspace $S^{(\bullet)}$ is the Lie subalgebra of $\mathcal{F}$ generated by $S$.*

Following [7] and [8], define the subspaces $I_k^\ell(\mathcal{F})$ by:

$$I_k^\ell(\mathcal{F}) = \sum_\lambda \mathcal{F}^{(\lambda_1)} \mathcal{F}^{(\lambda_2)} \cdots \mathcal{F}^{(\lambda_\ell)},$$

where the summation goes over all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in (\mathbb{Z}_{\geqslant 0})^\ell$ such that $\sum_{i=1}^{\ell} \lambda_i = k$. Denote also

$$I_k^{\leqslant \ell}(\mathcal{F}) := \sum_{1 \leqslant \ell' \leqslant \ell} I_k^{\ell'}(\mathcal{F}), \qquad I_k(\mathcal{F}) := I_k^{<\infty} = \sum_{\ell \geqslant 1} I_k^{\ell}(\mathcal{F}). \tag{1.2}$$

Clearly, $\mathcal{F} I_k^\ell(\mathcal{F}), I_k^\ell(\mathcal{F})\mathcal{F} \subset I_k^{\ell+1}(\mathcal{F})$. Therefore, $I_k(\mathcal{F})$ is a two-sided ideal in $\mathcal{F}$. Taking into account that $\mathcal{F}^{(k)}\mathcal{F} + \mathcal{F}^{(k)} = \mathcal{F}\mathcal{F}^{(k)} + \mathcal{F}^{(k)}$ for all $k$, it is easy to see that $I_k(\mathcal{F}) = I_k^k(\mathcal{F}) + \mathcal{F} I_k^k(\mathcal{F})$ for all $k$.

**Lemma 1.2.** *For each $k, \ell \geqslant 1$ one has:*

(a) $I_k^\ell(\mathcal{F}) \subset I_{k-1}^\ell(\mathcal{F})$, $I_k^{\leqslant \ell}(\mathcal{F}) \subset I_{k-1}^{\leqslant \ell}(\mathcal{F})$.
(b) $[\mathcal{F}, I_{k-1}^\ell(\mathcal{F})] \subset I_k^\ell(\mathcal{F})$, $[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})] \subset I_k^{\leqslant \ell}(\mathcal{F})$.
(c) $I_k^{\leqslant \ell+1}(\mathcal{F}) = \mathcal{F}[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leqslant \ell+1}(\mathcal{F})]$.

**Proof.** To prove (a) and (b), we need the following obvious recursion for $I_k^\ell(\mathcal{F})$, $\ell > 1$:

$$I_k^\ell(\mathcal{F}) = \sum_{i \geqslant 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) \tag{1.3}$$

(with the natural convention that $I_{k'}^{\ell'}(\mathcal{F}) = 0$ if $k' < 0$). Then we prove (a) by induction in $\ell$. If $\ell = 1$, the assertion becomes $\mathcal{F}^{(k)} \subset \mathcal{F}^{(k-1)}$. Iterating this inclusion and using the inductive hypothesis, we obtain for $\ell > 1$

$$\begin{aligned} I_k^\ell(\mathcal{F}) &= \sum_{i \geqslant 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) = \mathcal{F} I_k^{\ell-1}(\mathcal{F}) + \sum_{i>0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) \\ &\subset \mathcal{F} I_{k-1}^{\ell-1}(\mathcal{F}) + \sum_{i>0} \mathcal{F}^{(i-1)} I_{k-i}^{\ell-1}(\mathcal{F}) = \sum_{i \geqslant 0} \mathcal{F}^{(i)} I_{k-i-1}^{\ell-1}(\mathcal{F}) = I_{k-1}^\ell(\mathcal{F}). \end{aligned}$$

This proves (a).

Prove (b) also by induction in $\ell$. If $\ell = 1$, the assertion becomes $[\mathcal{F}, \mathcal{F}^{(k-1)}] \subset \mathcal{F}^{(k)}$, which is obvious. Using the inductive hypothesis, we obtain

$$\begin{aligned} \big[\mathcal{F}, I_{k-1}^\ell(\mathcal{F})\big] &= \sum_{i \geqslant 1} \big[\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\ell-1}(\mathcal{F})\big] \subset \sum_{i \geqslant 1} \big[\mathcal{F}, \mathcal{F}^{(i-1)}\big] I_{k-i}^{\ell-1}(\mathcal{F}) + \mathcal{F}^{(i-1)} \big[\mathcal{F}, I_{k-i}^{\ell-1}(\mathcal{F})\big] \\ &\subset \sum_{i \geqslant 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) = I_k^\ell(\mathcal{F}). \end{aligned}$$

This proves (b).

Prove (c). Obviously, $I_k^{\leqslant \ell+1}(\mathcal{F}) \supset \mathcal{F}[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leqslant \ell+1}(\mathcal{F})]$ by (b). Therefore, it suffices to prove the opposite inclusion

$$I_k^{\leqslant \ell+1}(\mathcal{F}) \subset \mathcal{F}\big[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})\big] + \big[\mathcal{F}, I_{k-1}^{\leqslant \ell+1}(\mathcal{F})\big].$$

We will use the following obvious consequence of (1.3):

$$I_k^{\leqslant \ell+1}(\mathcal{F}) = \sum_{i \geqslant 0} \mathcal{F}^{(i)} I_{k-i}^{\leqslant \ell}(\mathcal{F}).$$

Therefore, it suffices to prove that

$$\mathcal{F}^{(i)} I_{k-i}^{\leqslant \ell}(\mathcal{F}) \subset \mathcal{F}\big[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})\big] + \big[\mathcal{F}, I_{k-1}^{\leqslant \ell+1}(\mathcal{F})\big] \tag{1.4}$$

for all $i \geqslant 0$, $\ell \geqslant 1$, $k \geqslant 1$. We prove (1.4) by induction in all pairs $(\ell, i)$ ordered lexicographically. Indeed, suppose that the assertion is proved for all $(\ell', i') < (\ell, i)$. The base of induction is when $\ell = 1$, $i = 0$. Indeed, $I_k^{\leqslant 1}(\mathcal{F}) = \mathcal{F}^{(k)}$ for all $k$ and (1.4) becomes $\mathcal{F}\mathcal{F}^{(k)} \subset \mathcal{F}[\mathcal{F}, \mathcal{F}^{(k-1)}] + [\mathcal{F}, I_{k-1}^{\leqslant 2}(\mathcal{F})]$, which is obviously true since $[\mathcal{F}, \mathcal{F}^{(k-1)}] = \mathcal{F}^{(k)}$.

If $\ell \geqslant 1$, $i > 0$, we obtain, using the Leibniz rule, the following inclusion:

$$\mathcal{F}^{(i)} I_{k-i}^{\leqslant \ell}(\mathcal{F}) = \big[\mathcal{F}, \mathcal{F}^{(i-1)}\big] I_{k-i}^{\leqslant \ell}(\mathcal{F}) \subset \big[\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\leqslant \ell}(\mathcal{F})\big] + \mathcal{F}^{(i-1)}\big[\mathcal{F}, I_{k-i}^{\leqslant \ell}(\mathcal{F})\big].$$

Therefore, by (b)

$$\mathcal{F}^{(i)} I_{k-i}^{\leqslant \ell}(\mathcal{F}) \subset \big[\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\leqslant \ell}(\mathcal{F})\big] + \mathcal{F}^{(i-1)} I_{k+1-i}^{\leqslant \ell}(\mathcal{F}).$$

Finally, using the inductive hypothesis for $(\ell, i-1)$ and taking into account that $\mathcal{F}^{(i-1)} I_{k-i}^{\leqslant \ell}(\mathcal{F}) \subset I_{k-1}^{\leqslant \ell+1}(\mathcal{F})$, and, therefore,

$$\big[\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\leqslant \ell}(\mathcal{F})\big] \subset \big[\mathcal{F}, I_{k-1}^{\leqslant \ell+1}(\mathcal{F})\big],$$

we obtain the inclusion (1.4).

If $\ell \geqslant 2$, $i = 0$, then using the inductive hypothesis for all pairs $(\ell - 1, i')$, $i' \geqslant 0$, we obtain:

$$I_k^{\leqslant \ell}(\mathcal{F}) = \mathcal{F}\big[\mathcal{F}, I_{k-1}^{\leqslant \ell-1}(\mathcal{F})\big] + \big[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})\big].$$

Multiplying by $\mathcal{F}$ on the left we obtain:

$$\mathcal{F} I_k^{\leqslant \ell}(\mathcal{F}) = \mathcal{F}^2\big[\mathcal{F}, I_{k-1}^{\leqslant \ell-1}(\mathcal{F})\big] + \mathcal{F}\big[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})\big] = \mathcal{F}\big[\mathcal{F}, I_{k-1}^{\leqslant \ell}(\mathcal{F})\big]$$

because $\mathcal{F}^2 \subset \mathcal{F}$ and $I_{k-1}^{\leqslant \ell-1}(\mathcal{F}) \subset I_{k-1}^{\leqslant \ell}(\mathcal{F})$. This immediately implies (1.4).

Part (c) is proved. The lemma is proved. □

**Lemma 1.3.** *For any $k', k \geqslant 0$, and any $\ell, \ell' \geqslant 1$ one has:*

(a) $I_k^{\ell}(\mathcal{F}) I_{k'}^{\ell'}(\mathcal{F}) \subset I_{k+k'}^{\ell+\ell'}(\mathcal{F})$, $I_k^{\leqslant \ell}(\mathcal{F}) I_{k'}^{\leqslant \ell'}(\mathcal{F}) \subset I_{k+k'}^{\leqslant \ell+\ell'}(\mathcal{F})$.
(b) $[I_k^{\ell}(\mathcal{F}), I_{k'}^{\ell'}(\mathcal{F})] \subset [\mathcal{F}, I_{k+k'}^{\ell+\ell'-1}(\mathcal{F})]$, $[I_k^{\leqslant \ell}(\mathcal{F}), I_{k'}^{\leqslant \ell'}(\mathcal{F})] \subset [\mathcal{F}, I_{k+k'}^{\leqslant \ell+\ell'-1}(\mathcal{F})]$.

**Proof.** Part (a) follows from the obvious fact that

$$\big(\mathcal{F}^{(\lambda_1)}\mathcal{F}^{(\lambda_2)}\cdots\mathcal{F}^{(\lambda_{\ell_1})}\big)\big(\mathcal{F}^{(\mu_1)}\mathcal{F}^{(\mu_2)}\cdots\mathcal{F}^{(\mu_{\ell_2})}\big)\subset I_k^{\ell_1+\ell_2}(\mathcal{F}),$$

where $k=\lambda_1+\lambda_2+\cdots+\lambda_{\ell_1}+\mu_1+\mu_2+\cdots+\mu_{\ell_2}$.

Prove (b). First, we prove the first inclusion for $\ell=1$. We proceed by induction on $k$. The base of induction, $k=0$, is obvious because $I_0^1(\mathcal{F})=\mathcal{F}$. Assume that the assertion is proved for all $k_1<k$, i.e., we have:

$$\big[\mathcal{F}^{(k_1)},I_{k'}^{\ell'}(\mathcal{F})\big]\subset\big[\mathcal{F},I_{k_1+k'}^{\ell'}(\mathcal{F})\big].$$

Then, using the fact that $\mathcal{F}^{(k)}=[\mathcal{F},\mathcal{F}^{(k-1)}]$ and the Jacobi identity, we obtain:

$$\begin{aligned}\big[\mathcal{F}^{(k)},I_{k'}^{\ell'}(\mathcal{F})\big]&=\big[\big[\mathcal{F},\mathcal{F}^{(k-1)}\big],I_{k'}^{\ell'}(\mathcal{F})\big]\\&\subset\big[\mathcal{F},\big[\mathcal{F}^{(k-1)},I_{k'}^{\ell'}(\mathcal{F})\big]\big]+\big[\mathcal{F}^{(k-1)},\big[\mathcal{F},I_{k'}^{\ell'}(\mathcal{F})\big]\big]\\&\subset\big[\mathcal{F},\big[\mathcal{F},I_{k'+k-1}^{\ell'}(\mathcal{F})\big]\big]+\big[\mathcal{F}^{(k-1)},I_{k'+1}^{\ell'}(\mathcal{F})\big]\subset\big[\mathcal{F},I_{k'+k}^{\ell'}(\mathcal{F})\big]\end{aligned}$$

by the inductive hypothesis and Lemma 1.2(b). This proves the first inclusion of (b) for $\ell=1$.

Furthermore, we will proceed by induction on $\ell$. Now $\ell>1$, assume that the assertion is proved for all $\ell_1<\ell$, i.e., we have the inductive hypothesis in the form:

$$\big[I_k^{\ell_1}(\mathcal{F}),I_{k'}^{\ell'}(\mathcal{F})\big]\subset\big[\mathcal{F},I_{k+k'}^{\ell_1+\ell'-1}(\mathcal{F})\big]$$

for all $k,k'\geqslant 0$.

We need the following useful Jacobi–Leibniz type identity in $\mathcal{F}$:

$$[ab,c]+[bc,a]+[ca,b]=0 \tag{1.5}$$

for all $a,b,c\in\mathcal{F}$. (The identity was communicated to the authors by C. Reutenauer and was used in a different context in the recent paper [5].)

Using (1.3) and (1.5) with all $a\in\mathcal{F}^{(i)}$, $b\in I_{k-i}^{\ell-1}(\mathcal{F})$, $c\in I_{k'}^{\ell'}(\mathcal{F})$, we obtain for all $i\geqslant 0$:

$$\begin{aligned}\big[\mathcal{F}^{(i)}I_{k-i}^{\ell-1}(\mathcal{F}),I_{k'}^{\ell'}(\mathcal{F})\big]&\subset\big[\mathcal{F}^{(i)},I_{k-i}^{\ell-1}(\mathcal{F})I_{k'}^{\ell'}(\mathcal{F})\big]+\big[I_{k-i}^{\ell-1}(\mathcal{F}),I_{k'}^{\ell'}(\mathcal{F})\mathcal{F}^{(i)}\big]\\&\subset\big[\mathcal{F}^{(i)},I_{k+k'-i}^{\ell+\ell'-1}(\mathcal{F})\big]+\big[I_{k-i}^{\ell-1}(\mathcal{F}),I_{k'+i}^{\ell'+1}(\mathcal{F})\big]\subset\big[\mathcal{F},I_{k+k'}^{\ell+\ell'-1}(\mathcal{F})\big]\end{aligned}$$

by the already proved (a) and inductive hypothesis. This finishes the proof of the first inclusion of (b). The second inclusion of (b) also follows. □

Generalizing (1.3), for any subset $S$ of $\mathcal{F}$ denote by $I_k^\ell(\mathcal{F},S)$ the image of *Span* $S^{\otimes(k+\ell)}$ under the canonical map $\mathcal{F}^{\otimes(k+\ell)}\twoheadrightarrow I_k^\ell(\mathcal{F})$, i.e.,

$$I_k^\ell(\mathcal{F},S)=\sum_\lambda S^{(\lambda_1)}S^{(\lambda_2)}\cdots S^{(\lambda_\ell)}, \tag{1.6}$$

$\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)\in(\mathbb{Z}_{\geqslant 0})^\ell$ such that $\sum_{i=1}^{\ell}\lambda_i=k$.

In particular, $I_k^1(\mathcal{F}, S) = S^{(k)}$ and $I_0^\ell = S^\ell$.

The following result is obvious.

**Lemma 1.4.** *Let $\mathcal{F}$ be an object of $\mathcal{N}$ and $S \subset \mathcal{F}$. Then:*

(a) *For any $k \geqslant 0$, $\ell \geqslant 2$ one has*

$$I_k^\ell(\mathcal{F}, S) = \sum_{i=0}^{k} S^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}, S).$$

(b) *For any $k', k \geqslant 0$, and any $\ell, \ell' \geqslant 1$ one has:*

$$I_k^\ell(\mathcal{F}, S) I_{k'}^{\ell'}(\mathcal{F}, S) \subset I_{k+k'}^{\ell+\ell'}(\mathcal{F}, S), \qquad \left[I_k^\ell(\mathcal{F}, S), I_{k'}^{\ell'}(\mathcal{F}, S)\right] \subset I_{k+k'+1}^{\ell+\ell'-1}(\mathcal{F}, S).$$

*In particular,*

$$S^{(i)} I_k^\ell(\mathcal{F}, S) \subset I_{k+i}^{\ell+1}(\mathcal{F}, S), \qquad \left[S^{(i)}, I_k^\ell(\mathcal{F}, S)\right] \subset I_{k+i+1}^{\ell}(\mathcal{F}, S). \tag{1.7}$$

## 2. $\mathcal{N}$-Lie algebras and $\mathcal{N}$-current Lie algebras

Given objects $\mathcal{F}$ and $\mathcal{A}$ of **Alg**, we refer to a morphism $\iota : \mathcal{F} \to \mathcal{A}$ in **Alg** as an *$\mathcal{F}$-algebra structure on $\mathcal{A}$* (we will also refer to $\mathcal{A}$ as an *$\mathcal{F}$-algebra*).

Note that each $\mathcal{F}$-algebra structure on $\mathcal{A}$ turns $\mathcal{A}$ into an algebra in the category of $\mathcal{F}$-bimodules (i.e., $\mathcal{A}$ admits two $\mathcal{F}$-actions $\mathcal{F} \otimes \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \otimes \mathcal{F} \to \mathcal{A}$ via $f \otimes a \mapsto \iota(f) \cdot a$ and $a \otimes f \mapsto a \cdot \iota(f)$ respectively).

We fix an arbitrary sub-category $\mathcal{N}$ of **Alg** throughout the section. In most cases we take $\mathcal{N} = \mathbf{Alg}$.

**Definition 2.1.** An $\mathcal{N}$-Lie algebra is a triple $(\mathcal{F}, \mathcal{L}, \mathcal{A})$, where $\mathcal{F}$ is an object of $\mathcal{N}$, $\mathcal{A}$ is an $\mathcal{F}$-algebra, and $\mathcal{L}$ is an *$\mathcal{F}$-Lie subalgebra of* $\mathcal{A}$, i.e., if $\mathcal{L}$ is a Lie subalgebra (under the commutator bracket) of $\mathcal{A}$ invariant under the adjoint action of $\mathcal{F}$ on $\mathcal{A}$ given by $(f, a) \mapsto \iota(f) \cdot a - a \cdot \iota(f)$ for all $f \in \mathcal{F}$, $a \in \mathcal{A}$.

A morphism $(\mathcal{F}_1, \mathcal{L}_1, \mathcal{A}_1) \to (\mathcal{F}_2, \mathcal{L}_2, \mathcal{A}_2)$ of $\mathcal{N}$-Lie algebras is a pair $(\varphi, \psi)$, where $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism in $\mathcal{N}$ and $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ is a morphism in **Alg** such that $\psi(\mathcal{L}_1) \subset \mathcal{L}_2$ and $\psi \circ \iota_1 = \iota_2 \circ \varphi$.

