BATH '23  Minicourse on Webs

Goal of minicourse: try to understand the diagrammatic toolkit.

Modern diagram revolution (Kuperberg, Khovanov, ...): There are some really complex categories we want to understand where it is hard to do computations (e.g., category $O$, $Perv$ (quiver), etc.). But if you choose carefully a combinatorial monoidal subcategory, then you can completely describe it (if you're lucky) by just relying on diagrams. Suddenly you can compute!

Examples: Webs, KLR algebras, Soergel diagrams, ... (ignore for the moment)

We'll focus on most approachable example: $Rep^+ GL_n(C)$. Try to explain philosophy + methodology. Go present a new category of your own (I have ideas).

If you took a class on Rep Thy (if you didn't, don't worry) then maybe you think you know everything about $Rep GL_n$ (f.d. $C$-rep of $GL_n(C)$)

- Classification of irreducible reps by highest weight

  $\text{Irr } GL_n \leftrightarrow \Lambda_{\text{wt}} \leftrightarrow \text{later}$

- * Oops *

  $\lambda \leftrightarrow \lambda$

- Formulas for duality and character $\chi_{\lambda}: GL_n \to C$

  $g \mapsto Tr_{GL_n}(g)$ (or equiv wt space multiplicity)

- Even formulas for $\otimes$ decomposition:

  $L_{\lambda} \otimes L_{\mu} = \bigoplus L_{\nu}$

  (It's got a $\otimes$!)

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To Semisimple! Any $X \in Rep GL_n$ is $X = \bigoplus L_{\lambda}$ for some $\lambda \in \Lambda_{\text{wt}}$.
All these statements are about objects. What about morphisms?

* Schur's lemma: \( \text{Hom}(L_\lambda, L_\mu) = \begin{cases} \mathbb{C} & \text{if } \lambda \neq \mu \\ 0 & \text{if } \lambda = \mu \end{cases} \)

So if \( X = \bigoplus L_{\lambda_i}^{\otimes x_i} \), \( Y = \bigoplus L_{\lambda_i}^{\otimes y_i} \) then
\[
\text{Hom}(X, Y) = \text{Hom}\left( \bigoplus L_{\lambda_i}^{\otimes x_i}, \bigoplus L_{\lambda_i}^{\otimes y_i} \right) = \bigoplus \text{Hom}(L_{\lambda_i}^{\otimes x_i}, L_{\lambda_i}^{\otimes y_i})
\]

This is canonical, \( L_{\lambda_i}^{\otimes x_i} \otimes X \) is isotypic component, this canonical. (ie factorize)

Whereas \( L_{\lambda_i} \otimes X \) isn't canonical, like a line of choice of basis inside \( \mathbb{C} \)

\[
\text{Now } \left( \text{Hom}_C(X, Y) \right)_{\lambda_i} = \text{Hom}_C(L_{\lambda_i}^{\otimes x_i}, L_{\lambda_i}^{\otimes y_i}) \cong \text{Hom}_C(C^{\otimes x_i}, C^{\otimes y_i}) = \text{Mat}_{y_i \times x_i}(C)
\]

if you fix a decomposition \( \otimes \leftrightarrow \) choice of basis, then a matrix entry \( i \in \text{Mat}_{y_i \times x_i}(C) \)

on RHS matches \( L_{\lambda_i}^{\otimes x_i} \) \( \otimes \) \( L_{\lambda_i}^{\otimes y_i} \)

Composition is matrix mult. Artin-Wedderburn basis of \( \text{Hom}(X, Y) \).

BUT there'll also \( \otimes \) on morphisms. Oh no! AW basis is "incompatible"

with \( \otimes \) in that tensoring matrix entries gives difficult uncontrollable (?) monove

The \( \otimes \) has surprising amount of structure and describing \( \text{Rep}(G) \) as

a monoidal cat much more interesting then descriing as just an additive one.
Think: $\text{Vec}_C \cong \text{Mat}_{nm}(C)$-mod $\cong \text{subcat of Rep}_C$ wrt objects $L_\lambda$ for fixed $\lambda$.

\[ \mathbb{C} \leftrightarrow C^n \leftrightarrow L_\lambda \]

Math equivalence, an equiv of additive cats. Hom $(L_\lambda, L_\lambda) = \text{End}_C$ for simple $L_\lambda$.