Denote by $\mathbf{LieAlg}_{\mathcal{N}}$ the category of $\mathcal{N}$-Lie algebras.

For an $\mathcal{N}$-Lie algebra $(\mathcal{F}, \mathcal{L}, \mathcal{A})$, let $L_i(\mathcal{F}, \mathcal{L}, \mathcal{A}) := (\mathcal{F}, \mathcal{L}^{(i)}, \mathcal{A})$, $0 \leqslant i \leqslant j \leqslant 3$, where $\mathcal{L}^{(i)}$, $i = 1, 2, 3$, are given by:

- $\mathcal{L}^{(1)}$ is the normalizer Lie algebra of $\mathcal{L}$ in $\mathcal{A}$.
- $\mathcal{L}^{(2)}$ is the Lie subalgebra of $\mathcal{A}$ generated by $\iota(\mathcal{F}) \subset \mathcal{A}$ and by the semigroup $\mathcal{S} = \{s \in \mathcal{A}{:}\ s \cdot \mathcal{L} = \mathcal{L} \cdot s\}$.
- $\mathcal{L}^{(3)}$ is the Lie subalgebra of $\mathcal{A}$ generated by $\mathcal{G}(\iota(\mathcal{F})) \subset \mathcal{A}$, where $\mathcal{G}$ is the stabilizer of $\mathcal{L}$ in the group $Aut_{\Bbbk}(\mathcal{A})$, i.e.,

$$\mathcal{G} = \left\{g \in Aut_{\Bbbk}(\mathcal{A}){:}\ g(\mathcal{L}) = \mathcal{L}\right\}. \tag{2.1}$$

The following result is obvious.

**Lemma 2.2.** *For each $\mathcal{N}$-Lie algebra $(\mathcal{F}, \mathcal{L}, \mathcal{A})$ and $i = 1, 2, 3$ the triple $L_i(\mathcal{F}, \mathcal{L}, \mathcal{A})$ is also an $\mathcal{N}$-Lie algebra.*

Therefore, we can construct a number of new $\mathcal{N}$-Lie algebras by combining the operations $L_i$ for a given $\mathcal{N}$-Lie algebra.

**Remark 2.3.** In general, none of $L_i$ defines a functor $\mathbf{LieAlg}_{\mathcal{N}} \to \mathbf{LieAlg}_{\mathcal{N}}$. However, for each $i = 1, 2, 3$ one can find an appropriate subcategory $\mathcal{C}$ of $\mathbf{LieAlg}_{\mathcal{N}}$ such the restriction of $L_i$ to $\mathcal{C}$ is a functor $\mathcal{C} \to \mathbf{LieAlg}_{\mathcal{N}}$.

**Remark 2.4.** The operation $L_3$ is interesting only when $\mathcal{F}$ is noncommutative because for any object $(\mathcal{F}, \mathcal{L}, \mathcal{A})$ of $\mathbf{LieAlg}_{\mathcal{N}}$ such that $\mathcal{F}$ is commutative and all automorphisms of $\mathcal{A}$ are inner, one obtains $\mathcal{L}^{(3)} = \iota(\mathcal{F})$ and therefore, $L_3(\mathcal{F}, \mathcal{L}, \mathcal{A}) = (\mathcal{F}, \iota(\mathcal{F}), \mathcal{A})$.

Denote by $\pi$ the natural (forgetful) projection functor $\mathbf{LieAlg}_{\mathcal{N}} \to \mathcal{N}$ such that $\pi(\mathcal{F}, \mathcal{L}, \mathcal{A}) = \mathcal{F}$ and $\pi(\varphi, \psi) = \varphi$.

**Definition 2.5.** A *noncommutative current Lie algebra* (*$\mathcal{N}$-current Lie algebra*) is a functor $\mathfrak{s} : \mathcal{N} \to \mathbf{LieAlg}_{\mathcal{N}}$ such that $\pi \circ \mathfrak{s} = Id_{\mathcal{N}}$ (i.e., $\mathfrak{s}$ is a section of $\pi$).

Note that if $\mathcal{N} = (\mathcal{F}, Id_{\mathcal{F}})$ has only one object $\mathcal{F}$ and only the identity arrow $Id_{\mathcal{F}}$, then the $\mathcal{N}$-current Lie algebra is simply any object of $\mathbf{LieAlg}_{\mathbf{Alg}}$ of the form $(\mathcal{F}, \mathcal{L}, \mathcal{A})$. In this case, we will sometimes refer to the Lie algebra $\mathcal{L}$ as an *$\mathcal{F}$-current Lie algebra*.

In principle, we can construct a number of $\mathcal{F}$-current Lie algebras by twisting a given one with operations $L_i$ from Lemma 2.2. However, the study of such "derived" $\mathcal{F}$-current Lie algebras is beyond the scope of the present paper.

In what follows we will suppress the tensor sign in expressions like $\mathcal{F} \otimes A$ and write $\mathcal{F} \cdot A$ instead. Note that for any object $A$ of $\mathbf{Alg}_1$ and any object $\mathcal{F}$ of $\mathbf{Alg}$ the product $\mathcal{F} \otimes A$ is naturally an $\mathcal{F}$-algebra via the embedding $\mathcal{F} \hookrightarrow \mathcal{F} \cdot A$ $(f \mapsto f \cdot 1)$.

The following is a first obvious example of $\mathcal{N}$-current Lie algebras.

**Lemma 2.6.** *For any object algebra $A$ of $\mathbf{Alg}_1$ and any object $\mathcal{F}$ of $\mathcal{N}$ define the object $\mathfrak{s}_A(\mathcal{F}) = (\mathcal{F}, \mathcal{F} \cdot A, \mathcal{F} \cdot A)$ of $\mathbf{LieAlg}_{\mathcal{N}}$. Then the association $\mathcal{F} \mapsto \mathfrak{s}_A(\mathcal{F})$ defines a noncommutative current Lie algebra $\mathfrak{s}_A : \mathcal{N} \to \mathbf{LieAlg}_{\mathcal{N}}$.*

The main object of our study will be a refinement of the above example. Given an object $A$ of $\mathbf{Alg}_1$, and a subspace $\mathfrak{g} \subset A$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ (i.e., $\mathfrak{g}$ is a Lie subalgebra of $A$), we say that $(\mathfrak{g}, A)$ is a *compatible pair*. For any compatible pair $(\mathfrak{g}, A)$ and an object $\mathcal{F}$ of $\mathcal{N}$, denote by $(\mathfrak{g}, A)(\mathcal{F})$ the Lie subalgebra of the $\mathcal{F} \cdot A = \mathcal{F} \otimes A$ (under the commutator bracket) generated by $\mathcal{F} \cdot \mathfrak{g}$, that is, $(\mathfrak{g}, A)(\mathcal{F}) = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$ in notation (1.1).

**Proposition 2.7.** *For any compatible pair $(\mathfrak{g}, A)$ the association*

$$\mathcal{F} \mapsto \big(\mathcal{F}, (\mathfrak{g}, A)(\mathcal{F}), \mathcal{F} \cdot A\big)$$

*defines the $\mathcal{N}$-current Lie algebra*

$$(\mathfrak{g}, A) : \mathcal{N} \to \mathbf{LieAlg}_{\mathcal{N}}.$$

**Proof.** It suffices to show that any arrow $\varphi$ in $\mathcal{N}$, i.e., any algebra homomorphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ defines a homomorphism of Lie algebras $(\mathfrak{g}, A)(\mathcal{F}_1) \to (\mathfrak{g}, A)(\mathcal{F}_2)$. We need the following obvious fact.

**Lemma 2.8.** *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be objects of* **Alg** *and let $\varphi : A_1 \to A_2$ be a morphism in* **Alg**. *Let $S_1 \subset \mathcal{A}_1$ and $S_2 \subset \mathcal{A}_2$ be two subsets such that $\varphi(S_1) \subset S_2$. Then the restriction of $\varphi$ to the Lie algebra $S_1^{(\bullet)}$ (in notation* (1.1)*) is a homomorphism of Lie algebras $S_1^{(\bullet)} \to S_2^{(\bullet)}$.*

Applying Lemma 2.8 with $\mathcal{A}_i = \mathcal{F}_i \cdot A$, $S_i = \mathcal{F}_i \cdot \mathfrak{g}$, $i = 1, 2$, $\varphi = f \otimes id_A : \mathcal{F}_1 \cdot A \to \mathcal{F}_2 \cdot A$, we obtain a Lie algebra homomorphism $(\mathfrak{g}, A)(\mathcal{F}_1) = (\mathcal{F}_1 \cdot \mathfrak{g})^{(\bullet)} \to (\mathcal{F}_2 \cdot \mathfrak{g})^{(\bullet)} = (\mathfrak{g}, A)(\mathcal{F}_2)$.

It remains to show that the action of $\mathcal{F}$ on $\mathcal{L} = (\mathfrak{g}, A)(\mathcal{F}) = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$ is stable under the commutator bracket with $\mathcal{F}$. Indeed, $S = \mathcal{F} \cdot \mathfrak{g}$ is invariant under the adjoint action of $\mathcal{F}$ on $\mathcal{F} \cdot A$. By induction and the Jacobi identity $\mathcal{L} = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$ is also invariant under this action of $\mathcal{F}$. The proposition is proved. □

If $\mathcal{F}$ is commutative, then $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g}$ is the $\mathcal{F}$-current algebra. Therefore, if $\mathcal{F}$ is an arbitrary object of $\mathcal{N}$, the Lie algebra $(\mathfrak{g}, A)(\mathcal{F})$ deserves a name of the *$\mathcal{N}$-current Lie algebra associated with the compatible pair* $(\mathfrak{g}, A)$.

If $A = U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, then we abbreviate $\mathfrak{g}(\mathcal{F}) := (\mathfrak{g}, U(\mathfrak{g}))(\mathcal{F})$. Another natural choice of $A$ is algebra $End(V)$, where $V$ is a faithful $\mathfrak{g}$-module. In this case, we will sometimes abbreviate $(\mathfrak{g}, V)(\mathcal{F}) := (\mathfrak{g}, End(V))(\mathcal{F})$.

The following result provides an estimation of $(\mathfrak{g}, A)(\mathcal{F})$ from below. Set

$$\langle \mathfrak{g} \rangle = \sum_{k \geqslant 1} \mathfrak{g}^k, \tag{2.2}$$

i.e., $\langle \mathfrak{g} \rangle$ is the associative subalgebra of $A$ generated by $\mathfrak{g}$.

**Proposition 2.9.** *Let $(\mathfrak{g}, A)$ be a compatible pair and $\mathcal{F}$ be an object of $\mathcal{N}$. Then*

(a) $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \subset (\mathfrak{g}, A)(\mathcal{F})$ *and* $\mathcal{F}\mathcal{F}^{(k)} \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] \subset (\mathfrak{g}, A)(\mathcal{F})$ *for all* $k \geqslant 0$.
(b) *If $\mathfrak{g}$ is abelian, i.e., $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = 0$, then*

$$(\mathfrak{g}, A)(\mathcal{F}) = \sum_{k \geqslant 0} \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1}. \tag{2.3}$$

(c) *If $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$ (i.e., $\mathcal{F}$ is perfect as a Lie algebra), then $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \langle \mathfrak{g} \rangle$.*

**Proof.** Prove (a). We need the following technical result.

**Lemma 2.10.** *Let $(\mathfrak{g}, A)$ be a compatible pair. For all $m \geqslant 2$ denote by $\widetilde{\mathfrak{g}^m}$ the $\Bbbk$-linear span of all powers $g^m$, $g \in \mathfrak{g}$. Then for any $m \geqslant 2$ one has*

$$\widetilde{\mathfrak{g}^m} + \left(\mathfrak{g}^{m-1} \cap \mathfrak{g}^m\right) = \mathfrak{g}^m. \tag{2.4}$$

**Proof.** Since $\mathfrak{g}^{i-1}\mathfrak{g}'\mathfrak{g}^{m-i-1} \subset \mathfrak{g}^{m-1}$ for all $i \leqslant m-1$, we obtain the following congruence for any $c = (c_1, \dots, c_m)$, $c_i \in (\Bbbk - \{0\})$ and $x = (x_1, \dots, x_m)$, $x_i \in \mathfrak{g}$, $i = 1, 2, \dots, m$:

$$(c_1x_1 + \cdots + c_mx_m)^m \equiv \sum_\lambda \binom{m}{\lambda} c^\lambda x^\lambda \mod \left(\mathfrak{g}^{m-1} \cap \mathfrak{g}^m\right),$$

where the summation is over all partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of $m$ and we abbreviated $c^\lambda = c_1^{\lambda_1} \cdots c_m^{\lambda_m}$ and $x^\lambda = x_1^{\lambda_1} \cdots x_m^{\lambda_m}$. Varying $c = (c_1, \dots, c_m)$, the above congruence implies that each monomial $x^\lambda$ belongs to $\widetilde{\mathfrak{g}^m} + (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m)$. In particular, taking $\lambda = (1, 1, \dots, 1)$, we obtain $\mathfrak{g}^m \subseteq \widetilde{\mathfrak{g}^m} + (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m)$. Taking into account that $\widetilde{\mathfrak{g}^m} \subseteq \mathfrak{g}^m$, we obtain (2.4). The lemma is proved. □

We also need the following useful identity in $\mathcal{F} \cdot A$:

$$[sE, tF] = st \cdot [E, F] + [s, t] \cdot FE = ts \cdot [E, F] + [s, t] \cdot EF \tag{2.5}$$

for any $s, t \in \mathcal{F}$, $E, F \in A$.

We prove the first inclusion of (a) by induction on $k$. If $k = 0$, one obviously has $\mathcal{F}^{(0)}\mathfrak{g}^1 = \mathcal{F} \cdot \mathfrak{g} \subset (\mathfrak{g}, A)(\mathcal{F})$. Assume now that $k > 0$. Then for $g \in \mathfrak{g}$ we obtain by using (2.5):

$$\left[\mathcal{F} \cdot g, \mathcal{F}^{(k-1)} \cdot g^k\right] = \left[\mathcal{F}, \mathcal{F}^{(k-1)}\right] \cdot g^{k+1} = \mathcal{F}^{(k)} \cdot g^{k+1}$$

which implies that $\mathcal{F}^{(k)} \cdot \widetilde{\mathfrak{g}^{k+1}} \subset (\mathfrak{g}, A)(\mathcal{F})$ (in the notation of Lemma 2.10). Using Lemma 2.10, we obtain

$$\mathcal{F}^{(k)} \cdot \widetilde{\mathfrak{g}^{k+1}} \equiv \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \mod \mathcal{F}^{(k)} \cdot \left(\mathfrak{g}^k \cap \mathfrak{g}^{k+1}\right).$$

Taking into account that $\mathcal{F}^{(k)} \cdot (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \subset \mathcal{F}^{(k-1)} \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$ by the inductive hypothesis (here we used the inclusion $\mathcal{F}^{(k)} \subset \mathcal{F}^{(k-1)}$), the above relation implies that $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1}$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This proves the first inclusion of (a). To prove the second inclusion, we compute using (2.5)

$$\left[\mathcal{F} \cdot \mathfrak{g}, \mathcal{F}^{(k-1)} \cdot \mathfrak{g}^k\right] \equiv \mathcal{F}\mathcal{F}^{(k-1)} \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right] \mod \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1}.$$

Therefore, using the already proved inclusion $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \subset (\mathfrak{g}, A)(\mathcal{F})$, we see that $\mathcal{F}\mathcal{F}^{(k-1)} \cdot [\mathfrak{g}, \mathfrak{g}^k]$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This finishes the proof of (a).

Prove (b). Clearly, (a) implies that $(\mathfrak{g}, A)(\mathcal{F})$ contains the right-hand side of (2.3). Therefore, it suffices to prove that the latter space is closed under the commutator. Indeed, since $\mathfrak{g}$ is abelian, one has

$$\begin{aligned}\left[\mathcal{F}^{(k_1)} \cdot \mathfrak{g}^{k_1+1}, \mathcal{F}^{(k_2)} \cdot \mathfrak{g}^{k_2+1}\right] &= \left[\mathcal{F}^{(k_1)}, \mathcal{F}^{(k_2)}\right] \cdot \mathfrak{g}^{k_1+k_2+2} \\ &\subset \mathcal{F}^{(k_1+k_2+1)} \cdot \mathfrak{g}^{k_1+k_2+2} \subset (\mathfrak{g}, A)(\mathcal{F})\end{aligned}$$

because $[\mathcal{F}^{(k_1)}, \mathcal{F}^{(k_2)}] \subset \mathcal{F}^{(k_1+k_2+1)}$. This finishes the proof of (b).

Prove (c). Since $\mathcal{F}' = \mathcal{F}$, the already proved part (a) implies $\mathcal{F} \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$ for all $k \geqslant 1$, therefore, $\mathcal{F} \cdot \langle \mathfrak{g} \rangle \subseteq (\mathfrak{g}, A)(\mathcal{F})$. Since $\langle \mathfrak{g} \rangle$ is an associative subalgebra of $A$ containing $\mathfrak{g}$, we obtain an opposite inclusion $(\mathfrak{g}, A)(\mathcal{F}) \subseteq \mathcal{F} \cdot \langle \mathfrak{g} \rangle$. This finishes the proof of (c).

The proposition is proved. □

**Remark 2.11.** Proposition 2.9(c) shows that the case when $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$ is not of much interest. This happens, for example, when $\mathcal{F}$ is a Weyl algebra or the quantum torus. In these cases a natural anti-involution on $\mathcal{F}$ can be taken into account. We will discuss it in a separate paper.

**Definition 2.12.** We say that a compatible pair $(\mathfrak{g}, A)$ is of *finite type* if there exist $m > 0$ such that $\mathfrak{g} + \mathfrak{g}^2 + \cdots + \mathfrak{g}^m = A$, and we call such minimal $m$ the *type* of $(\mathfrak{g}, A)$. If such $m$ does not exists, we say that $(\mathfrak{g}, A)$ is of infinite type.

Note that $(\mathfrak{g}, A)$ is of type 1 if and only if $\mathfrak{g} = A$, which, in its turn, implies that $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot A$ for all objects $\mathcal{F}$ of $\mathcal{N}$. Note also that if $\langle g \rangle = A$ and $A$ is finite-dimensional over $\Bbbk$, then $(\mathfrak{g}, A)$ is always of finite type.