It doesn't matter what $L_\lambda$ is! This is good in some ways.

Indeed, thinking categorically, you lose all info about objects? What is dualy?

What kind of thing is an object? "Who cares—I know morphisms" says screeching cat, the end. But $\text{Rep } GL_2 \cong \text{Rep } GL_3 \cong \text{Rep } SL_2 \cong \ldots$ ad alepot cats!

This is fixed by keeping track of monadal structure. $\text{Rep } G \neq \text{Rep } G'$ as monadal cats.

That course on rep theory is only the beginning! Gotta work monadically.

\underline{Basics of $\text{Rep } GL_n$}: My favorite rep: $V = C^n$ $GL_n$.

\[ \text{Bas } \{ e_1, e_2, \ldots, e_n \} \]

From $V$ we can get lots of other reps!

Recall: $G$ a group, $\text{G} \times V$. This $G \times V \otimes V'$ where $g(v \otimes v') = g(v) \otimes g(v')$.

So $G \times V \otimes V$, etc. This action commutes w/ action of $S_3$ on $V$, permuting the tensor factors.

$\Rightarrow G \times S^3 V$ and $G \times A^3 V$. Eg: $g(v_1 \otimes v_2 \otimes v_3) = g(v_1)g(v_2)g(v_3)$.

So we have reps $L_i = \Lambda^i V$ for all $0 \leq i \leq n$.

$L_0 = C$ where $G$ acts trivially, monadal identity.

$L_0 \otimes X \cong X$ for all $X$. (Generically)

$L_n = \text{Det}$ is also 1D, $g$ acts by mult by det $g$.
Also have

\[ \Lambda^3 V \otimes \Lambda^5 V \otimes \Lambda^7 V = L_2 \otimes L_2 \otimes L_2 = \Lambda_i \text{ for } i = (3, 3, 2) \]

\[ L_i \text{ is irreducible, but } L_i \otimes L_j \text{ usually isn't.} \]

**How do I tell if a linear map } f: \Lambda^i V \otimes \Lambda^j V \to \Lambda^k V \text{ is a } G\text{-morphism?}**

Lots of elements of } G \text{ to check? Often better to rephrase using Lie algebra. Lie } \mathfrak{gl}_n = \mathfrak{gl}_n \text{ vs. } \text{Mat}_n. \text{ acting on } V \text{ in usual way.}

Enough to check if commutes w/ generators of } \mathfrak{gl}_n, \text{ namely }

\[ x_i = (\ldots, 0, 0, 1_{i-1}) \quad \text{ and } \quad x_i(e_{ij}) = e_i, \quad x_i(e_j) = 0 \text{ etc.} \]

\[ y_i = (0, 0, 0, \ldots, 1_{i-1}) \quad \text{ and } \quad y_i(e_{ij}) = e_{i+1}, \quad y_i(e_j) = 0 \text{ etc.} \]

\[ 1 \leq i < n-1 \]

\[ \mathfrak{gl}_n \text{ action } \Lambda^i V \text{ via } x_i(\Lambda^j V) = \Lambda^j V + \Lambda^{j+1} \]

So e.g.

\[ y_i(e_1 e_2 e_3) = e_2 e_3 e_1 + e_1 \]

\[ y_i(e_1 e_2 e_3) = e_2 e_3 e_1 + e_1 e_2 e_3 \]

Similarly.

\[ y_1(e_1 e_2 e_3) = e_2 e_3 e_1 + e_1 e_2 e_3 \]

Now, \[ \Lambda^* V \text{ is a graded algebra w/ mult } \Lambda. \text{ Given } \]

\[ L_i \otimes L_j \to L_{ij} \text{ lets give more tools to compute.} \]

\[ \Lambda^i \otimes \Lambda^j \to \Lambda^{ij} \]

\[ G\text{-morphisms by defn of action } \Lambda^*. \]
For $S \in \{1,2,\ldots,n\}$ let $e_S = e_{s_1} e_{s_2} \ldots e_{s_k} \in \Lambda^k V$.

The $\{e_S\}$ form a basis for $L_k$.

$e_S e_{S'} = \begin{cases} 0 & \text{if } S \neq S' \\ (-1)^{l(S,S')} e_{S \cup S'} & \text{if } S \subset S' \end{cases}$

$e_S e_{S'} = \begin{cases} 0 & \text{if } S \neq S' \\ (-1)^{l(S,S')} e_{S \cup S'} & \text{if } S \subset S' \end{cases}$

$\left\{ (S,S') \in S \times S | S \subset S' \right\} = \left\{ (S,S') \in S \times S | S \subset S' \right\}$

Exercise: From this formula verify that $\Lambda$ is a graded $\mathbb{G}_l$-algebra.

$\Delta : \mathbb{G}_l \to \mathbb{G}_l \otimes \mathbb{G}_l$ is adjoint to mult.

Now $\Lambda^* V$ is also a graded $\mathbb{G}_l$-algebra.

$\Delta(e_i) = \sum_{S \subset S^*} (-1)^{l(S,S')} e_S e_{S^*}$. Exercise: Same goes for $l_{ij}$.

First real computation: $\mu(\Delta(e_i)) = \sum_{S \subset S^*} (-1)^{l(S,S')} e_S e_{S^*} = (e_i^i) e_i$.

So $\mu \Delta = (e_i^i) \cdot \text{id} \mathbb{G}_l$.

Indication: set combinatorics governs intertwiners between $\otimes$ of $\Lambda^k V$.

Now let's use diagrammatic to encode these morphisms.

Notation: $i = (i_1, i_2, \ldots, i_d)$ a sequence with $i_j \in \mathbb{G}_l$.

$\text{Notation: } i = (i_1, i_2, \ldots, i_d)$ a sequence with $i_j \in \mathbb{G}_l$.

Draw $i$ as $i_1 \cdots i_d$.

$\otimes$ is huzzy concept.
Draw $\text{id}_i$ as $\text{i}$ and $\text{id}_j = \text{dir}(\text{id}_j)$ as $\text{ii}$.

\[ \text{\(\otimes\) is still homogeneous now on morphisms.} \]

Draw $\text{mor}_i \rightarrow \text{mor}_j$ as $\begin{array}{c}
\text{target} \\
\text{sara} \\
\end{array}$.

Composition is vertical concatenation $\text{fog} \Rightarrow \frac{p}{q}$ with $\text{target}(q) = \text{sara}(f)$.

Ex: $\begin{array}{c}
\text{i} \\
\text{li} \\
\end{array} \rightarrow \begin{array}{c}
\text{w} \\
\text{m} \\
\end{array} \Rightarrow \frac{\Delta}{\text{\(\otimes\)}} = \text{moa} = (\text{cyl}) \text{id}_i \Rightarrow (\text{cyl}) \text{id}_j \Rightarrow (\text{cyl}) \text{ili} \Rightarrow (\text{cyl}) \text{jil}$

Note: If a picture represents a morphism, so draw a linear combo of pictures like RHS (w/ same src & tgt)

Call $\begin{array}{c}
\text{i} \\
\text{oj} \\
\end{array} = (\text{w}) \text{I}$ the bugon relation.

Ex: $\begin{array}{c}
\text{wijk} \\
\text{i} \\
\end{array} \Rightarrow \begin{array}{c}
\text{wijk} \\
\text{i} \\
\end{array} \Rightarrow \text{Mij} \text{ik} \Rightarrow (\text{Mij} \text{id}_k)$ split into rectangles to parse!

Equality is associativity of $\otimes$.

Ex: $\begin{array}{c}
\text{wij} \\
\text{i} \\
\end{array} \Rightarrow \begin{array}{c}
\text{wij} \\
\text{i} \\
\end{array}$ coassociativity.
Ex of a diagrammatic computation: "parallelogram squash"

It's a map $\Lambda \nabla \nabla \nabla \nabla \rightarrow \Lambda \nabla \nabla \nabla \nabla$

You could do this w/ vectors, but it wouldn't be nearly as easy and there would be a lot of case by case analysis!

Idea: the diagram efficiently encodes an enormous matrix and obviates a complicated operation on vectors. Can do hard matrix mult by easy diagrammatic rules. It's nice NOT knowing what the objects are...