**Proposition 2.13.** *Assume that* $(\mathfrak{g}, A)$ *is of type* 2*, i.e.,* $\mathfrak{g} \neq A$ *and* $\mathfrak{g} + \mathfrak{g}^2 = A$*. Then*

$$(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \mathcal{F}' \cdot A + \mathcal{F}\mathcal{F}' \cdot [A, A], \tag{2.6}$$

*where* $\mathcal{F}' = [\mathcal{F}, \mathcal{F}]$.

**Proof.** Since Furthermore, Proposition 2.9(a) guarantees that

$$\mathcal{F} \cdot \mathfrak{g} + \mathcal{F}' \cdot \mathfrak{g}^2 + \mathcal{F}\mathcal{F}' \cdot \left[\mathfrak{g}, \mathfrak{g}^2\right] \subset (\mathfrak{g}, A)(\mathcal{F}).$$

Clearly, $\mathcal{F} \cdot \mathfrak{g} + \mathcal{F}' \cdot \mathfrak{g}^2 = \mathcal{F} \cdot \mathfrak{g} + \mathcal{F}' \cdot A$ (because $\mathcal{F}' \subset \mathcal{F}$). Let us now prove that $[\mathfrak{g}, A] = [A, A]$. Obviously, $[\mathfrak{g}, A] \subseteq [A, A]$. The opposite inclusion immediately follows from inclusion $[\mathfrak{g}^2, \mathfrak{g}^2] \subseteq [\mathfrak{g}, \mathfrak{g}^3]$, which, in its turn follows from (1.5): taking any $a \in \mathfrak{g}$, $b \in \mathfrak{g}$, $c \in \mathfrak{g}^2$ in (1.5), we obtain $[ab, c] \in [\mathfrak{g}, \mathfrak{g}^3]$.

Using the equation $[\mathfrak{g}, A] = [A, A]$ we obtain $\mathcal{F}\mathcal{F}' \cdot [A, A] \subset (\mathfrak{g}, A)(\mathcal{F})$. This proves that $(\mathfrak{g}, A)(\mathcal{F})$ contains the right-hand side of (2.6).

To finish the proof, it suffices to show that the latter subspace is closed under the commutator. Indeed, abbreviating $A' = [A, A]$, we obtain

$$[\mathcal{F} \cdot \mathfrak{g}, \mathcal{F}' \cdot A] \subset \mathcal{F}\mathcal{F}' \cdot [\mathfrak{g}, A] + [\mathcal{F}, \mathcal{F}'] \cdot A\mathfrak{g} \subset \mathcal{F}\mathcal{F}' \cdot A' + \mathcal{F}' \cdot A,$$

$$[\mathcal{F} \cdot \mathfrak{g}, \mathcal{F}\mathcal{F}' \cdot A'] \subset \mathcal{F}^2\mathcal{F}' \cdot [\mathfrak{g}, A'] + [\mathcal{F}, \mathcal{F}\mathcal{F}'] \cdot A'\mathfrak{g} \subset \mathcal{F}\mathcal{F}' \cdot A' + \mathcal{F}' \cdot A,$$

$$[\mathcal{F}' \cdot A, \mathcal{F}' \cdot A] \subset (\mathcal{F}')^2 \cdot A' + [\mathcal{F}', \mathcal{F}'] \cdot A^2 \subset \mathcal{F}\mathcal{F}' \cdot A' + \mathcal{F}' \cdot A,$$

$$[\mathcal{F}' \cdot A, \mathcal{F}\mathcal{F}' \cdot A'] \subset \mathcal{F}'\mathcal{F}\mathcal{F}' \cdot [A, A'] + [\mathcal{F}', \mathcal{F}\mathcal{F}'] \cdot A'A$$

because $\mathcal{F}'\mathcal{F}\mathcal{F}' \cdot [A, A'] \subset \mathcal{F}\mathcal{F}' \cdot A' \subset (\mathfrak{g}, A)(\mathcal{F})$ and $[\mathcal{F}', \mathcal{F}\mathcal{F}'] \cdot A'A \subset \mathcal{F}' \cdot A \subset (\mathfrak{g}, A)(\mathcal{F})$. Finally,

$$[\mathcal{F}'\mathcal{F} \cdot A', \mathcal{F}\mathcal{F}' \cdot A'] \subset (\mathcal{F}'\mathcal{F})^2 \cdot [A', A'] + [\mathcal{F}'\mathcal{F}, \mathcal{F}\mathcal{F}'] \cdot (A')^2$$

because $(\mathcal{F}'\mathcal{F})^2 \cdot [A', A'] \subset \mathcal{F}'\mathcal{F} \cdot A'$ and $[\mathcal{F}'\mathcal{F}, \mathcal{F}\mathcal{F}'] \cdot [A, A]^2 \subset \mathcal{F}' \cdot A$. The proposition is proved. □

For any $\Bbbk$-vector space $V$ and any object $\mathcal{F}$ of $\mathcal{N}$ we abbreviate $sl(V,\mathcal{F}) := (sl(V), End(V))(\mathcal{F})$ and $gl(V,\mathcal{F}) := (End(V), End(V))(\mathcal{F}) = \mathcal{F} \cdot End(V)$. The following result shows that for $n = \dim V \geqslant 2$ the commutator Lie algebra $sl_n(\mathcal{F}) = [gl_n(\mathcal{F}), gl_n(\mathcal{F})]$ is, in fact, $sl(V,\mathcal{F})$.

**Corollary 2.14.** *Let $V$ be a finite-dimensional $\Bbbk$-vector space such that* $\dim V > 1$. *Then* $(\mathfrak{g}, A) = (sl(V), End(V))$ *is of type* 2 *and*

$$sl(V,\mathcal{F}) = \mathcal{F}' \cdot 1 + \mathcal{F} \cdot sl(V).$$

Hence $sl(V,\mathcal{F})$ is the set of all $X \in gl(V,\mathcal{F})$ such that $Tr(X) \in \mathcal{F}' = [\mathcal{F},\mathcal{F}]$ (where $Tr : gl(V,\mathcal{F}) = \mathcal{F} \cdot End(V) \to \mathcal{F}$ is the trivial extension of the ordinary trace $End(V) \to \Bbbk$).

**Proof.** Let us prove that the pair $(\mathfrak{g}, A) = (sl(V), End(V))$ is of type 2, i.e., $sl(V) + sl(V)^2 = End(V)$. It suffices to show that $1 \in sl(V)^2$. To prove it, choose a basis $e_1, \ldots, e_n$ in $V$ so that $V \cong \Bbbk^n$, $sl(V) \cong sl_n(\Bbbk)$ and $A = End(V) \cong M_n(\Bbbk)$. Indeed, for any indices $i \neq j$ both $E_{ij}$ and $E_{ji}$ belong to $sl(V)$, and $E_{ij}E_{ji} = E_{ii} \in sl(V)^2$. Therefore, $1 = \sum_{i=1}^n E_{ii}$ also belongs to $sl(V)^2$. Applying Proposition 2.13 and using the obvious fact that $[A, A] = sl(V)$, we obtain

$$sl(V,\mathcal{F}) = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot A + \mathcal{F}\mathcal{F}'[A, A] = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot 1.$$

This proves the first assertion. The second one follows from the obvious fact that the trace $Tr : \mathcal{F} \cdot End(V) \to \mathcal{F}$ is the projection to the second summand of the direct sum decomposition

$$\mathcal{F} \cdot End(V) = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot 1.$$

The corollary is proved. □

We can construct more pairs of type 2 as follows. Let $V$ be a $\Bbbk$-vector space and $\Phi : V \times V \to \Bbbk$ be a bilinear form on $V$. Denote by $o(\Phi)$ the orthogonal Lie algebra of $\Phi$, i.e.,

$$o(\Phi) = \bigl\{M \in End(V):\ \Phi\bigl(M(u), v\bigr) + \Phi\bigl(u, M(v)\bigr) = 0\ \forall u, v \in V\bigr\}.$$

Denote by $K_\Phi \subset V$ the sum of the left and the right kernels of $\Phi$ (if $\Phi$ is symmetric or skew-symmetric, then $K_\Phi$ is the left kernel of $\Phi$). Finally, denote by $End(V, K_\Phi)$ the parabolic subalgebra of $End(V)$ which consists of all $M \in End(V)$ such that $M(K_\Phi) \subset K_\Phi$. Clearly, $o(\Phi) \subset End(V, K_\Phi)$, i.e., $(o(\Phi), End(V, K_\Phi))$ is a compatible pair. For any object $\mathcal{F}$ of $\mathcal{N}$ we abbreviate $o(\Phi,\mathcal{F}) := (o(\Phi), End(V, K_\Phi))(\mathcal{F})$.

Denote by $sl(V, K_\Phi)$ the set of all $M$ in $End(V, K_\Phi)$ such that $Tr(M) = 0$ and $Tr(M_{K_\Phi}) = 0$, where $M_{K_\Phi} : K_\Phi \to K_\Phi$ is the restriction of $M$ to $K_\Phi$ and $1_{K_\Phi} \in End(V, K_\Phi)$ is any element such that $\Bbbk \cdot 1 + \Bbbk \cdot 1_{K_\Phi} + sl(V, K_\Phi) = End(V, K_\Phi)$. If $K_\Phi = 0$, we set $1_{K_\Phi} = 0$.

**Corollary 2.15.** *Let $V \neq 0$ be a finite-dimensional $\Bbbk$-vector space and $\Phi$ be a symmetric or skew-symmetric bilinear form on $V$. Then $(o(\Phi), End(V, K_\Phi))$ is of type 2 and*

$$o(\Phi, \mathcal{F}) = \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot 1 + \mathcal{F}' \cdot 1_K + (\mathcal{F}\mathcal{F}' + \mathcal{F}') \cdot sl(V, K_\Phi). \tag{2.7}$$

**Proof.** We will write $K$ instead of $K_\Phi$. First prove that $(\mathfrak{g}, A) = (o(\Phi), End(V, K))$ is of type 2. We pass to the algebraic closure of the involved objects, i.e., replace both $V$ and $K$ with $\overline{V} = \bar{\Bbbk} \cdot V = \bar{\Bbbk} \otimes V$, $\overline{K} = \bar{\Bbbk} \cdot K$, etc., where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$. Using the obvious fact that $\overline{U + U'} = \overline{U} + \overline{U'}$ and $\overline{U \otimes U'} = \overline{U} \otimes \overline{U'}$ for any subspaces of $End(V)$ and $\overline{o(\Phi)} = o(\overline{\Phi})$, we see that it suffices to show that the pair $(o(\overline{\Phi}), End(\overline{V}, \overline{K}))$ is of type 2.

Furthermore, without loss of generality we consider the case when $K = 0$, i.e., form $\Phi$ is nondegenerate. One can do it by using the block matrix decomposition with respect to a choice of compliment of $K$ over which $\Phi$ is nondegenerate.

We will prove the lemma when $\dim V > 2$ and leave the rest to the reader. If $\Phi$ is symmetric, one can choose a basis of $\overline{V}$ so that $\overline{V} \cong \bar{\Bbbk}^n$, and $\overline{\Phi}$ is the standard dot product on $\bar{\Bbbk}^n$. In this case $o(\overline{\Phi})$ is $o_n(\bar{\Bbbk})$, the Lie algebra of orthogonal matrices, which is generated by all elements $E_{ij} - E_{ji}$ where $E_{ij}$ is the corresponding elementary matrix. Using the identity $(E_{ij} - E_{ji})^2 = -(E_{ii} + E_{jj})$ for $i \neq j$, we see that $o_n(\bar{\Bbbk})^2$ contains all diagonal matrices. Furthermore, if $i, j, k$ are pairwise distinct indices then $(E_{ij} - E_{ji})(E_{jk} - E_{kj}) = E_{ik}$. Thus we have shown that $o_n(\bar{\Bbbk})^2 = M_n(\bar{\Bbbk}) = End(\overline{V}, \overline{K})$. Therefore, $o_n(\Bbbk)^2 = M_n(\Bbbk) = End(V, K)$. This proves the assertion for the symmetric $\Phi$.

If $\Phi$ is skew-symmetric and nondegenerate, then $n = 2m$ and one can choose a basis of $\overline{V}$ such that $V$ is identified with $\bar{\Bbbk}^n$ and $o(\overline{\Phi})$ is identified with the symplectic Lie algebra $sp_{2m}(\bar{\Bbbk})$. Recall that a basis in $sp_{2m}(\bar{\Bbbk})$ can be chosen as follows. It consists of elements $E_{ij} - E_{j+m,i+m}$, $E_{i,m+j} + E_{j,m+i}$, $E_{m+i,j} + E_{m+j,i}$, for $i, j \leqslant m$. Using the identity $(E_{i,i+m} + E_{i+m,i})^2 = E_{ii} + E_{i+m,i+m}$ and the fact that $(E_{ii} - E_{i+m,i+m}) \in sp_{2m}(\bar{\Bbbk})$, we see that all diagonal matrices belong $sp_{2m}(\bar{\Bbbk}) + sp_{2m}(\bar{\Bbbk})^2$.

Also, the identity $(E_{ii} - E_{i+m,i+m})(E_{ij} - E_{j+m,i+m}) = E_{ij}$ for $i \neq j$ implies that $E_{ij} \in sp_{2m}(\bar{\Bbbk})^2$ for all $i, j \leqslant m$. Similarly, one can prove that $E_{ij} \in sp_{2m}(\bar{\Bbbk})^2$ for $i, j \geqslant m$.

Furthermore, the identity $(E_{ii} - E_{i+m,i+m})(E_{i\ell} + E_{i+m,\ell-m}) = E_{i\ell} - E_{i+m,\ell-m}$ implies that $sp_{2m}(\bar{\Bbbk}) + sp_{2m}(\bar{\Bbbk})^2$ contains all $E_{ik}$ for $i \leqslant m, k > m$ and for $i > m, k \leqslant m$. Thus we have shown that $sp_{2m}(\bar{\Bbbk}) + sp_{2m}(\bar{\Bbbk})^2 = M_n(\bar{\Bbbk}) = End(\overline{V}, \overline{K})$. Therefore, $sp_{2m}(\Bbbk) + sp_{2m}(\Bbbk)^2 = M_n(\Bbbk) = End(V, K)$. This proves the assertion for the skew-symmetric $\Phi$. Prove (2.7) now. We abbreviate $A = End(V, K)$. Recall that $[A, A] = sl(V, K)$ and, if $K \neq \{0\}$, then $\Bbbk \cdot 1 + sl(V, K)$ is of codimension 1 in $A$, i.e., $1_K$ always exists. Therefore, applying Proposition 2.13, we obtain

$$\begin{aligned} o(\Phi, \mathcal{F}) &= \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot A + \mathcal{F}\mathcal{F}'[A, A] \\ &= \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot 1 + \mathcal{F}' \cdot 1_K + (\mathcal{F}\mathcal{F}' + \mathcal{F}') \cdot sl(V, K). \end{aligned}$$

This finishes the proof of Corollary 2.15. □

Note that our Lie algebras $o(\mathcal{F})$ and $sp(\mathcal{F})$ do not coincide with the usual orthogonal and symplectic Lie algebras, which are defined when the ring $\mathcal{F}$ possesses an involution.

## 3. Upper bounds of $\mathcal{N}$-current Lie algebras

For any compatible pair $(\mathfrak{g}, A)$ define two subspaces $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ of $\mathcal{F} \cdot A$ by:

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geqslant 1} I_k(\mathcal{F}) \cdot \left[\mathfrak{g}, \mathfrak{g}^{k+1}\right] + \left[\mathcal{F}, I_{k-1}(\mathcal{F})\right] \cdot \mathfrak{g}^{k+1}, \tag{3.1}$$

where $I_k(\mathcal{F})$ is defined in (1.3); and

$$\overline{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum I_{k_1}^{\ell_1+1} I_{k_2}^{\ell_2+1} \cdot \left[J_{\ell_1}^{k_1+1}, J_{\ell_2}^{k_2+1}\right] + \left[I_{k_1}^{\ell_1+1}, I_{k_2}^{\ell_2+1}\right] \cdot J_{\ell_2}^{k_2+1} J_{\ell_1}^{k_1+1}, \tag{3.2}$$

where the summation is over all $k_1, k_2 \geqslant 0$, $\ell_1, \ell_2 \geqslant 0$, and we abbreviated $I_k^\ell := I_k^\ell(\mathcal{F})$, $J_k^\ell := I_k^\ell(A, \mathfrak{g})$ in notation (1.6).

We will refer to $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ as the *upper bound* of $(\mathfrak{g}, A)(\mathcal{F})$ and to $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ as the *refined upper bound* of $(\mathfrak{g}, A)(\mathcal{F})$.

It is easy to see that the assignments $\mathcal{F} \mapsto \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and $\mathcal{F} \mapsto \overline{(\mathfrak{g}, A)}(\mathcal{F})$ are functors $\widetilde{(\mathfrak{g}, A)}$ and $\overline{(\mathfrak{g}, A)}$ from $\mathcal{N}$ to the category $Vect_{\Bbbk}$ of $\Bbbk$-vector spaces.

The following lemma is obvious.

**Lemma 3.1.** *If* $(\mathfrak{g}, A)$ *is a compatible pair of type* $m$ *(see Definition* 2.12*), then*

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k=1}^{m-1} I_k(\mathcal{F}) \cdot \left[\mathfrak{g}, \mathfrak{g}^{k+1}\right] + \left[\mathcal{F}, I_{k-1}(\mathcal{F})\right] \cdot \mathfrak{g}^{k+1}. \tag{3.3}$$

Now we formulate the main result of this section, which explains our terminology and proves that both $\widetilde{(\mathfrak{g}, A)}$ and $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ define $\mathcal{N}$-current Lie algebras $\mathcal{N} \to \mathbf{LieAlg}_{\mathcal{N}}$.

**Theorem 3.2.** *For any compatible pair* $(\mathfrak{g}, A)$ *and any object* $\mathcal{F}$ *of* $\mathcal{N}$ *one has:*

(a) *The subspace* $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ *is a Lie subalgebra of* $\mathcal{F} \cdot A$.
(b) *The subspace* $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ *is a Lie subalgebra of* $\mathcal{F} \cdot A$.
(c) $(\mathfrak{g}, A)(\mathcal{F}) \subseteq \overline{(\mathfrak{g}, A)}(\mathcal{F}) \subseteq \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$.

**Proof.** We need the following two lemmas.

**Lemma 3.3.** *For any compatible pair* $(\mathfrak{g}, A)$ *one has*

$$\left[\mathfrak{g}^{k+1}, \mathfrak{g}^m\right] \subset \left[\mathfrak{g}, \mathfrak{g}^{k+m}\right]$$

*for any* $k, m \geqslant 1$.

This is an obvious consequence of (1.5).