Here's how to do the computation with vectors though! Matrix coeff by matrix coeff.

Pick $S_1$ of size 11, $S_2$ of size 2, $T_1$ of size 5, $T_2$ of size 8.

$f(e_{s_1}, e_{s_2}) \in \Lambda \nabla \nabla \nabla \nabla$ coeff of $e_{t_1} \otimes e_{t_2}$ is a matrix entry.

Lemma: Coeff of $e_{t_1} \otimes e_{t_2}$ in $f(e_{s_1}, e_{s_2})$ is a signed count of \# of ways to label all non-boundary strands w/ subsets of $I \ldots n$ s.t. $A \cup B$ and $A \cap B$ at each vertex. $S_{\cap i} (\pm 1)^{\lambda_i}$.
To check equality we'll check all matrix coeffs.

**RHS:**

\[
\begin{bmatrix}
5 \\
6 \\
2 \\
1 \\
\end{bmatrix}
\]

We need \(X_1 S_2 = T_2\), \(X_2 T_1 = S_1\).

If \(X_1 S_2 \not\in S_1\) (not \(\otimes\)), coeff = zero.

If \(S_2 \otimes T_2, T_1 \otimes S_1\), \(T_2 \setminus S_2 = S_1 \setminus T_1\), then the only option is \(X = T_2 S_2\).

And coeff is \(l(x, s_2) + \overline{l(t_1, x)}\).

**LHS:**

\[
\begin{bmatrix}
T_1 \\
T_2 \\
Z_2 \\
Z_1 \\
\end{bmatrix}
\]

How many ways to label \(y_1, y_2, y_3, y_4, y_5, y_6\)?

**Claim:** If \(\otimes\), no valid label, so call 0.

**Claim:** If \(\otimes\), then \(y_1 + y_2 = T_2 S_2\) in any valid label, and any splitting of \(X = T_2 S_2\) into sets of size 2 and 4

(\(\text{size 6}\) gives a valid labeling and \(\text{1-by-definition} B_1 B_2\)).

\(\text{Moreover, } (-1)^{l(z_1 y_1) + l(T_1 Y_2) + l(Y_3 S_1) + l(y_2 Z_1)} = (-1)^{l(x, s_2) + \overline{l(T_1, x)}}\)

It's a worthy exercise to do! "State sum model" for evaluating webs on vectors.

(But diagrammatics is much easier!)
More relns:

\[
\begin{align*}
\text{Ex:} & \quad \begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{l}
\end{array}
\end{array} & = & \begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{l}
\end{array}
\end{array} & \text{This is one of the axioms of } \otimes \text{ (interchange-by)} \\
& & & \text{fog} = (f \circ \text{oid}) \cdot (\text{id} \circ \text{og}) \\
& & & = (\text{id} \circ \text{og}) \cdot (f \circ \text{oid})
\end{align*}
\]

"rectilinear isotopy" move rectangles past each other like robot arms.

\[
\text{Ex: The hardest one! Square flap}
\]

\[
\begin{align*}
\text{a-s+t} & \quad \text{b-s-t} \\
\text{a-s} & \quad \text{b-s} \\
\text{a} & \quad \text{b}
\end{align*}
\]

\[
\sum_{r \geq 0} \left( (a \cdot b) - (b \cdot s) \right)
\]

\[
\text{so really } \sum_{r \geq 0} \text{min}(st)
\]

\[
\text{Conventions: negative label on any strand } \Rightarrow \text{ diagram } = 0
\]

\[
\text{No valid state sum}
\]

\[
\text{Note: } r=0 \text{ gives } 1.
\]

\[
\text{Ex:}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{1}
\end{array}
\end{array} & = & \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{1} \\
\text{1} \\
\text{1}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{1} \\
\text{1} \\
\text{0}
\end{array}
\end{array} & \text{(3-2)} \\
& & & \text{r=1}
\end{align*}
\]

\[
\text{or label can be removed (normal identity)}
\]
Beautiful + difficult exercise: Use the state sum model to verify the square flop relation. You'll need Chu-Vandermonde.