For any subsets $X$ and $Y$ of an object $\mathcal{A}$ of **Alg** and $\varepsilon \in \{0, 1\}$ denote

$$X \bullet_\varepsilon Y := \begin{cases} X \cdot Y & \text{if } \varepsilon = 0, \\ [X, Y] & \text{if } \varepsilon = 1. \end{cases}$$

**Lemma 3.4.** *Let $\Gamma$ be an abelian group and let $A$ and $\mathcal{F}$ be objects of* **Alg**. *Assume that $E_\alpha \subset \mathcal{F}$ and $B_\alpha \subset A$ are two families of subspaces labeled by $\Gamma$ such that*

$$E_\alpha \bullet_\varepsilon E_\beta \subseteq E_{\alpha+\beta+\varepsilon\cdot v}, \qquad B_\beta \bullet_\varepsilon B_\alpha \subseteq B_{\alpha+\beta-\varepsilon\cdot v} \tag{3.4}$$

*for all $\alpha, \beta \in \Gamma$, $\varepsilon \in \{0, 1\}$, where $v$ is a fixed element of $\Gamma$. Then for any $\alpha_0 \in \Gamma$ the subspace*

$$\mathfrak{h} = E_{\alpha_0} \cdot B_{\alpha_0+v} + \sum_{\alpha,\beta\in\Gamma,\, \varepsilon\in\{0,1\}} (E_\alpha \bullet_{1-\varepsilon} E_\beta) \cdot (B_{\beta+v} \bullet_\varepsilon B_{\alpha+v})$$

*is a Lie subalgebra of $\mathcal{F} \cdot A = \mathcal{F} \otimes A$.*

**Proof.** Eq. (2.5) implies that

$$[E \cdot B, E' \cdot B'] \subset (E \bullet_{1-\delta} E') \cdot (B' \bullet_\delta B)$$

for each $\delta \in \{0, 1\}$.

(i) Set $E = E_\alpha \bullet_{1-\varepsilon} E_\beta$, $B = B_{\beta+v} \bullet_\varepsilon B_{\alpha+v}$, $E' = E_{\alpha'} \bullet_{1-\varepsilon'} E_{\beta'}$, $B' = B_{\beta'+v} \bullet_{\varepsilon'} B_{\alpha'+v}$. Define $\alpha'' = \alpha + \beta + (1 - \varepsilon) \cdot v$ and $\beta'' = \alpha' + \beta' + (1 - \varepsilon') \cdot v$.

Taking into account that $E \subseteq E_{\alpha''}$, $E' \subseteq E_{\beta''}$, $B \subseteq B_{\alpha''+v}$, and $B' \subseteq B_{\beta''+v}$ by (3.4), we obtain for each $\delta \in \{0, 1\}$:

$$[E \cdot B, E' \cdot B'] \subset (E_{\alpha''} \bullet_{1-\delta} E_{\beta''}) \cdot (B_{\beta''+v} \bullet_\delta B_{\alpha''+v}) \subset \mathfrak{h}.$$

(ii) Set $E = E_{\alpha_0}$, $B = B_{\alpha_0+v}$, $E' = E_{\alpha'} \bullet_{1-\varepsilon'} E_{\beta'}$, $B' = B_{\beta'+v} \bullet_{\varepsilon'} B_{\alpha'+v}$. Define $\beta''$ as above. Taking into account that $E' \subseteq E_{\beta''}$ and $B' \subseteq B_{\beta''+v}$ by (3.4), where $\beta'' = \alpha' + \beta' + (1 - \varepsilon') \cdot v$, we obtain for each $\delta \in \{0, 1\}$:

$$[E \cdot B, E' \cdot B'] \subset (E_{\alpha_0} \bullet_{1-\delta} E_{\beta''}) \cdot (B_{\beta''+v} \bullet_\delta B_{\alpha_0+v}) \subset \mathfrak{h}.$$

(ii) Taking $E = E' = E_{\alpha_0}$, $B = B' = B_{\alpha_0+v}$, we obtain for each $\delta \in \{0, 1\}$:

$$[E \cdot B, E' \cdot B'] \subset (E_{\alpha_0} \bullet_{1-\delta} E_{\alpha_0}) \cdot (B_{\alpha_0+v} \bullet_\delta B_{\alpha_0+v}) \subset \mathfrak{h}.$$

The lemma is proved. □

Now we are going to prove the theorem part-by-part.

Prove (a). Using (2.5), we obtain

$$[\mathcal{F} \cdot \mathfrak{g}, \mathcal{F} \cdot \mathfrak{g}] \subset \mathcal{F}^2 \cdot [\mathfrak{g}, \mathfrak{g}] + [\mathcal{F}, \mathcal{F}] \cdot \mathfrak{g}^2 \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$$

because $\mathcal{F}^2 \subset \mathcal{F}$, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, and $I_0(\mathcal{F}) = \mathcal{F}$. Furthermore,

$$\big[\mathcal{F} \cdot \mathfrak{g}, I_k(\mathcal{F}) \cdot \big[\mathfrak{g}, \mathfrak{g}^{k+1}\big]\big] \subset \mathcal{F} I_k(\mathcal{F}) \cdot \big[\mathfrak{g}, \big[\mathfrak{g}, \mathfrak{g}^{k+1}\big]\big] + \big[\mathcal{F}, I_k(\mathcal{F})\big] \cdot \big[\mathfrak{g}, \mathfrak{g}^{k+1}\big]\mathfrak{g} \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$$

because $\mathcal{F}I_k(\mathcal{F}) \subset I_k(\mathcal{F})$, $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^{k+1}]] \subset [\mathfrak{g}, \mathfrak{g}^{k+1}]$, and $[\mathfrak{g}, \mathfrak{g}^{k+1}]\mathfrak{g} \subset \mathfrak{g}^{k+2}$. Finally, set $J_k := [\mathcal{F}, I_{k-1}(\mathcal{F})]$. Since $I_{k-1}(\mathcal{F})$ is a two-sided ideal in $\mathcal{F}$, we have $\mathcal{F}J_k \subset \mathcal{F}I_{k-1}(\mathcal{F}) \subset I_{k-1}(\mathcal{F})$. Lemma 1.2(b) taken with $\ell = \infty$, implies $J_k \subset I_k(\mathcal{F})$. Therefore, $[\mathcal{F}, J_k] \subset [\mathcal{F}, I_k(\mathcal{F})]$ and

$$\left[\mathcal{F} \cdot \mathfrak{g}, J_k \cdot \mathfrak{g}^{k+1}\right] \subset \mathcal{F} \cdot J_k \cdot \left[\mathfrak{g}, \mathfrak{g}^{k+1}\right] + [\mathcal{F}, J_k] \cdot \mathfrak{g}^{k+2} \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F}).$$

Note that for any $k, m \geqslant 1$ one has:

$$\left[I_k(\mathcal{F}) \cdot \mathfrak{g}^{k+1}, I_m(\mathcal{F}) \cdot \mathfrak{g}^{m+1}\right] \subset I_k(\mathcal{F})I_m(\mathcal{F}) \cdot \left[\mathfrak{g}^{k+1}, \mathfrak{g}^{m+1}\right] + \left[I_k(\mathcal{F}), I_m(\mathcal{F})\right] \cdot \mathfrak{g}^{k+m+2} \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$$

because $I_k(\mathcal{F})I_m(\mathcal{F}) \subset I_{k+m}(\mathcal{F})$ by Lemma 1.3(a), $[\mathfrak{g}^{k+1}, \mathfrak{g}^{m+1}] \subset [\mathfrak{g}, \mathfrak{g}^{k+m}]$ by Lemma 3.3, and $[I_k(\mathcal{F}), I_m(\mathcal{F})] \subset [\mathcal{F}, I_{k+m-1}(\mathcal{F})]$ by Lemma 1.3(b) taken with $\ell = \infty$. Therefore, taking into account that

$$[\mathbf{I}_k, \mathbf{I}_m] \subset \left[I_k(\mathcal{F}) \cdot \mathfrak{g}^{k+1}, I_m(\mathcal{F}) \cdot \mathfrak{g}^{m+1}\right] \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$$

for $\mathbf{I}_r$ standing for any of the spaces $I_r(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{r+1}], [\mathcal{F}, I_{r-1}(\mathcal{F})] \cdot \mathfrak{g}^{r+1}$ we finish the proof of (a).

Prove (b). Taking in Lemma 3.4: $\Gamma = \mathbb{Z}^2$, $\alpha = (k, \ell + 1) \in \mathbb{Z}^2$, $v = (1, -1)$, one can see that $E_\alpha$ equals to $I_k^{\ell+1}(\mathcal{F})$ if $k, \ell \geqslant 0$ and zero otherwise. Also, $B_{\alpha+v}$ equals to $I_\ell^{k+1}(A, \mathfrak{g})$ if $k, \ell \geqslant 0$ and zero otherwise.

Lemma 1.4 implies that (3.4) holds for all $\alpha, \beta \in \mathbb{Z}^2$, $\varepsilon \in \{0, 1\}$. Therefore, applying Lemma 3.4 with $\alpha_0 = (0, 1)$, we finish the proof of the assertion that $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F} \cdot A$. Prove (c). The first inclusion $(\mathfrak{g}, A)(\mathcal{F}) \subset \overline{(\mathfrak{g}, A)}(\mathcal{F})$ is obvious because $\mathcal{F} \cdot \mathfrak{g} \subset \overline{(\mathfrak{g}, A)}(\mathcal{F})$ and $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F} \cdot A$.

Let us prove the second inclusion $(\mathfrak{g}, A)(\mathcal{F}) \subset \overline{(\mathfrak{g}, A)}(\mathcal{F})$ of (c).

Rewrite the result of Lemma 1.3 (with $\ell_1 = \ell_2 = \infty$) in the form of (3.4) as:

$$I_{k_1}^{\ell_1+1}(\mathcal{F}) \bullet_{1-\varepsilon} I_{k_2}^{\ell_2+1}(\mathcal{F}) \subset I_{k_1}(\mathcal{F}) \bullet_{1-\varepsilon} I_{k_2}(\mathcal{F}) \subset \begin{cases} I_{k_1+k_2}(\mathcal{F}) & \text{if } \varepsilon = 1, \\ [\mathcal{F}, I_{k_1+k_2-1}(\mathcal{F})] & \text{if } \varepsilon = 0. \end{cases}$$

Using the obvious inclusion $J_k^{\ell+1} = I_k^{\ell+1}(A, \mathfrak{g}) \subset \mathfrak{g}^{k+1}$ for all $k, \ell \geqslant 0$ and Lemma 3.3, we obtain

$$J_{\ell_2}^{k_2+1} \bullet_\varepsilon J_{\ell_1}^{k_1+1} \subset \mathfrak{g}^{k_2+1} \bullet_\varepsilon \mathfrak{g}^{k_1+1} \subset \begin{cases} \mathfrak{g}^{k_1+k_2+2} & \text{if } \varepsilon = 0, \\ [\mathfrak{g}, \mathfrak{g}^{k_1+k_2+1}] & \text{if } \varepsilon = 1 \end{cases}$$

for all $k_1, k_2, \ell_1, \ell_2 \geqslant 0$, $\varepsilon \in \{0, 1\}$. Therefore, we get the inclusion:

$$\left(I_{k_1}^{\ell_1+1} \bullet_{1-\varepsilon} I_{k_2}^{\ell_2+1}\right) \cdot \left(J_{\ell_2}^{k_2+1} \bullet_\varepsilon J_{\ell_1}^{k_1+1}\right) \subset \begin{cases} I_{k_1+k_2}(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k_1+k_2+1}] & \text{if } \varepsilon = 1 \\ [\mathcal{F}, I_{k_1+k_2-1}(\mathcal{F})] \cdot \mathfrak{g}^{k_1+k_2+2} & \text{if } \varepsilon = 0 \end{cases} \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F}).$$

This proves the inclusion $\overline{(\mathfrak{g}, A)}(\mathcal{F}) \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and finishes the proof of (c).

Therefore, Theorem 3.2 is proved. □

Now we will refine Theorem 3.2 by introducing a natural filtration on each Lie algebra involved and proving the "filtered" version of the theorem.

For any compatible pair $(\mathfrak{g}, A)$, any object $\mathcal{F}$ of $\mathcal{N}$, and each $m \geqslant 1$ we define the subspaces $\mathcal{F} \cdot \langle \mathfrak{g} \rangle_m$, $(\mathfrak{g}, A)_m(\mathcal{F})$, $\widetilde{(\mathfrak{g}, A)}_m(\mathcal{F})$ and $\overline{(\mathfrak{g}, A)}_m(\mathcal{F})$ of $\mathcal{F} \cdot A$ by:

$$\mathcal{F} \cdot \langle \mathfrak{g} \rangle_m = \sum_{1 \leqslant k \leqslant m} \mathcal{F} \cdot \mathfrak{g}^k,$$

$$(\mathfrak{g}, A)_m(\mathcal{F}) = \sum_{0 \leqslant k < m} (\mathcal{F} \cdot \mathfrak{g})^{(k)}, \tag{3.5}$$

$$\widetilde{(\mathfrak{g}, A)}_m(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{1 \leqslant k < m} I_k^{\leqslant m-k}(\mathcal{F}) \cdot \left[\mathfrak{g}, \mathfrak{g}^{k+1}\right] + \left[\mathcal{F}, I_{k-1}^{\leqslant m-k}(\mathcal{F})\right] \cdot \mathfrak{g}^{k+1}, \tag{3.6}$$

where $I_k^{\leqslant \ell}(\mathcal{F})$ is defined in (1.3) and

$$\overline{(\mathfrak{g}, A)}_m(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum I_{k_1}^{\ell_1+1} I_{k_2}^{\ell_2+1} \cdot \left[J_{\ell_1}^{k_1+1}, J_{\ell_2}^{k_2+1}\right] + \left[I_{k_1}^{\ell_1+1}, I_{k_2}^{\ell_2+1}\right] \cdot J_{\ell_2}^{k_2+1} J_{\ell_1}^{k_1+1}, \tag{3.7}$$

where the summation is over all $k_1, k_2 \geqslant 0$, $\ell_1, \ell_2 \geqslant 0$ such that $k_1 + k_2 + \ell_1 + \ell_2 + 2 \leqslant m$, and we abbreviated $I_k^\ell := I_k^\ell(\mathcal{F})$, $J_k^\ell := I_k^\ell(A, \mathfrak{g})$ in the notation (1.6).

Recall that a Lie algebra $\mathfrak{h} = (\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \cdots)$ is called a *filtered Lie algebra* if $[\mathfrak{h}_{k_1}, \mathfrak{h}_{k_2}] \subset \mathfrak{h}_{k_1+k_2}$ for all $k_1, k_2 \geqslant 0$.

Taking into account that $[\mathfrak{g}^{k_1+1}, \mathfrak{g}^{k_2+1}] \subset \mathfrak{g}^{k_1+k_2+1}$, one can see that $\mathfrak{h}_m = \mathcal{F} \cdot \langle \mathfrak{g} \rangle_m$, $m \geqslant 1$, defines an increasing filtration on the Lie algebra $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$ (where $\langle \mathfrak{g} \rangle$ is as in (2.2)).

The following result is a filtered version of Theorem 3.2.

**Theorem 3.5.** *For any compatible pair* $(\mathfrak{g}, A)$ *and an object* $\mathcal{F}$ *of* $\mathcal{N}$

(a) $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ *is a filtered Lie subalgebra of* $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$.
(b) $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ *is a filtered Lie subalgebra of* $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$.
(c) *There is a chain of inclusions of filtered Lie algebras:*

$$(\mathfrak{g}, A)(\mathcal{F}) \subseteq \overline{(\mathfrak{g}, A)}(\mathcal{F}) \subseteq \widetilde{(\mathfrak{g}, A)}(\mathcal{F}).$$

The proof of Theorem 3.5 is almost identical to that of Theorem 3.2.

## 4. Perfect pairs and achievable upper bounds

Below we lay out some sufficient conditions on the compatible pair $(\mathfrak{g}, A)$ which guarantee that the upper bounds are achievable.

**Definition 4.1.** We say that a compatible pair $(\mathfrak{g}, A)$ is *perfect* if

$$\left[\mathfrak{g}, \mathfrak{g}^k\right]\mathfrak{g} + \left(\mathfrak{g}^k \cap \mathfrak{g}^{k+1}\right) = \mathfrak{g}^{k+1} \tag{4.1}$$

for all $k \geqslant 2$.

**Definition 4.2.** We say that a Lie algebra $\bar{\mathfrak{g}}$ over an algebraically closed field $\bar{\Bbbk}$ is *strongly graded* if there exists an element $h_0 \in \bar{\mathfrak{g}}$ such that:

(i) The operator $ad\, h_0$ on $\bar{\mathfrak{g}}$ is diagonalizable, i.e.,

$$\bar{\mathfrak{g}} = \bigoplus_{c \in \bar{\Bbbk}} \bar{\mathfrak{g}}_c, \tag{4.2}$$

where $\bar{\mathfrak{g}}_c \subset \bar{\mathfrak{g}}$ is an eigenspace of $ad\, h_0$ with the eigenvalue $c$.

(ii) The nullspace $\bar{\mathfrak{g}}_0$ of $ad\, h_0$ is spanned by $[\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_{-c}]$, $c \in \bar{\Bbbk} \setminus \{0\}$.

The class of strongly graded Lie algebras is rather large; it includes all semisimple and Kac–Moody Lie algebras, as well as the Virasoro algebra.

**Main Theorem 4.3.** *Let* $(\mathfrak{g}, A)$ *be a compatible pair. Then*

(a) *If* $(\mathfrak{g}, A)$ *is perfect, then for any object* $\mathcal{F}$ *of* $\mathcal{N}$ *one has*

$$(\mathfrak{g}, A)(\mathcal{F}) = \widetilde{(\mathfrak{g}, A)}(\mathcal{F}),$$

*i.e., the* $\mathcal{N}$*-current Lie algebras* $(\mathfrak{g}, A), \widetilde{(\mathfrak{g}, A)} : \mathcal{N} \to \mathbf{LieAlg}_{\mathcal{N}}$ *are equal.*

(b) *If* $\bar{\mathfrak{g}} = \bar{\Bbbk} \otimes \mathfrak{g}$ *is strongly graded, then* $(\mathfrak{g}, A)$ *is perfect.*

(c) *If* $\mathfrak{g}$ *is semisimple over* $\Bbbk$*, then for any object* $\mathcal{F}$ *of* $\mathcal{N}$ *one has*

$$(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geqslant 2} I_{k-1}(\mathcal{F}) \cdot \left(\mathfrak{g}^k\right)_+ + \left[\mathcal{F}, I_{k-2}(\mathcal{F})\right] \cdot Z_k(\mathfrak{g}), \tag{4.3}$$

*where* $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$ *is the "centerless" part of* $\mathfrak{g}^k$*,* $Z_k(\mathfrak{g}) = Z(\langle \mathfrak{g} \rangle) \cap \mathfrak{g}^k$*, and* $Z(\langle \mathfrak{g} \rangle)$ *is the center of* $\langle \mathfrak{g} \rangle = \sum_{k \geqslant 1} \mathfrak{g}^k$.

**Proof.** To prove the theorem we need a proposition and two lemmas.

**Proposition 4.4.** *Let* $(\mathfrak{g}, A)$ *be a compatible pair and* $\mathcal{F}$ *be an object of* $\mathcal{N}$.

(a) *Assume that for some* $k \geqslant 1$ *one has*

$$I \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right] \subset (\mathfrak{g}, A)(\mathcal{F})$$

*where* $I$ *is a left ideal in* $\mathcal{F}$*. Then:*

$$[\mathcal{F}, I] \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right]\mathfrak{g} \subset (\mathfrak{g}, A)(\mathcal{F}). \tag{4.4}$$

(b) *Assume that for some* $k \geqslant 1$ *one has*

$$J \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$$

*where $J$ is a subset of $\mathcal{F}$ such that $[\mathcal{F}, J] \subset J$. Then:*

$$[\mathcal{F}, J] \cdot \mathfrak{g}^{k+1} + (\mathcal{F}J + J) \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right] \subset (\mathfrak{g}, A)(\mathcal{F}). \tag{4.5}$$

**Proof.** Prove (a). Indeed,

$$\left[\mathcal{F} \cdot \mathfrak{g}, I \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right]\right] \equiv [\mathcal{F}, I] \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right]\mathfrak{g} \mod \mathcal{F}I \cdot \left[\mathfrak{g}, \left[\mathfrak{g}, \mathfrak{g}^k\right]\right].$$

Since $\mathcal{F}I \subset I$ and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^k]] \subset [\mathfrak{g}, \mathfrak{g}^k]$, and, therefore, $\mathcal{F}I \cdot [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^k]] \subset I \cdot [\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}, A)(\mathcal{F})$, the above congruence implies that $[\mathcal{F}, I] \cdot [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g}$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This proves (a).

Prove (b). For any $g \in \mathfrak{g}$ we obtain:

$$\left[\mathcal{F} \cdot g, J \cdot g^k\right] = [\mathcal{F}, J] \cdot g^{k+1}$$

which implies that $[\mathcal{F}, J] \cdot \widetilde{\mathfrak{g}^{k+1}} \subset (\mathfrak{g}, A)(\mathcal{F})$ (in the notation of Lemma 2.10). Using Lemma 2.10, we obtain

$$[\mathcal{F}, J] \cdot \widetilde{\mathfrak{g}^{k+1}} \equiv [\mathcal{F}, J] \cdot \mathfrak{g}^{k+1} \mod [\mathcal{F}, J] \cdot \left(\mathfrak{g}^k \cap \mathfrak{g}^{k+1}\right).$$

Taking into account that $[\mathcal{F}, J] \cdot (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \subset [\mathcal{F}, J] \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$, the above formula implies that $[\mathcal{F}, J] \cdot \mathfrak{g}^{k+1}$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. Furthermore,

$$\left[\mathcal{F} \cdot \mathfrak{g}, J \cdot \mathfrak{g}^k\right] \equiv \mathcal{F}J \cdot \left[\mathfrak{g}, \mathfrak{g}^k\right] \mod [\mathcal{F}, J] \cdot \mathfrak{g}^{k+1}.$$

Therefore, using the already proved inclusion $[\mathcal{F}, J] \cdot \mathfrak{g}^{k+1} \subset (\mathfrak{g}, A)(\mathcal{F})$, we see that $\mathcal{F}J \cdot [\mathfrak{g}, \mathfrak{g}^k]$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. Finally, using the fact that $[\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k$, we obtain $J \cdot [\mathfrak{g}, \mathfrak{g}^k] \subset J \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$. This proves (b).

Proposition 4.4 is proved. □

**Lemma 4.5.** *Let $(\mathfrak{g}, A)$ be a compatible pair. Assume that $h_0 \in \bar{\mathfrak{g}} = \bar{\Bbbk} \otimes \mathfrak{g}$ is such that $ad\, h_0$ is diagonalizable, i.e., has a decomposition* (4.2)*. Then:*

(a) *For each $k \geqslant 1$ and each $\mathbf{c} = (c_1, \ldots, c_{k+1}) \in \bar{\Bbbk}^{k+1} \setminus \{0\}$ the subspace $\bar{\mathfrak{g}}_{c_1} \cdots \bar{\mathfrak{g}}_{c_{k+1}}$ of $\bar{\mathfrak{g}}^{k+1}$ belongs to $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k]\bar{\mathfrak{g}} + (\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1})$.*
(b) *If $\bar{\mathfrak{g}} = \bar{\Bbbk} \otimes \mathfrak{g}$ is a strongly graded Lie algebra, then one has (in the notation of Definition* 4.2*):*

$$\bar{\mathfrak{g}}_0^{k+1} \subset \left[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k\right]\bar{\mathfrak{g}} + \left(\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1}\right).$$

**Proof.** Prove (a). Clearly, under the adjoint action of $h_0$ on $\bar{\mathfrak{g}}^k$ each vector of $x \in \bar{\mathfrak{g}}_{c_1} \cdots \bar{\mathfrak{g}}_{c_k}$ satisfies $[h_0, x] = (c_1 + \cdots + c_k)x$. Therefore, for any $(c_1, \ldots, c_k) \in \bar{\Bbbk}^k$ such that $c_1 + \cdots + c_k \neq 0$ the subspace $\bar{\mathfrak{g}}_{c_1} \cdots \bar{\mathfrak{g}}_{c_k}$ belongs to $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k]$. Clearly,

$$\bar{\mathfrak{g}}_{c_1} \cdots \bar{\mathfrak{g}}_{c_{k+1}} \equiv \bar{\mathfrak{g}}_{c_{\sigma(1)}} \cdots \bar{\mathfrak{g}}_{c_{\sigma(k)}} \bar{\mathfrak{g}}_{c_{\sigma(k+1)}} \mod \bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1}$$

for any permutation $\sigma \in S_{k+1}$.

It is also easy to see that for any $\mathbf{c} = (c_1, \dots, c_{k+1}) \in \bar{\Bbbk}^{k+1} \setminus \{0\}$ there exists a permutation $\sigma \in S_{k+1}$ such that $c_{\sigma(1)} + \cdots + c_{\sigma(k)} \neq 0$ and, therefore,

$$\bar{\mathfrak{g}}_{c_1} \cdots \bar{\mathfrak{g}}_{c_{k+1}} \subset (\bar{\mathfrak{g}}_{c_{\sigma(1)}} \cdots \bar{\mathfrak{g}}_{c_{\sigma(k)}}) \bar{\mathfrak{g}}_{c_{\sigma(k+1)}} + \bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1} \in [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k] \bar{\mathfrak{g}} + (\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1}).$$

This proves (a).

Prove (b) now. There exists $c \in \bar{\Bbbk} \setminus \{0\}$ such that $\bar{\mathfrak{h}}_c = [\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_{-c}] \neq 0$. Then we obtain the following congruence:

$$[\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_0^{k-1} \bar{\mathfrak{g}}_{-c}] \mathfrak{g}_0 \equiv \bar{\mathfrak{g}}_0^{k-1} \bar{\mathfrak{h}}_c \bar{\mathfrak{g}}_0 \mod [\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_0^{k-1}] \bar{\mathfrak{g}}_{-c} \bar{\mathfrak{g}}_0.$$

Taking into account that $[\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_0] \subset \bar{\mathfrak{g}}_c$, we obtain:

$$[\bar{\mathfrak{g}}_c, \bar{\mathfrak{g}}_0^{k-1}] \bar{\mathfrak{g}}_{-c} \bar{\mathfrak{g}}_0 \subset \sum_{i=1}^{k-1} \bar{\mathfrak{g}}_0^{i-1} \bar{\mathfrak{g}}_c \bar{\mathfrak{g}}_0^{k-1-i} \bar{\mathfrak{g}}_{-c} \bar{\mathfrak{g}}_0 \subset [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k] \bar{\mathfrak{g}} + (\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1})$$

by the already proved part (a). Therefore, $\bar{\mathfrak{g}}_0^{k-1} \bar{\mathfrak{h}}_c \bar{\mathfrak{g}}_0 \subset [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k] \bar{\mathfrak{g}} + (\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1})$. Since $\bar{\mathfrak{g}}$ is strongly graded, the subspaces $\mathfrak{h}_c$, $c \in \bar{\Bbbk} \setminus \{0\}$ span $\bar{\mathfrak{g}}_0$, and therefore, $\bar{\mathfrak{g}}_0^{k+1} \subset [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^k] \bar{\mathfrak{g}} + (\bar{\mathfrak{g}}^k \cap \bar{\mathfrak{g}}^{k+1})$. This proves (b).

The lemma is proved. □

**Lemma 4.6.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$. Then for any compatible pair $(\mathfrak{g}, A)$ one has the following decomposition of the $\mathfrak{g}$-module $\mathfrak{g}^k$, $k \geqslant 2$:*

$$\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^k] + Z_k(\mathfrak{g}), \qquad [\mathfrak{g}, \mathfrak{g}^k] \cap Z_k(\mathfrak{g}) = \{0\},$$

*where $Z_k(\mathfrak{g}) = Z(\langle \mathfrak{g} \rangle) \cap \mathfrak{g}^k$, and $Z(\langle \mathfrak{g} \rangle)$ is the center of $\langle \mathfrak{g} \rangle = \sum_{k \geqslant 0} \mathfrak{g}^k$.*

**Proof.** Clearly, $\mathfrak{g}^k$ is a semisimple finite-dimensional $\mathfrak{g}$-module (under the adjoint action). Therefore, it uniquely decomposes into isotypic components one of which, the component of invariants, is $Z_k(\mathfrak{g})$. Denote the sum of all noninvariant isotypic components by $(\mathfrak{g}^k)_+$. By definition, $\mathfrak{g}^k = (\mathfrak{g}^k)_+ + Z_k(\mathfrak{g})$ and $(\mathfrak{g}^k)_+ \cap Z_k(\mathfrak{g}) = 0$. It remains to prove that $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$. Indeed, $[\mathfrak{g}, \mathfrak{g}^k] \subseteq (\mathfrak{g}^k)_+$. On the other hand, each nontrivial irreducible $\mathfrak{g}$-submodule $V \subset \mathfrak{g}^k$ is faithful, i.e., $[\mathfrak{g}, V] = V$ (since $[\mathfrak{g}, V]$ is always a $\mathfrak{g}$-submodule of $V$). Therefore, $[\mathfrak{g}, \mathfrak{g}^k]$ contains all noninvariant isotypic components, i.e., $[\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}^k)_+$. The double inclusion obtained implies that $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$. The lemma is proved. □

Now we are ready to prove the theorem part-by-part.

Prove (a). In view of Theorem 3.2(c), it suffices to prove that $\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) \subset (\mathfrak{g}, A)(\mathcal{F})$, that is,

$$I_k(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] \subset (\mathfrak{g}, A)(\mathcal{F}), \qquad [\mathcal{F}, I_k(\mathcal{F})] \cdot \mathfrak{g}^{k+2} \subset (\mathfrak{g}, A)(\mathcal{F}) \tag{4.6}$$

for $k \geqslant 0$.

We will prove (4.6) by induction on $k$. First, verify the base of induction at $k = 0$. Obviously, $I_0(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}] \subset FF \cdot [\mathfrak{g}, \mathfrak{g}] \subset (\mathfrak{g}, A)(\mathcal{F})$. Furthermore, Proposition 4.4(b) taken with $k = 1$, $J = \mathcal{F}$ implies that $[\mathcal{F}, \mathcal{F}] \cdot \mathfrak{g}^2 \subset (\mathfrak{g}, A)(\mathcal{F})$.

Now assume that $k>0$. Using a part of the inductive hypothesis in the form $[\mathcal{F}, I_{k-1}(\mathcal{F})]\cdot \mathfrak{g}^{k+1}\subset(\mathfrak{g},A)(\mathcal{F})$ and applying Proposition 4.4(b) with $J=[\mathcal{F}, I_{k-1}(\mathcal{F})]$, we obtain $(\mathcal{F}J+J)\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\subset(\mathfrak{g},A)(\mathcal{F})$. In its turn, Lemma 1.2(c) taken with $\ell=\infty$ implies that $\mathcal{F}J+J=I_k(\mathcal{F})$. Therefore, we obtain

$$I_k(\mathcal{F})\cdot\left[\mathfrak{g},\mathfrak{g}^{k+1}\right]\subset(\mathfrak{g},A)(\mathcal{F}),$$

which is the first inclusion of (4.6). To prove the second inclusion (4.6), we will use Proposition 4.4(a) with $I=I_k(\mathcal{F})$:

$$(\mathfrak{g},A)(\mathcal{F})\supset\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\left[\mathfrak{g},\mathfrak{g}^{k+1}\right]\mathfrak{g}.$$

On the other hand, using the perfectness of the pair $(\mathfrak{g},A)$, we obtain:

$$\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\left[\mathfrak{g},\mathfrak{g}^{k+1}\right]\mathfrak{g}\equiv\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\mathfrak{g}^{k+2}\mod\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\mathfrak{g}^{k+1}\cup\mathfrak{g}^{k+2}.$$

But Lemma 1.2(a) taken with $\ell=\infty$ implies that $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$, therefore,

$$\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\left(\mathfrak{g}^{k+1}\cap\mathfrak{g}^{k+2}\right)\subset\left[\mathcal{F},I_k(\mathcal{F})\right]\cdot\mathfrak{g}^{k+1}\subset\left[\mathcal{F},I_{k-1}(\mathcal{F})\right]\cdot\mathfrak{g}^{k+1}\subset(\mathfrak{g},A)(\mathcal{F})$$

by the inductive hypothesis. This gives the second inclusion of (4.6). Therefore, Theorem 4.3(a) is proved.

Prove (b) now. Lemma 4.5 guarantees that for any strongly graded Lie algebra $\mathfrak{g}$ one has:

$$[\bar{\mathfrak{g}},\bar{\mathfrak{g}}^k]\bar{\mathfrak{g}}+\left(\bar{\mathfrak{g}}^k\cap\bar{\mathfrak{g}}^{k+1}\right)=\bar{\mathfrak{g}}^{k+1}$$

for all $k\geqslant 2$, where $\bar{\mathfrak{g}}=\bar{\Bbbk}\otimes\mathfrak{g}$ is the "algebraic closure" of $\mathfrak{g}$. Since the "algebraic closure" commutes with the multiplication and the commutator bracket in $A$, the restriction of the above equation to $\mathfrak{g}^{k+1}\subset\bar{\mathfrak{g}}^{k+1}$ becomes (4.1).

This finishes the proof of Theorem 4.3(b).

Prove (c) now. Since for each semisimple Lie algebra $\mathfrak{g}$ the compatible pair $(\mathfrak{g},A)$ is perfect by the already proved Theorem 4.3(b), Theorem 4.3(a) implies that $(\mathfrak{g},A)(\mathcal{F})=\widetilde{(\mathfrak{g},A)}(\mathcal{F})$. Therefore, in order to finish the proof of Theorem 4.3(c), it suffices to show that

$$\widetilde{(\mathfrak{g},A)}(\mathcal{F})=\mathcal{F}\cdot\mathfrak{g}+\sum_{k\geqslant 2}I_{k-1}(\mathcal{F})\cdot\left(\mathfrak{g}^k\right)_+ +\left[\mathcal{F},I_{k-2}(\mathcal{F})\right]\cdot Z_k(\mathfrak{g}).\tag{4.7}$$

Using Lemma 4.6 and the definition (3.1) of $\widetilde{(\mathfrak{g},A)}(\mathcal{F})$, we obtain

$$\begin{aligned}\widetilde{(\mathfrak{g},A)}(\mathcal{F})&=\mathcal{F}\cdot\mathfrak{g}+\sum_{k\geqslant 1}I_k(\mathcal{F})\cdot\left[\mathfrak{g},\mathfrak{g}^{k+1}\right]+\left[\mathcal{F},I_{k-1}(\mathcal{F})\right]\cdot\mathfrak{g}^{k+1}\\&=\mathcal{F}\cdot\mathfrak{g}+\sum_{k\geqslant 1}\left(I_k(\mathcal{F})+\left[\mathcal{F},I_{k-1}(\mathcal{F})\right]\right)\cdot\left[\mathfrak{g},\mathfrak{g}^{k+1}\right]+\left[\mathcal{F},I_{k-1}(\mathcal{F})\right]\cdot Z_{k+1}(\mathfrak{g}),\end{aligned}$$

which, after taking into account that $[\mathcal{F},I_{k-1}(\mathcal{F})]\subset I_k(\mathcal{F})$ (and shifting the index of summation), becomes the right-hand side of (4.7). This finishes the proof of Theorem 4.3(b).

Therefore, Theorem 4.3 is proved. □

The following is a direct corollary of Theorem 4.3.

**Corollary 4.7.** *Assume that a compatible pair $(\mathfrak{g}, A)$ is perfect and $\mathcal{F}$ is a $\Bbbk$-algebra satisfying $I_1(\mathcal{F}) = \mathcal{F}$. Then*

$$(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \mathcal{F} \cdot \big[\mathfrak{g}, \langle\mathfrak{g}\rangle\big] + [\mathcal{F}, \mathcal{F}] \cdot \langle\mathfrak{g}\rangle \tag{4.8}$$

*(where $\langle\mathfrak{g}\rangle = \sum_{k\geqslant 1} \mathfrak{g}^k$).*

**Proof.** First, show by induction that $I_k(\mathcal{F}) = \mathcal{F}$ for all $k \geqslant 1$. It follows immediately from Lemma 1.2(c) implying that $I_{k+1}(\mathcal{F}) = \mathcal{F}[\mathcal{F}, I_k(\mathcal{F})] + [\mathcal{F}, I_k(\mathcal{F})]$.

This and (3.1) imply that

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k\geqslant 1} \mathcal{F} \cdot \big[\mathfrak{g}, \mathfrak{g}^{k+1}\big] + [\mathcal{F}, \mathcal{F}] \cdot \mathfrak{g}^{k+1} = \mathcal{F} \cdot \mathfrak{g} + \mathcal{F} \cdot \big[\mathfrak{g}, \langle\mathfrak{g}\rangle\big] + [\mathcal{F}, \mathcal{F}] \cdot \langle\mathfrak{g}\rangle.$$

This and Theorem 4.3 finish the proof. □

**Remark 4.8.** The condition $I_1(\mathcal{F}) = \mathcal{F}$ holds for each noncommutative simple unital algebra $\mathcal{F}$, e.g., for each noncommutative skew-field $\mathcal{F}$ containing $\Bbbk$. Therefore, for all such algebras and any perfect pair $(\mathfrak{g}, A)$, the Lie algebra $(\mathfrak{g}, A)(\mathcal{F})$ is given by the relatively simple formula (4.8), which also complements (2.6).