Exercise:\n\[
\begin{array}{c}
V' \\
V \\
V \\
V'
\end{array}
\times
\begin{array}{c}
V' \\
V \\
V \\
V'
\end{array}
\rightarrow
\begin{array}{c}
V' \\
V \\
V \\
V'
\end{array}
\]

When $V=V'=L_1$, find as linear combo of $11$ and $21$.

When $V=L_1$, $V'=L_2$:

\[
\begin{array}{c}
2 \\
4 \\
2 \\
1 \\
1 \\
2 \\
2
\end{array}
\]

When $V=V'=L_2$:

\[
\begin{array}{c}
2 \\
4 \\
2 \\
1 \\
2 \\
2
\end{array}
\]

Generalize and prove w/ state sum model.

Now let's state a theorem.

Defn: FundRep$G$ is the full subcategory whose objects have the form $L_i$ for words $i$ in $[1, N]^*$.

This is the category whose morphism we're modeling. It's a strict monoidal cat: objects are words and $\otimes$ is concatenation. Diagrams are for strict monoidal cats.

Soon: how much of $\text{FundRepG}$ does $\text{Frob}$ know.

Defn: Let $\text{Webs}^+$ be the monoidal $\mathbb{Z}$-linear category w/ presentation:

Ob: gen by $i \in \mathbb{N}$, i.e. objects are words $i$.

Mor: gen by $\begin{array}{c}
\vdash \\
\vdash
\end{array}$. Diagrams built from these $= \text{Webs}$.

Relns: Assoc., Coassoc., Bigon, Square Flop.

Note: Interchange law is free as part of what a monoidal presentation means.
That is, \( \text{Hom}_{\text{Webs}^+}(\mathcal{E}, f) \) = linear combo of diagrams w/ bottom \( i \),

\[ \text{top } j, \text{ modulo relation applied locally to subdiagrams.} \]

**Def:** \( \text{Webs}_n^+ \) is quotient of \( \text{Webs}^+ \) by further relation that \( I_k = 0 \)

for all \( k > n \). (Amazing: other relation independent of \( n \)!!!)

**Def:** \( \text{eval}_n: \text{Webs}^+ \rightarrow \text{Find} \otimes \text{O}_n \) the obvious functor \( (\otimes, \mathbb{Z}-\text{linear}) \)

obviously, \( \text{eval}_n \) descends to \( \text{Webs}_n^+ \).

**Thm:** \( \text{eval}_{n,C}: \text{Webs}_n^+ \otimes \mathbb{Z} \rightarrow \text{Find} \otimes \text{O}_n \) is an equiv. of monoidal cats.

due to Gaitsgory-Kauffmann-Morrison'14.

That is:

- All morphisms are in the span of webs, and

- We found all the relations!

**Various Remarks:**

1) Proof uses Steinberg Duality. We'll unravel a different proof

in these talks: SHD won't work in other types/situations.

2) Both sides have a \( q \)-deformation: \( \text{Find} U_q(\mathfrak{g}_n) \)

Replace \((\mathcal{E})\) with \([a, b]_q^\mathcal{E}\) and \((-\mathcal{O}^{SS})\) with \((-\mathcal{O}^{SS})^q\).

3) (E 16 arXiv) gives meaning to \( \text{Webs}^+ \) before \( -\mathcal{O} \rightarrow \mathbb{C} \), ie, to the integral form of webs. Matches Tilt \( \text{gl}_n \) in finite characteristic.

4) CKM technically do \( \text{SL}_n \) not \( \text{O}_n \), but it's basically the same story, \( \text{O}_n \) has fewer technical details.
5) History: $S\mathcal{L}_3$ 1925 Rumer-Teller-Weyl
"Temperley-Lieb algebra"
$S\mathcal{L}_3$ and other rank 2 groups: 1996 Kuperberg. Poses general question.
Morrison's thesis 07: relation, but no proof of equivalence.
CKM '14: Skew-HD saved the day.
BERT '21 arXiv: Type C, i.e. $\text{Rep } S_{2n}$.
other types: still open, but progress being made. Ask me.