The following result is a specialization of Theorem 4.3 to the case when $\mathfrak{g} = sl_2(\Bbbk)$.

**Theorem 4.9.** *Let $A$ an object of* $\mathbf{Alg}_1$ *containing $sl_2(\Bbbk)$ as a Lie subalgebra. Then*

$$\big(sl_2(\Bbbk), A\big)(\mathcal{F}) = \mathcal{F} \cdot sl_2(\Bbbk) + \big[\mathcal{F} \cdot 1, Z_1(A, \mathcal{F})\big] + \sum_{i\geqslant 1} Z_i(A, \mathcal{F}) \cdot V_{2i}, \tag{4.9}$$

*where*

$$Z_i(A, \mathcal{F}) = \sum_{j\geqslant 0} I_{i+2j-1}(\mathcal{F}) \cdot \Delta^j,$$

*$\Delta = 2EF + 2FE + H^2$ is the Casimir element, and $V_{2i}$ is the $sl_2(\Bbbk)$-submodule of $A$ generated by $E^i$. In particular, if $A = End(V)$, where $V$ is a simple $(m+1)$-dimensional $sl_2(\Bbbk)$-module, then*

$$\big(sl_2(\Bbbk), A\big)(\mathcal{F}) = [\mathcal{F}, \mathcal{F}] \cdot 1 + \sum_{k=1}^{m} I_{k-1}(\mathcal{F}) \cdot V_{2k}. \tag{4.10}$$

**Proof.** Prove (4.9). Clearly, each $\mathfrak{g}^k$ is a finite-dimensional $sl_2(\Bbbk)$-module generated by the highest weight vectors $\Delta^j E^i$, $i, j \geqslant 0$, $i + 2j \leqslant k$. That is, in notation of (4.3), one has

$$\big(\mathfrak{g}^k\big)_+ = \sum_{i>0,\, j\geqslant 0,\, i+2j\leqslant k} \Delta^j \cdot V_{2i},$$

where the sum is direct (but some summands may be zero) and

$$V_{2i}=\sum_{r=-i}^{i}\Bbbk\cdot(ad\,F)^{i+r}\bigl(E^i\bigr)$$

is the corresponding simple $sl_2(\Bbbk)$-module; and

$$Z_k(\mathfrak{g})=\sum_{1\leqslant j\leqslant k/2}\Bbbk\cdot\Delta^j,$$

where the sum is direct. Therefore, taking into account that $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$, the equation (4.3) simplifies to

$$\begin{aligned}\bigl(sl_2(\Bbbk),A\bigr)(\mathcal{F})&=\mathcal{F}\cdot\mathfrak{g}+\sum_{i>0,j\geqslant 0}I_{i+2j-1}(\mathcal{F})\cdot\Delta^jV_{2i}+\sum_{j\geqslant 1}\bigl[\mathcal{F},I_{2j-2}(\mathcal{F})\bigr]\cdot\Delta^j\\&=\mathcal{F}\cdot sl_2(\Bbbk)+\bigl[\mathcal{F}\cdot 1,Z_1(A,\mathcal{F})\bigr]+\sum_{i\geqslant 1}Z_i(A,\mathcal{F})\cdot V_{2i}.\end{aligned}$$

This finishes the proof of (4.9).

Prove (4.10). Indeed, now $\Delta\in\Bbbk\setminus\{0\}$, $E^k=0$ for $k>m$. Therefore

$$Z_k(A,\mathcal{F})=\sum_{j\geqslant 0}I_{k+2j-1}(\mathcal{F})\cdot\Delta^j=\sum_{j\geqslant 0}I_{k+2j-1}(\mathcal{F})\cdot 1=I_{k-1}(\mathcal{F})\cdot 1$$

because $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$. Finally, using (4.9), we obtain:

$$\begin{aligned}\bigl(sl_2(\Bbbk),A\bigr)(\mathcal{F})&=\mathcal{F}\cdot sl_2(\Bbbk)+\bigl[\mathcal{F},Z_1(A,\mathcal{F})\bigr]\cdot 1+\sum_{i\geqslant 1}Z_i(A,\mathcal{F})\cdot V_{2i}\\&=[\mathcal{F},\mathcal{F}]\cdot 1+\sum_{1\leqslant k\leqslant m}I_{k-1}(\mathcal{F})\cdot V_{2k}.\end{aligned}$$

Theorem 4.9 is proved. □

## 5. $\mathcal{N}$-groups

Throughout the section we assume that each object of $\mathcal{N}$ is a unital $\Bbbk$-algebra, i.e., $\mathcal{N}$ is a sub-category of $\mathbf{Alg}_1$.

### 5.1. *From $\mathcal{N}$-Lie algebras to $\mathcal{N}$-groups and generalized $K_1$-theories*

In this section we use $\mathcal{F}$-algebras and the category $\mathbf{LieAlg}_{\mathcal{N}}$ defined in Section 2.

**Definition 5.1.** An *affine $\mathcal{N}$-group* is a triple $(\mathcal{F},\mathcal{G},\mathcal{A})$, where $\mathcal{F}$ is an object of $\mathcal{N}$, $\mathcal{A}$ is an $\mathcal{F}$-algebra in $\mathbf{Alg}_1$ (i.e., $\iota:\mathcal{F}\to\mathcal{A}$ respects the unit), and $\mathcal{G}$ is a subgroup of the group of units $\mathcal{A}^\times$ such that $\mathcal{G}$ contains the image $\iota(\mathcal{F}^\times)=\iota(\mathcal{F})^\times$. A morphism $(\mathcal{F}_1,\mathcal{G}_1,\mathcal{A}_1)\to(\mathcal{F}_2,\mathcal{G}_2,\mathcal{A}_2)$

of affine $\mathcal{N}$-groups is a pair $(\varphi, \psi)$, where $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism in $\mathcal{N}$ and $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is a group homomorphism such that $\psi \circ \iota_1|_{\mathcal{F}_1^\times} = \iota_2|_{\mathcal{F}_1^\times} \circ \psi$.

Denote by $\mathbf{Gr}_{\mathcal{N}}$ the category of affine $\mathcal{N}$-groups.

Next, we will construct a number of affine $\mathcal{N}$-groups out of a given affine $\mathcal{N}$-group or a given $\mathcal{N}$-algebra as follows.

Let $\mathbf{LieAlg}_{\mathcal{N};1}$ be the sub-category of $\mathbf{LieAlg}_{\mathcal{N}}$ whose objects are triples $(\mathcal{F}, \mathcal{L}, \mathcal{A})$, where $\mathcal{F}$ is an object of $\mathbf{Alg}_1$, $\mathcal{A}$ is an $\mathcal{F}$-algebra in $\mathbf{Alg}_1$, and $\mathcal{L}$ is a Lie subalgebra of $\mathcal{A}$ invariant under the adjoint action of $\iota(\mathcal{F})$; morphisms in $\mathbf{LieAlg}_{\mathcal{N};1}$ are those morphisms in $\mathbf{LieAlg}_{\mathcal{N}}$ which respect the unit.

Given an object $(\mathcal{F}, \mathcal{L}, \mathcal{A})$ of $\mathbf{LieAlg}_{\mathcal{N};1}$, define the triple $Exp(\mathcal{F}, \mathcal{L}, \mathcal{A}) := (\mathcal{F}, \mathcal{G}, \mathcal{A})$, where $\mathcal{G}$ is the subgroup of $\mathcal{A}^\times$ generated by $\iota(\mathcal{F})^\times$ and by the stabilizer $\{g \in \mathcal{A}^\times\colon\ g\mathcal{L}g^{-1} = \mathcal{L}\}$ of $\mathcal{L}$ in $\mathcal{A}^\times$.

Given an object $(\mathcal{F}, \mathcal{G}, \mathcal{A})$ of $\mathbf{Gr}_{\mathcal{N}}$, define the triple $Lie(\mathcal{F}, \mathcal{G}, \mathcal{A}) := (\mathcal{F}, \mathcal{L}, \mathcal{A})$, where $\mathcal{L}$ is the Lie subalgebra of $\mathcal{A}$ generated (over $\Bbbk$) by the set $\{g \cdot \iota(f) \cdot g^{-1}\colon\ g \in \mathcal{G}, f \in \mathcal{F}\}$, that is, $\mathcal{L}$ is the smallest Lie subalgebra of $\mathcal{A}$ containing $\iota(\mathcal{F})$ and invariant under conjugation by $\mathcal{G}$ (therefore, $\mathcal{L}$ is invariant under the adjoint action of the subalgebra $\iota(\mathcal{F})$).

The following result is obvious.

**Lemma 5.2.** *For each object $(\mathcal{F}, \mathcal{L}, \mathcal{A})$ of* $\mathbf{LieAlg}_{\mathcal{N};1}$ *the triple $Exp(\mathcal{F}, \mathcal{L}, \mathcal{A})$ is an affine $\mathcal{N}$-group; and for any affine $\mathcal{N}$-group $(\mathcal{F}, \mathcal{G}, \mathcal{A})$ the triple $Lie(\mathcal{F}, \mathcal{G}, \mathcal{A})$ is an object of* $\mathbf{LieAlg}_{\mathcal{N};1}$.

**Remark 5.3.** The operations *Exp* and *Lie* are analogues of the Lie correspondence (between Lie algebras and Lie groups). However, similarly to the operations $L_i$ from Lemma 2.2, in general they are not functors.

Composing these two operations with each other and the operations $L_i$, we can obtain a number of affine $\mathcal{N}$-groups out of a given $\mathcal{N}$-Lie algebra and vice versa.

By definition, one has a natural (forgetful) projection functor $\pi : \mathbf{Gr}_{\mathcal{N}} \to \mathcal{N}$ by $\pi(\mathcal{F}, \mathcal{G}, \mathcal{A}) = \mathcal{F}$ and $\pi(\varphi, \psi) = \varphi|_{\mathcal{F}_1}$.

A *noncommutative current group* (or simply *$\mathcal{N}$-current group*) is any functor $\mathfrak{G} : \mathcal{N} \to \mathbf{Gr}_{\mathcal{N}}$ such that $\pi \circ \mathfrak{G} = Id_{\mathcal{N}}$ (i.e., $\mathfrak{G}$ is a section of $\pi$).

Note that if $\mathcal{N} = (\mathcal{F}, Id_{\mathcal{F}})$ has only one object $\mathcal{F}$ and only the identity arrow $Id_{\mathcal{F}}$, then the $\mathcal{N}$-current group is simply any object of $\mathbf{Gr}_{\mathbf{Alg}_1}$ of the form $(\mathcal{F}, \mathcal{G}, \mathcal{A})$. In this case, we will sometimes refer to $\mathcal{G}$ an *$\mathcal{F}$-current group*.

The above arguments allow for constructing a number of $\mathcal{F}$-current groups out of $\mathcal{F}$-current Lie algebras and vice versa. In a different situation, for any subcategory $\mathcal{N}$ of $\mathbf{Alg}_1$, we will construct below a class of $\mathcal{N}$-current groups associated with compatible pairs $(\mathfrak{g}, A)$. More general $\mathcal{N}$-current groups will be considered elsewhere.

Similarly to Section 2, given an object $\mathcal{F}$ of $\mathbf{Alg}_1$ and a group $G$, we refer to a group homomorphism $\iota : \mathcal{F}^\times \to G$ in $\mathbf{Alg}$ as an *$\mathcal{F}$-group structure on $G$* (we will also refer to $G$ an *$\mathcal{F}$-group*).

**Definition 5.4.** A *decorated group* is a pair $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F}$ is an object of $\mathbf{Alg}_1$ and $\mathcal{G}$ is an $\mathcal{F}$-group.

We denote by $\mathbf{DecGr}$ the category whose objects are decorated groups and morphisms are pairs $(\varphi, \psi) : (\mathcal{F}_1, \mathcal{G}_1) \to (\mathcal{F}_2, \mathcal{G}_2)$, where $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism in $\mathbf{Alg}_1$ and $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is a group homomorphism such that $\psi \circ \iota_1 = \iota_2 \circ \varphi|_{\mathcal{F}_1^\times}$.

In particular, one has a natural (forgetful) projection functor $\pi : \mathbf{DecGr} \to \mathbf{Alg}_1$ by $\pi(\mathcal{F}, G) = \mathcal{F}$ and $\pi(\varphi, \psi) = \varphi$. Note also that for any object $(\mathcal{F}, \mathcal{G}, \mathcal{A})$ of $\mathbf{Gr}_{\mathcal{N}}$ the pair $(\mathcal{F}, \mathcal{G})$ is a decorated group, therefore, the projection $(\mathcal{F}, \mathcal{G}, \mathcal{A}) \mapsto (\mathcal{F}, \mathcal{G})$ defines a (forgetful) functor $\mathbf{Gr}_{\mathcal{N}} \to \mathbf{DecGr}$.

**Definition 5.5.** A *generalized $K_1$-theory* is a functor $K : \mathcal{N} \to \mathbf{DecGr}$ such that $\pi \circ K = \pi|_{\mathcal{N}}$.

In what follows we will construct a number of generalized $K_1$-theories as compositions of an $\mathcal{N}$-current group $\mathfrak{s} : \mathcal{N} \to \mathbf{Gr}_{\mathcal{N}}$ with a certain functors $\kappa$ from $\mathbf{Gr}_{\mathcal{N}}$ to the category of *decorated groups*.

Given an object $(\mathcal{F}, \mathcal{G}, \mathcal{A})$ of $\mathbf{Gr}_{\mathcal{N}}$, we define $\kappa_{com}(\mathcal{F}, \mathcal{G}, \mathcal{A}) := \mathcal{G}/[\mathcal{G}, \mathcal{G}]$, where $[\mathcal{G}, \mathcal{G}]$ is the (normal) commutator subgroup of $G$.

**Lemma 5.6.** *The correspondence $(\mathcal{F}, \mathcal{G}, \mathcal{A}) \mapsto \kappa_{com}(\mathcal{F}, \mathcal{G}, \mathcal{A})$ defines a functor $\kappa_{com}$ from $\mathbf{Gr}_{\mathcal{N}}$ to the category of decorated abelian groups. In particular, for any $\mathcal{N}$-current group $\mathfrak{G} : \mathcal{N} \to \mathbf{Gr}_{\mathcal{N}}$ the composition $\kappa_{com} \circ \mathfrak{G}$ is a generalized $K_1$-theory.*

Note that each $K_1$-theory defined by Lemma 5.6 is still commutative. Below we will construct a number of noncommutative *nilpotent* $K_1$-theories in a similar manner.

**Definition 5.7.** Let $\mathcal{A}$ be an $\mathcal{F}$-algebra. Recall that a subset $S$ of $\mathcal{A}$ is called a nilpotent if $S^n = 0$ for some $n$. In particular, an element $e \in \mathcal{A}$ is nilpotent if $\{e\}$ is nilpotent, i.e., $e^n = 0$. We denote by $\mathcal{A}_{nil}$ the set of all nilpotent elements in $\mathcal{A}$. Note that $\mathcal{A}_{nil}$ is invariant under adjoint action of the group of units $\mathcal{A}^\times$. We say that an element $e \in \mathcal{A}$ is an *$\mathcal{F}$-stable nilpotent* if $(\mathcal{F} \cdot e \cdot \mathcal{F})^n = 0$ for some $n > 0$. Denote by $(\mathcal{F}, \mathcal{A})_{nil}$ the set of all $\mathcal{F}$-stable nilpotent element of $\mathcal{A}$. Denote also by $\mathcal{A}_{nil}^{\mathcal{F}^\times}$ and $(\mathcal{F}, \mathcal{A})_{nil}^{\mathcal{F}^\times}$ the centralizers of $\mathcal{F}^\times$ in $\mathcal{A}_{nil}$ and $(\mathcal{F}, \mathcal{A})_{nil}$ respectively.

In particular, taking $\mathcal{A} = \mathcal{F}$, we see that $f \in (\mathcal{F}, \mathcal{F})_{nil}$ if and only if the ideal $\mathcal{F} f \mathcal{F} \subset \mathcal{F}$ is nilpotent. Note also that if the image $\iota(\mathcal{F})$ is in the center of $\mathcal{A}$, then $(\mathcal{F}, \mathcal{A})_{nil} = \mathcal{A}_{nil}$.

Let $(\mathcal{F}, \mathcal{G}, \mathcal{A})$ be an object of $\mathbf{Gr}_{\mathcal{N}}$ and let $S$ be a subset of $\mathcal{A}_{nil}$, denote by $E_S = E_S(\mathcal{F}, \mathcal{G}, \mathcal{A})$ the subgroup of $\mathcal{G}$ generated by all $1 + xsx^{-1}$, $s \in S$, $x \in \mathcal{G}$. Clearly, $E_S$ is a normal subgroup of $\mathcal{G}$. Then denote the quotient group $\mathcal{G}/E_S$ by:

$$\begin{cases} \kappa_{nil}(\mathcal{F}, \mathcal{G}, \mathcal{A}) & \text{if } S = \mathcal{A}_{nil}, \\ \kappa_{stnil}(\mathcal{F}, \mathcal{G}, \mathcal{A}) & \text{if } S = (\mathcal{F}, \mathcal{A})_{nil}, \\ \kappa_{nil,inv}(\mathcal{F}, \mathcal{G}, \mathcal{A}) & \text{if } S = \mathcal{A}_{nil}^{\mathcal{F}^\times}, \\ \kappa_{stnil,inv}(\mathcal{F}, \mathcal{G}, \mathcal{A}) & \text{if } S = (\mathcal{F}, \mathcal{A})_{nil}^{\mathcal{F}^\times}. \end{cases} \tag{5.1}$$

**Lemma 5.8.** *Each of the four correspondences*

$$(\mathcal{F}, \mathcal{L}, \mathcal{A}) \mapsto \kappa(\mathcal{F}, \mathcal{L}, \mathcal{A}),$$

*where $\kappa = \kappa_{nil}, \kappa_{stnil}, \kappa_{nil,inv}, \kappa_{stnil,inv}$, defines a functor*

$$\kappa : \mathbf{Gr}_{\mathcal{N}} \to \mathbf{DecGr}.$$

*In particular, for any $\mathcal{N}$-current group $\mathfrak{G} : \mathcal{N} \to \mathbf{Gr}_{\mathcal{N}}$ the composition $\kappa \circ \mathfrak{G}$ is a generalized $K_1$-theory $\mathcal{N} \to \mathbf{DecGr}$.*

**Proof.** Clearly, in each case of (5.1), the association $\mathcal{A} \mapsto S = S_{\mathcal{A}}$ is functorial, i.e., commutes with morphisms of $\mathcal{F}$-algebras. Therefore, the association $\mathcal{A} \mapsto E_S$ is also functorial in all four cases (5.1). This finishes the proof of the lemma. □

We will elaborate examples of generalized (noncommutative) $K_1$-groups in a separate paper.

*5.2. $\mathcal{N}$-current groups for compatible pairs*

Here we keep the notation of Section 2. Since both $\mathcal{F}$ and $A$ are now unital algebras, so is $\mathcal{F} \otimes A = \mathcal{F} \cdot A$.

Note that for the $\mathcal{N}$-Lie algebra $\mathfrak{s}: \mathcal{F} \mapsto (\mathcal{F}, (\mathfrak{g}, A)(\mathcal{F}), \mathcal{F} \cdot A)$ the corresponding $\mathcal{N}$-current group is of the form $(\mathcal{F}, G_{\mathfrak{g},A}(\mathcal{F}), \mathcal{F} \cdot A)$, where $G_{\mathfrak{g},A}(\mathcal{F})$ is the normalizer of $(\mathfrak{g}, A)(\mathcal{F})$ in $(\mathcal{F} \cdot A)^\times$ (i.e., $G_{\mathfrak{g},A}(\mathcal{F}) = \{g \in (\mathcal{F} \cdot A)^\times \colon\ g \cdot (\mathfrak{g}, A)(\mathcal{F}) \cdot g^{-1} = (\mathfrak{g}, A)(\mathcal{F})\}$).

The following facts are obvious.

**Lemma 5.9.** *Let $(\mathfrak{g}, A)$ be a compatible pair and $S \subset \mathfrak{g}$ be a generating set of $\mathfrak{g}$ as a Lie algebra. Then for any object $\mathcal{F}$ of $\mathbf{Alg}_1$ an element $g \in (\mathcal{F} \cdot A)^\times$ belongs to $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if:*

$$g(u \cdot x)g^{-1} \subset (\mathfrak{g}, A)(\mathcal{F}) \tag{5.2}$$

*for all $x \in S$, $u \in \mathcal{F}$.*

**Lemma 5.10.** *For each compatible pair of the form $(\mathfrak{g}, A) = (sl_n(\Bbbk), M_n(\Bbbk))$ and an object $\mathcal{F}$ of $\mathcal{N}_1$ one has: $G_{\mathfrak{g},A}(\mathcal{F}) = GL_n(\mathcal{F}) = (\mathcal{F} \cdot A)^\times$.*

In what follows we will consider compatible pairs of the form $(\mathfrak{g}, End(V))$ where $V$ is a simple finite-dimensional $\mathfrak{g}$-module. By choosing an appropriate basis in $V$, we identify $A = End(V)$ with $M_n(\Bbbk)$ so that $G_{\mathfrak{g},A}(\mathcal{F}) \subset GL_n(\mathcal{F})$. In all cases to be considered, we will compute the "Cartan subgroup" $(\mathcal{F}^\times)^n \cap G_{\mathfrak{g},A}(\mathcal{F})$ of $G_{\mathfrak{g},A}(\mathcal{F})$.

Let $\Phi_0$ be the bilinear form on the $\Bbbk$-vector space $V = \Bbbk^n$ given by:

$$\Phi_0(x, y) = x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1.$$

Also define the bilinear form $\Phi_1$ on $\Bbbk^{2m}$ by:

$$\Phi_1(x, y) = x_1 y_{2m} + x_2 y_{2m-1} + \cdots + x_m y_m - x_{m+1} y_{m-1} \cdots - x_{2m} y_1.$$

**Proposition 5.11.** *Let $\mathcal{F}$ be an object of $\mathbf{Alg}_1$, $A = M_n(\Bbbk)$, and suppose that either $\mathfrak{g} = o(\Phi_0)$ or $\mathfrak{g} = o(\Phi_1)$ and $n = 2m$. Then an invertible diagonal matrix $D = diag(f_1, \ldots, f_n) \in GL_n(\mathcal{F})$ belongs to $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if*

$$f_i f_{n-i+1} - f_1 f_n \in I_1(\mathcal{F}) = \mathcal{F}[\mathcal{F}, \mathcal{F}]$$

*for $i = 1, \ldots, n$.*

**Proof.** We will prove the proposition for $\mathfrak{g} = o(\Phi_0)$ (the proof for $\mathfrak{g} = o(\Phi_1)$ is nearly identical). It is easy to see that $\mathfrak{g}$ is a Lie subalgebra of $sl_n(\Bbbk)$ generated by all $e_{ij} := E_{ij} - E_{n-j+1,n-i+1}$,

$i, j = 1, \ldots, n$. therefore, Lemma 5.9 (with $S = \{e_{ij}\}$) guarantees that $D = (f_1, \ldots, f_n) \in (\mathcal{F}^\times)^n$ belongs to $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if $D(u \cdot e_{ij})D^{-1} \subset (\mathfrak{g}, A)(\mathcal{F})$ for all $u \in \mathcal{F}$, $i, j = 1, \ldots, n$. Note that

$$D(u \cdot e_{ij})D^{-1} = f_i u f_j^{-1} E_{ij} - f_{j'} u f_{i'}^{-1} E_{j',i'} = f_i u f_j^{-1} e_{ij} + \delta_{ij}(u) E_{j',i'},$$

where $\delta_{ij}(u) = f_i u f_j^{-1} - f_{j'} u f_{i'}^{-1}$ and $i' = n + 1 - i$, $j' = n + 1 - j$. Therefore, taking into account that $(\mathfrak{g}, A)(\mathcal{F}) = [\mathcal{F}, \mathcal{F}] \cdot 1 + \mathcal{F} \cdot \mathfrak{g} + I_1(\mathcal{F}) \cdot sl_n(\Bbbk)$ by Corollary 2.15, we see that $D(u \cdot e_{ij})D^{-1} \in (\mathfrak{g}, A)(\mathcal{F})$ if and only if $\delta_{ij}(u) \in I_1(\mathcal{F})$. Note that $\delta_{ij}(u) \equiv u\delta_{ij}(1) \bmod I_1(\mathcal{F})$. Since $I_1(\mathcal{F})$ is an ideal, $u\delta_{ij}(1) \in I_1(\mathcal{F})$ for all $u \in \mathcal{F}$ if and only if $\delta_{ij}(1) \in I_1(\mathcal{F})$, i.e., $f_i f_j^{-1} - f_{j'} f_{i'}^{-1} \bmod I_1(\mathcal{F})$. Taking into account that $f_i f_j^{-1} - f_{j'} f_{i'}^{-1} \equiv (f_i f_{i'} - f_j f_{j'}) f_j^{-1} f_{i'}^{-1} \bmod I_1(\mathcal{F})$, we see that $D \in G_{\mathfrak{g},A}(\mathcal{F})$ if and only if $f_i f_{n+1-i} - f_j f_{n+1-j} \in I_1(\mathcal{F})$ for all $i, j$. Clearly, it suffices to take $j = 1$. The proposition is proved. □

To formulate the main result of this section we need the following notation.

For any $\ell \geqslant 0$ and any $m_1, \ldots, m_{\ell+1} \in \mathcal{F}$ denote

$$\Delta^{(\ell)}(m_1, \ldots, m_{\ell+1}) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} m_{k+1} \tag{5.3}$$

and refer to it as the $\ell$th difference derivative. Clearly,

$$\Delta^{(\ell)}(m_1, \ldots, m_{\ell+1}) = \Delta^{(\ell-1)}(m_1, \ldots, m_\ell) - \Delta^{(\ell-1)}(m_2, \ldots, m_{\ell+1}).$$

Let $A_n = M_n(\Bbbk) = End(V_{n-1})$, where $V_{n-1}$ is the $n$-dimensional irreducible $sl_2(\Bbbk)$-module. Then $G_{sl_2(\Bbbk),A_n}(\mathcal{F})$ is naturally a subgroup of $GL_n(\mathcal{F})$.

**Main Theorem 5.12.** *For any object $\mathcal{F}$ of* $\mathbf{Alg}_1$*, the "Cartan subgroup" $(\mathcal{F}^\times)^n \cap G_{sl_2(\Bbbk),A_n}(\mathcal{F})$ consists of all $D = (f_1, \ldots, f_n) \in (\mathcal{F}^\times)^n$ such that:*

$$\Delta^{(k)}\big(f_1 f_2^{-1}, \ldots, f_{k+1} f_{k+2}^{-1}\big) \in I_k(\mathcal{F}) \tag{5.4}$$

*for $k = 1, \ldots, n-2$.*

**Proof.** We will prove the theorem in several steps. First, we prove Proposition 5.13 by using Lemmas 5.14 and 5.15. Then we prove Lemma 5.16 and Proposition 5.17. The proof of the Proposition is based on Lemma 5.18. The final step in the proof of Theorem 5.12 is Theorem 5.19. This theorem required Proposition 5.21, Lemma 5.23, and Proposition 5.24. Our proofs of Propositions 5.21 and 5.24 use Lemmas 5.22 and 5.25 correspondingly.

We start with a characterization of $(\mathcal{F}^\times)^n \cap G_{sl_2(\Bbbk),A_n}(\mathcal{F})$.

**Proposition 5.13.** *A diagonal matrix $D = (f_1, \ldots, f_n) \in (\mathcal{F}^\times)^n$ belongs to the group $G_{sl_2(\Bbbk),A_n}(\mathcal{F})$ if and only if:*

$$\Delta^{(k)}\big(f_1 u f_2^{-1}, \ldots, f_{k+1} u f_{k+2}^{-1}\big) \in I_k(\mathcal{F}), \tag{5.5}$$

$$\Delta^{(k)}\big(f_n u f_{n-1}^{-1}, \ldots, f_{n-k} u f_{n-1-k}^{-1}\big) \in I_k(\mathcal{F}). \tag{5.6}$$

*for* $k = 1, \ldots, n-2$ *and all* $u \in \mathcal{F}$.

**Proof.** Denote by $(A_n)_k$ the set of all $x \in A_n$ such that $[H, x] = kx$. Clearly, $(A_n)_k \neq 0$ if and only if $k$ is even and $-2(n-1) \leqslant k \leqslant 2(n-1)$. In fact, $(A_n)_{2k}$ is the span of all those $E_{ij}$ such that $2(j-i) = k$. In particular, $E \in (A_n)_2$. Denote also $(sl_2(\Bbbk), A_n)(\mathcal{F})_k := (sl_2(\Bbbk), A_n)(\mathcal{F}) \cap \mathcal{F} \cdot (A_n)_k$.

**Lemma 5.14.** *The components* $(sl_2(\Bbbk), A_n)(\mathcal{F})_j$, $j = -2, 2$ *are given by:*

$$\big(sl_2(\Bbbk), A_n\big)(\mathcal{F})_2 = \sum_{k=0}^{n-2} I_k(\mathcal{F}) \cdot E^{(k)}, \qquad \big(sl_2(\Bbbk), A_n\big)(\mathcal{F})_{-2} = \sum_{k=0}^{n-2} I_k(\mathcal{F}) \cdot F^{(k)}, \tag{5.7}$$

*where*

$$E^{(k)} = \sum_{i=k+1}^{n-1} i \binom{i-1}{k} E_{i,i+1}, \qquad F^{(k)} = \sum_{i=k+1}^{n-1} i \binom{i-1}{k} E_{n+1-i,n-i}$$

*for* $k = 0, 1, \ldots, n-2$ *form a basis for* $(A_n)_2$ *and* $(A_n)_{-2}$ *respectively.*

**Proof.** Let us prove the formula for $j = 2$. Let $p_0(x), \ldots, p_{n-2}(x)$ be any polynomials in $\Bbbk[x]$ such that $\deg p_k(x) = k$ for all $k$. Then it is easy to see that

$$\big(sl_2(\Bbbk), A_n\big)(\mathcal{F})_2 = \sum_{k=0}^{n-2} I_k(\mathcal{F}) \cdot \big(p_k(H) \cdot E\big).$$

Take $p_k(H) = \binom{H'}{k}$, where $H' = \frac{1}{2}(n \cdot 1 - H) = \sum_{i=1}^{n} (i-1) E_{ii}$. Then $p_k(H) = \sum_{i=1}^{n} \binom{i-1}{k} E_{ii}$ and

$$p_k(H) \cdot E = \left(\sum_{i=1}^{n} \binom{i-1}{k} E_{ii}\right)\left(\sum_{i=1}^{n} i E_{i,i+1}\right) = E^{(k)}.$$

To prove the formula for $j = -2$, it suffices to conjugate the formula for $j = 2$ with the matrix of the longest permutation $w_0 = \sum_{i=1}^{n} E_{i,n+1-i}$, i.e., apply the involution $E_{ij} \mapsto E_{n+1-i,n+1-j}$. $\square$

**Lemma 5.15.** *For each diagonal matrix* $D = (f_1, f_2, \ldots, f_n) \in (\mathcal{F}^\times)^n$ *and* $u \in \mathcal{F}$ *one has*

$$D(u \cdot E) D^{-1} = \sum_{k=0}^{n-2} \Delta^{(k)}\big(f_1 u f_2^{-1}, \ldots, f_{k+1} u f_{k+2}^{-1}\big) \cdot (-1)^k E^{(k)},$$

$$D(u\cdot F)D^{-1}=\sum_{k=0}^{n-2}\Delta^{(k)}\big(f_nuf_{n-1}^{-1},\dots,f_{n-k}uf_{n-1-k}^{-1}\big)\cdot(-1)^{k+1}F^{(k)},$$

*where $\Delta^{(k)}$ is the kth divided difference as in* (5.3).

**Proof.** It is easy to see that the elements $E^{(k)}$, $F^{(k)}$ satisfy:

$$iE_{i,i+1}=\sum_{k=i-1}^{n-2}(-1)^{k+1-i}\binom{k}{i-1}E^{(k)},$$
$$iE_{n+1-i,n-i}=\sum_{k=i-1}^{n-2}(-1)^{k+1-i}\binom{k}{i-1}F^{(k)}\tag{5.8}$$

for $i=1,\dots,n-1$.

Furthermore,

$$\begin{aligned}D(u\cdot E)D^{-1}&=\sum_{i=1}^{n-1}f_iuf_{i+1}^{-1}\cdot iE_{i,i+1}=\sum_{i=1}^{n-1}f_iuf_{i+1}^{-1}\cdot\sum_{k=0}^{n-2}(-1)^{k+1-i}\binom{k}{i-1}E^{(k)}\\&=\sum_{k=0}^{n-2}\sum_{i=1}^{n-1}(-1)^i\binom{k}{i-1}f_iuf_{i+1}^{-1}\cdot(-1)^{k+1}E^{(k)}\\&=\sum_{k=i-1}^{n-2}\Delta^{(k)}\big(f_1uf_2^{-1},\dots,f_{k+1}uf_{k+2}^{-1}\big)\cdot(-1)^{k+1}E^{(k)}.\end{aligned}$$

The formula for $D(u\cdot F)D^{-1}$ follows. The lemma is proved. □

Now we are ready to finish the proof of Proposition 5.13.

Since the set $S=\{E,F\}$ generates $sl_2(\Bbbk)$, Lemma 5.9 guarantees that $D\in GL_n(\mathcal{F})$ belongs to $G_{sl_2(\Bbbk),A_n}(\mathcal{F})$ if and only if $D(u\cdot E)D^{-1}, D(u\cdot F)D^{-1}\in(sl_2(k),A_n)(\mathcal{F})$ for all $u\in\mathcal{F}$. Using the obvious fact that $D(u\cdot E)D^{-1}\subset\mathcal{F}\cdot(A_n)_2$ for all $D\in(\mathcal{F}^\times)^n$, $u\in\mathcal{F}$, we see that $D(u\cdot E)D^{-1}\in(sl_2(\Bbbk),A_n)(\mathcal{F})$ if and only if $D(u\cdot E)D^{-1}\in(sl_2(\Bbbk),A_n)(\mathcal{F})_2$. In turn, using Lemmas 5.14 and 5.15, we see that this is equivalent to

$$D(u\cdot E)D^{-1}=\sum_{k=0}^{n-2}\Delta^{(k)}(m_1,\dots,m_{k+1})\cdot(-1)^{k+1}E^{(k)}\in\sum_{k=0}^{n-2}I_k(\mathcal{F})\cdot E^{(k)},$$

which, because the $E^{(0)},\dots,E^{(n-2)}$ are linearly independent, is equivalent to (5.5). Applying the above argument to $D(u\cdot F)D^{-1}$, we obtain

$$D(u\cdot E)D^{-1}=\sum_{k=0}^{n-2}\Delta^{(k)}\big(f_nuf_{n-1}^{-1},\dots,f_{n-k}uf_{n-1-k}^{-1}\big)\cdot(-1)^{k+1}F^{(k)}\in\sum_{k=0}^{n-2}I_k(\mathcal{F})\cdot F^{(k)},$$

which gives (5.6).

The proposition is proved. □

Furthermore, we need to establish some basic properties of inclusions (5.4).

**Lemma 5.16.** *Let $m_1, m_2, \dots, m_\ell$ be elements of $\mathcal{F}$. The following are equivalent:*

(a) $\Delta^{(k)}(m_1, \dots, m_{k+1}) \in I_k(\mathcal{F})$ *for all* $1 \leqslant k \leqslant \ell - 1$.
(b) $\Delta^{(j-i)}(m_i, \dots, m_j) \in I_{j-i}(\mathcal{F})$ *for all* $1 \leqslant i \leqslant j \leqslant \ell$.

**Proof.** The implication (b) ⇒ (a) is obvious. Prove the implication (a) ⇒ (b). Denote

$$m_{ij} := \Delta^{(j-i)}(m_i, \dots, m_j) \tag{5.9}$$

for all $1 \leqslant i \leqslant j \leqslant \ell$. In particular, $m_{ii} = m_i$ and:

$$m_{ij} = m_{i,j-1} - m_{i+1,j} \tag{5.10}$$

for all $1 \leqslant i \leqslant j \leqslant \ell$.

Prove that the inclusions $m_{1,k+1} \in I_k(\mathcal{F})$ for $0 \leqslant k \leqslant \ell$ imply inclusions $m_{ij} \in I_{j-i}(\mathcal{F})$ for all $i \leqslant j$ such that $j - i \leqslant \ell$. We proceed by induction on $i$. The basis of the induction, when $i = 1$, is obvious. Assume that $i > 1$ and $j \leqslant \ell - 1 - i$. Then the inclusions (the inductive hypothesis)

$$m_{i-1,j} = m_{i-1,j-1} - m_{ij} \in I_{j+1-i}(\mathcal{F}) \subset I_{j-i}(\mathcal{F})$$

and $m_{i-1,j-1} \in I_{j-i}(\mathcal{F})$ imply that $m_{ij} \in I_{j-i}(\mathcal{F})$. This finishes the proof of the implication (a) ⇒ (b). The lemma is proved. □

**Proposition 5.17.** *Let $m_1, m_2, \dots, m_\ell$ be invertible elements of $\mathcal{F}$. The following are equivalent:*

(a) $\Delta^{(j-i)}(m_i, \dots, m_j) \in I_{j-i}(\mathcal{F})$ *for all* $i \leqslant j$.
(b) $\Delta^{(j-i)}(m_i^{-1}, \dots, m_j^{-1}) \in I_{j-i}(\mathcal{F})$ *for all* $i \leqslant j$.