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**Find $G\mathfrak{n}$ vs. $\text{Rep } G\mathfrak{n}$**

Let $\Lambda = \mathbb{Z}^n$, the weight lattice. These parameterize the eigenvalues
for the abelian group $T = (\mathbb{C}^\times)^n$ which appear in $\text{Rep } G\mathfrak{n}$ reps.
We say $v\in\mathbb{C}^n$ is a weight vector of weight $a=\{a_1, a_2, \ldots, a_n\}\in\mathbb{Z}^n$ if $v$

$$t = \left( \begin{array}{c} t_1 \\ t_2 \\ \vdots \\ t_n \end{array} \right) \in T, \quad t\cdot v = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} v.$$

**Ex:** $e^{ie_1} e^{ie_2}$ is a $(000 \ 10000)$ weight vector $te_1^i$ $te_2^i$.

$e^{ie_1} e^{ie_2}$ is $(00010000)$
$e^{ie_1} e^{ie_2}$ is $(000020000)$ as $e^{ie_1} e^{ie_2} e^{ie_3}$.

$X[a]$ is subspace of all weight vectors.

**Big Thm:** $X = \bigoplus X[a]$ for any fid. $G\mathfrak{n}$ rep.

$\Lambda_{\text{wt}}$ 
$\text{wt}(x) = \{ a \in \Lambda \mid X[a] \ni x \}$
Now the weights appearing in \((\mathbb{C}^n)^{\otimes d}\) are all positive, they live in \(N^\ast \mathbb{C}^n\).

**Def.** A weight \(\omega\) is polynomial if \(\omega_i \geq 0\) \(\forall i\). A repn is poly if all weights, ie, repns have poly weights.

(Why? the map \(G \mapsto GL(X)\) is poly if all matrix entries in \(GL(X)\) are poly, in easier of \(G\). Poly repns have poly weights.)

**Ex.** \(L^\omega = \text{Det}^\omega\) is ID w/ weight \((\omega, 1, 1, \ldots, 1)\).

\(\text{Det}^\omega\) is ID w/ weight \((-\omega, -1, -1, \ldots, -1)\), \(\omega \in \text{det}(g)^{-1}\).

**Thm.** For any f.d. repn \(X\), \(X \otimes \text{Det}^{\otimes k}\) is poly for \(k \gg 0\).

So basically \(\text{Rep}^+ \mathbb{C}^n = \{\text{poly repns}\}\) understands all of \(\text{Rep} \mathbb{C}^n\), just apply invertible functors \(\otimes \text{Det}^{-1}\). We're happy if we understand the non-nilpotent category \(\text{Rep}^+ \mathbb{C}^n\).

**Thm.** Every repn in \(\text{Rep}^+ \mathbb{C}^n\) is a direct summand of some object in \(\text{Fund} \mathbb{C}^n\).

\(\text{Rep}^+ \mathbb{C}^n \cong \text{Kar(\text{Fund} \mathbb{C}^n)}\). Pf in a bit.

The Karoubi envelope is a formal way to add direct summands to a category. Will discuss soon.

**Cor.** \(\text{Kar(\text{Wells}^+ \mathbb{C}^n)} \cong \text{Rep}^+ \mathbb{C}^n\). We get it all.
Let's talk about irreps of $G_{an}$.

**Def:** A weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is **dominant** if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

\[ \Lambda^+ = \Lambda_+ = \Lambda_+^{\text{dom}} \quad \text{corresponds to partition with \# in rows} \]

\[ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0. \]

**Ex:** \( \text{wts}(L_1) \)

\[
\begin{array}{c}
(1000) \\
(0100) \\
(0010) \\
(0001)
\end{array}
\]

Do \( \text{wts}(L_1 \otimes L_1) \) next, merge add when $\neq \bigotimes$.

\( \text{wts}(L_2) \)

\[
\begin{array}{c}
(1100) \\
(1010) \\
(0110) \\
(1001) \\
(0101) \\
(0011)
\end{array}
\]

\( \text{wts}(S^2L_1) \)

\[
\begin{array}{c}
(0200) \\
(0020) \\
(0002)
\end{array}
\]

\[
\begin{array}{c}
(1100) \\
(1010) \\
(0110)
\end{array}
\]

\[
\begin{array}{c}
(0200) \\
(0020) \\
(0002)
\end{array}
\]

**Def:** $\lambda \geq \mu$ if $\lambda - \mu$ is in $\mathbb{N}$-span of \( (0001 - 1000) \) for $1 \leq i \leq n - 1$

\[ \text{dominance order.} \]

**Thm:** Any irrep of $G_{an}$ has a weight $\lambda$ which is maximal with $\geq$ and $\Lambda^+$. Moreover, $\lambda \rightarrow 1$ is by dominant weight.

\[ \begin{array}{c}
\text{Irreps of } G_{an} \\
\{ \Lambda^+ \} \\
\{ \bigotimes \}
\end{array} \]

Moreover, $\dim L_\lambda[\lambda] = 1$. A basis vector in $L_\lambda[\lambda]$ is usually written $v_\lambda$.