**Proof.** We need the following notation. Similarly to (5.9) denote

$$m_{ij} := \Delta^{(j-i)}(m_i, \dots, m_j), \qquad m_{ij}^* := \Delta^{(j-i)}\bigl(m_i^{-1}, \dots, m_j^{-1}\bigr) \tag{5.11}$$

for all $1 \leqslant i \leqslant j \leqslant \ell$. In particular, $m_{ii}^* = m_i^{-1}$ and $m_{12}^* = -m_1^{-1}(m_1 - m_2)m_2^{-1} = -m_{11}^* m_{12} m_{22}^*$.

We need the following recursive formula for $m_{ij}^*$.

**Lemma 5.18.** *In the above notation, we have for all* $1 \leqslant i < j \leqslant \ell$*:*

$$m_{ij}^* = \sum_{\substack{i \leqslant i_1 \leqslant j_1 \leqslant j,\, i \leqslant i_2 < j_2 \leqslant j,\, i \leqslant i_3 \leqslant j_3 \leqslant j \\ j_1 - i_1 + j_2 - i_2 + j_3 - i_3 = j - i}} c_{i,i_1,i_2,i_3}^{j,j_1,j_2,j_3} m_{i_1,j_1}^* m_{i_2,j_2} m_{i_3,j_3}^*, \tag{5.12}$$

*where the coefficients are translation-invariant integers:*

$$c_{i+1,i_1+1,i_2+1,i_3+1}^{j+1,j_1+1,j_2+1,j_3+1}=c_{i,i_1,i_2,i_3}^{j,j_1,j_2,j_3}.$$

**Proof.** We proceed by induction on $j-i$. If $j=i+1$, we obtain:

$$m_{i,i+1}^*=-m_i^{-1}(m_i-m_{i+1})m_{i+1}^{-1}=-m_{ii}^*m_{i,i+1}m_{i+1,i+1}^*=-m_{i+1,i+1}^*m_{i,i+1}m_{i,i}^*.$$

Next, assume that $j-i>1$. Then, using the translation invariance of the coefficients in (5.12) for $m_{i,j-1}^*$, we obtain

$$m_{ij}^*=m_{i,j-1}^*-m_{i+1,j}^*=\sum_{\substack{i\leqslant i_1\leqslant j_1\leqslant j-1,i\leqslant i_2<j_2\leqslant j-1,i\leqslant i_3\leqslant j_3\leqslant j-1\\ j_1-i_1+j_2-i_2+j_3-i_3=j-1-i}}c_{i,i_1,i_2,i_3}^{j-1,j_1,j_2,j_3}\delta_{i_1,i_2,i_3}^{j_1,j_2,j_3},$$

where $\delta_{i_1,i_2,i_3}^{j_1,j_2,j_3}=m_{i_1,j_1}^*m_{i_2,j_2}m_{i_3,j_3}^*-m_{i_1+1,j_1+1}^*m_{i_2+1,j_2+1}m_{i_3+1,j_3+1}^*$. Furthermore,

$$\begin{aligned}\delta_{i_1,i_2,i_3}^{j_1,j_2,j_3}&=m_{i_1,j_1+1}^*m_{i_2,j_2}m_{i_3,j_3}^*+m_{i_1+1,j_1+1}^*\left(m_{i_2,j_2}m_{i_3,j_3}^*-m_{i_2+1,j_2+1}m_{i_3+1,j_3+1}^*\right)\\&=m_{i_1,j_1+1}^*m_{i_2,j_2}m_{i_3,j_3}^*+m_{i_1+1,j_1+1}^*m_{i_2,j_2+1}m_{i_3,j_3}^*+m_{i_1+1,j_1+1}^*m_{i_2+1,j_2+1}m_{i_3,j_3+1}^*.\end{aligned}$$

This proves the formula (5.12) for $m_{ij}^*$. The lemma is proved. □

We are ready to finish the proof of Proposition 5.17 now.

Due to the symmetry, it suffices to prove only one implication, say (a) ⇒ (b). The desired implication follows inductively from (5.11). □

Note that, in the view of Lemma 5.16, the condition (a) (respectively the condition (b)) of Proposition 5.17 for $m_i=f_if_{i+1}^{-1}$, $i=1,\dots,n-2$, is a particular case of (5.5) (respectively of (5.6)) with $u=1$.

Furthermore, we need one more result in order to finish the proof of Theorem 5.12.

**Theorem 5.19.** *The inclusions* (5.4) *imply the inclusions* (5.5) *(and, therefore,* (5.6)*).*

**Proof.** To prove the theorem we will develop a formalism of homogeneous maps $\mathcal{F}\to\mathcal{F}$ (relative to the ideals $I_k(\mathcal{F})$).

**Definition 5.20.** We say that a $\Bbbk$-linear map $\partial:\mathcal{F}\to\mathcal{F}$ is *homogeneous of degree* $\ell$ if $\partial(I_k(\mathcal{F}))\subset I_{k+\ell}(\mathcal{F})$ for all $k\geqslant 0$; denote by $End^{(\ell)}(\mathcal{F})$ the set of all such maps.

Lemma 1.4 guarantees that for each $f_1\in I_{\ell_1}(\mathcal{F})$, $f_2\in I_{\ell_2}(\mathcal{F})$, the map $\mathcal{F}\to\mathcal{F}$ given by $u\mapsto f_1uf_2$ is homogeneous of degree $\ell_1+\ell_2$.

We construct a number of homogeneous maps of degree 1 as follows. For an invertible element $m\in\mathcal{F}^\times$ define $\partial_m:\mathcal{F}\to\mathcal{F}$ by

$$\partial_m(u)=mum^{-1}-u=\left[m,um^{-1}\right].$$

Clearly, $\partial_m:\mathcal{F}\to\mathcal{F}$ is homogeneous of degree 1.

**Proposition 5.21.** *Let $m_1, m_2, \dots, m_\ell$ be invertible elements of $\mathcal{F}$ such that, in the notation* (5.9)*, one has $m_{ij} \in I_{j-i}(\mathcal{F})$ for all $1 \leqslant i \leqslant j \leqslant \ell$. Then*

$$\Delta^{(j-i)}(\partial_{m_i}, \dots, \partial_{m_j}) \in End^{(j+1-i)}(\mathcal{F})$$

*for all $1 \leqslant i \leqslant j \leqslant \ell$.*

**Proof.** We need the following notation. Similarly to (5.9), denote

$$\partial_{ij} = \Delta^{(j-i)}(\partial_{m_i}, \dots, \partial_{m_j}) \tag{5.13}$$

for $1 \leqslant i \leqslant j \leqslant \ell$. By definition, $\partial_{ii} = \partial_{m_i}$ and $\partial_{i,i+1} = \partial_{m_i} - \partial_{m_{i+1}}$.

**Lemma 5.22.** *For each $u \in \mathcal{F}$ and $1 \leqslant i \leqslant j \leqslant \ell$ one has:*

$$\partial_{ij}(u) = \sum_{\substack{i \leqslant i_1 \leqslant j_1 \leqslant j, i \leqslant i_2 \leqslant j_2 \leqslant j \\ j_1 - i_1 + j_2 - i_2 = j - i}} c_{i,i_1,i_2}^{j,j_1,j_2} \left[m_{i_1,j_1}, um^*_{i_2,j_2}\right] \tag{5.14}$$

*in the notation* (5.11)*, where the coefficients are translation-invariant integers:*

$$c_{i+1,i_1+1,i_2+1}^{j+1,j_1+1,j_2+1} = c_{i,i_1,i_2}^{j,j_1,j_2}.$$

**Proof.** We proceed by induction on $j - i$. If $j = i$, we have $\partial_{ii}(u) = \partial_{m_i}(u) = [m_i, um_i^{-1}]$. Next, assume that $j - i > 0$. Then, using the translation-invariance of the coefficients in (5.14) for $\partial_{i,j-1}(u)$, we obtain

$$\partial_{ij}(u) = \partial_{i,j-1}(u) - \partial_{i+1,j}(u) = \sum_{\substack{i \leqslant i_1 \leqslant j_1 \leqslant j-1, i \leqslant i_2 < j_2 \leqslant j-1 \\ j_1 - i_1 + j_2 - i_2 = j-1-i}} c_{i,i_1,i_2}^{j-1,j_1,j_2} \delta_{i_1,i_2}^{j_1,j_2},$$

where $\delta_{i_1,i_2}^{j_1,j_2} = [m_{i_1,j_1}, um^*_{i_2,j_2}] - [m_{i_1+1,j_1+1}, um^*_{i_2+1,j_2+1}]$. Furthermore,

$$\begin{aligned}\delta_{i_1,i_2}^{j_1,j_2} &= \left[m_{i_1,j_1+1}, um^*_{i_2,j_2}\right] + \left[m_{i_1+1,j_1+1}, um^*_{i_2,j_2}\right] - \left[m_{i_1+1,j_1+1}, um^*_{i_2+1,j_2+1}\right] \\ &= \left[m_{i_1,j_1+1}, um^*_{i_2,j_2}\right] + \left[m_{i_1+1,j_1+1}, um^*_{i_2,j_2+1}\right].\end{aligned}$$

This proves the formula (5.14) for $\partial_{ij}(u)$. The lemma is proved. □

Now we can finish the proof of Proposition 5.21.

Since $m_{i_1,j_1} \in I_{j_1-i_1}(\mathcal{F})$, $m^*_{i_2,j_2} \in I_{j_2-i_2}(\mathcal{F})$, and $[m_{i_1,j_1}, um^*_{i_2,j_2}]$ belongs to the ideal $I_{k+j_1-i_1+j_2-i_2+1}(\mathcal{F})$ for all $u \in I_k(\mathcal{F})$, formula (5.14) guarantees inclusion $\partial_{ij}(I_k(\mathcal{F})) \subset I_{k+j+1-i}(\mathcal{F})$ for all $k \geqslant 0$. This proves Proposition 5.21. □

We continue to study "difference operators" in $\mathcal{F}$. Let $m_i, d_i : \mathcal{F} \to \mathcal{F}$, $i = 1, \dots, \ell$, be linear maps. Denote

$$\partial^{(i,j)} := \Delta^{(j-i)}(m_i d_{i+1} \cdots d_j, m_{i+1} d_{i+2} \cdots d_j, \dots, m_{j-1} d_j, m_j) \tag{5.15}$$

for all $1 \leqslant i \leqslant j \leqslant \ell$.

For instance, $\partial^{(i,i)} = m_i$, $\partial^{(i,i+1)} = m_i d_{i+1} - m_{i+1}$, and

$$\partial^{(i,i+2)} = m_i d_{i+1} d_{i+2} - 2m_{i+1} d_{i+2} + m_{i+2}.$$

**Lemma 5.23.** *Let $D = (f_1, \ldots, f_n) \in (\mathcal{F}^\times)^n$. Set $m_i = f_i f_{i+1}^{-1}$. Then for each $u \in \mathcal{F}$ one has*

$$\Delta^{(j-i)}\left(f_i u f_{i+1}^{-1}, \ldots, f_j u f_{j+1}^{-1}\right) = \partial^{(i,j)}(u'), \tag{5.16}$$

*where we abbreviated $u' = f_j u f_j^{-1}$ and $d_i := \partial_{m_i} + 1$.*

**Proof.** Indeed, for $i \leqslant k \leqslant j$ one has

$$\begin{aligned} f_k u f_{k+1}^{-1} &= m_k m_{k+1} \cdots m_j f_{j+1} u f_{j+1}^{-1} (m_{k+1} \cdots m_j)^{-1} = m_k (\partial_{m_{k+1} \cdots m_j} + 1)(u') \\ &= m_k (\partial_{m_k} + 1) \cdots (\partial_{m_j} + 1)(u') = m_k d_{k+1} \cdots d_j (u'). \end{aligned}$$

Substituting so computed $f_k u f_{k+1}^{-1}$ into (5.3), we obtain (5.16). □

Therefore, all we need to finish the proof of Theorem 5.19 is to prove the following result.

**Proposition 5.24.** *Let $m_1, m_2, \ldots, m_\ell$ be elements of $\mathcal{F}$ and $\ell \geqslant 0$ such that, in the notation* (5.9)*, one has $m_{ij} \in I_{j-i}(\mathcal{F})$ for all $1 \leqslant i \leqslant j \leqslant \ell$. Then*

$$\partial^{(i,j)} \in End^{(j-i)}(\mathcal{F})$$

*for all $1 \leqslant i \leqslant j \leqslant \ell$, where again we abbreviated $\partial_i := \partial_{m_i}$.*

**Proof.** We need the following notations. Let $M_I$ be the linear map $\mathcal{F} \to \mathcal{F}$ given by:

$$M_I = m_{i_0,i_1} \partial_{i_1+1,i_2} \partial_{i_2+1,i_3} \cdots \partial_{i_{k-1}+1,i_k},$$

where

$$m_{i',j'} := \Delta^{(j'-i')}(m_{i'}, \ldots, m_{j'}), \qquad \partial_{i',j'} = \begin{cases} d_{i'} - 1 & \text{if } i' = j', \\ \Delta^{(j'-i')}(d_{i'}, \ldots, d_{j'}) & \text{if } i' < j' \end{cases}$$

for all $1 \leqslant i' \leqslant j' \leqslant \ell$.

**Lemma 5.25.** *In the notation* (5.15) *one has*

$$\partial^{(i,j)} = \sum_I c_I M_I, \tag{5.17}$$

*where the summation is taken over all subsets $I = \{i_0 < i_1 < i_2 \cdots < i_k\}$ of $\{1, \ldots, \ell\}$ such that $i_0 = i$, $i_k = j$, and the coefficients $c_I \in \mathbb{Z}$ are translation-invariant:*

$$c_{I+1} = c_I,$$

*where for any subset* $I = \{i_0, \dots, i_k\} \subset \{1, \dots, \ell\}$*, we abbreviate* $I + 1 = \{i_0 + 1, \dots, i_k + 1\} \subset \{2, \dots, \ell + 1\}$.

**Proof.** We proceed by induction on $j - i$. The basis of the induction when $j = i$ is obvious because $\partial^{(i,i)} = M_I = m_i$ for all $i$, where $I = \{i\}$. Note that

$$\partial^{(i,j)} = \partial^{(i,j-1)} d_j - \partial^{(i+1,j)} = \partial^{(i,j-1)} \partial_{j,j} + \left(\partial^{(i,j-1)} - \partial^{(i+1,j)}\right)$$

for all $i, j$. Therefore, we have by the inductive hypothesis and the translation-invariance of the coefficients $c_I$:

$$\partial^{(i,j)} = \sum_I c_I M_{I \sqcup \{j\}} + \sum_I c_I (M_I - M_{I+1}),$$

where the summations are over all subsets $I = \{i_0 < i_1 < \dots < i_k\}$ of $\{1, \dots, \ell\}$ such that $i_0 = i$, $i_k = j - 1$ (we have used the fact that $M_I \partial_{j,j} = M_{I \sqcup \{j\}}$). It is easy to see that for any $I = \{i_0 < i_1 < \dots < i_k\}$ one has

$$M_I - M_{I+1} = M_{I_1} + M_{I_2} + \dots + M_{I_k},$$

where $I_j = \{i_0 < i_1 < i_2 < \dots < i_{j-1} < i_j + 1 < \dots < i_k + 1\}$ for $j = 1, 2, \dots, k$. Therefore,

$$\partial^{(i,j)} = \sum_I c_I M_{I \sqcup \{j\}} + \sum_{I,j} c_I M_{I_j},$$

i.e., $\partial^{(i,j)}$ is of the form (5.17). The lemma is proved. □

Now we can finish the proof Proposition 5.24.

Recall from (5.4) that $m_{ij} = \Delta^{(j-i)}(m_i, \dots, m_j) \in I_{j-i}(\mathcal{F})$ (hence the map $u \mapsto m_{ij} u$ belongs to $End^{(j-i)}(\mathcal{F})$) and from Proposition 5.21 that $\partial_{ij} = \Delta^{(j-i)}(\partial_{m_i}, \dots, \partial_{m_j}) \in End^{(j+1-i)}(\mathcal{F})$ for all $1 \leqslant i \leqslant j \leqslant \ell$. This implies that for $I = \{i_0 < i_1 < i_2 < \dots < i_k\}$ one has:

$$M_I \in End^{(i_1 - i_0)}(\mathcal{F}) \circ End^{(i_2 - i_1)}(\mathcal{F}) \circ \dots \circ End^{(i_k - i_{k-1})}(\mathcal{F}) \subset End^{(i_k - i_0)}(\mathcal{F}).$$

Therefore, Lemma 5.25 guarantees that $\partial^{(i,j)} \in End^{(j-i)}(\mathcal{F})$ for all $1 \leqslant i \leqslant j \leqslant \ell$. Proposition 5.24 is proved. □

We are ready now to finish the proof of Theorem 5.19.

It is enough to apply Proposition 5.24 to the identity (5.16) from Lemma 5.23.

Finally, we are able to finish the proof of Theorem 5.12.

Note that, according to Theorem 5.19, the inclusions (5.4) imply the inclusions (5.5). Lemma 5.16 and Proposition 5.17 show that (5.5) and (5.6) are equivalent (one can see it by replacing $f_i$ by $f_{n-i-1}$, i.e., passing from $m_i$ to $m_{n-i}^{-1}$ for all $i$). Now the proof follows from Proposition 5.21.

Theorem 5.12 is proved. □

We will finish the section with a natural (yet conjectural) generalization of Theorem 5.12.

**Conjecture 5.26.** *Let* $\mathfrak{g} = sl_2(\Bbbk)$ *and* $A = A_n$ *be as in Theorem* 5.12*, and* $\mathcal{F}$ *be an object of* $\mathbf{Alg}_1$*. Then a matrix* $g \in GL_n(\mathcal{F})$ *belongs to* $G_{\mathfrak{g},A}(\mathcal{F})$ *if and only if*

$$g \cdot \mathfrak{g} \cdot g^{-1} \subset (\mathfrak{g}, A)(\mathcal{F}). \tag{5.18}$$

**Remark 5.27.** More generally, we would expect that for any perfect pair $(\mathfrak{g}, A)$ an element $g \in (\mathcal{F} \cdot A)^{\times}$ belongs to $G_{\mathfrak{g},A}(\mathcal{F})$ if and only if (5.18) holds.

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