So, $L = L_{(1,000)}$, $L_2 = L_{(1100)}$, $S^2L_1 = L_{(2000)}$, etc.
Def: Let $\omega_k = \binom{\lambda^+_{-\alpha_k}}{\lambda^+_{-\alpha_k}} \in \Lambda^+_{\text{dom}}$.

Then $L_k = L_{\omega_k}$. Since $\omega_k^\infty$ is a base for $\Lambda = \Lambda^+_{\text{dom}}$.

Moreover, $\Lambda^+_{\text{dom}} = \bigoplus_{k=1}^n \Lambda^+_{\text{dom}}$ (That is, any $\lambda = \sum \lambda_i \omega_i$ and $\lambda \in \Lambda^+_{\text{dom}} \Rightarrow \lambda \geq 0 \quad \forall i$.

Note: $\omega_k^\infty (1111111)$ and doesn't affect dominance.

$\lambda \in \Lambda^+_{\text{dom}} \Rightarrow \lambda \geq 0 \quad \forall i < n$. (But maybe $\omega_n < 0$)

Finally: to prove that: Let $\lambda \in \Lambda^+_{\text{dom}}$ and $\lambda = \sum \lambda_i \omega_i$.

Guess: $i = (1112222\ldots)$ and look at $L_i$.

Then $\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n$ has weight $\lambda$ and all other weights are $< \lambda$.

$L_i = \bigoplus \lambda_i^{\text{mp}}$ and only way to get $\lambda$ is if $\lambda_1 = 1$.

(Also, $\lambda_1 = 0$ unless $\mu < 1$)

So $L_i \in \mathcal{L}_i$.

For $\lambda \in \Lambda^+_{\text{dom}}$

Def: Let $PL(\lambda) = \{L_i = (i_1,\ldots,i_d) \mid \sum \omega_k = \lambda^+\}$ (just reorder $\lambda$ above).

Cor: For any $i \in PL(\lambda), L_i \not\in \mathcal{L}_i$ and all other summands are $L_j$ for $\mu < 1$. (mult. one)
So, what's Kar and how do we study it using webs?

If $L \in \mathcal{L}$ then $\exists L \in \mathcal{L} \text{ st. } p_i \in \mathcal{L}$. 

$\Rightarrow i_{op} = e \in \text{End}(X)$ is idempotent.

For any $y$, \[ \text{Hom}(X, y) \xrightarrow{e \cdot i_{op}} \text{Hom}(L, y) \text{, and} \]
\[ \text{Hom}(L, y) = \{ \alpha \in \text{Hom}(X, y) \mid e \alpha = \alpha \} \]
\[ = \text{Hom}(X, y) \cdot e \]

Similarly, $\text{Hom}(y, L) \cong e \cdot \text{Hom}(y, X)$

**Def:** \( Kar(C) \) has Ob: pairs $\langle x, e \rangle$ w/ $e \in \text{End}(X)$ idempotent

$\text{Hom}(\langle x, e \rangle, y, f) = f \cdot \text{Hom}(X, y) \cdot e$

**Note:** $e \rightarrow \text{Kar}(C)$

$x \rightarrow (x, e)$

we still call it $x$.

Less standard: what if something is a summand of multiple objects.

\[ X \xrightarrow{f_X} Y \xrightarrow{g_Y} Z \]

\[ y_{Y_X} = g_Y \circ f_X \]

\[ x \circ y \circ y_X = e_X = x \circ y_X \]

More generally, \[ z \circ y \circ y_X = z \circ y_X \]

Only remember $\xi_X(X, e_X)$ to a bunch of summands \( (X, e_X) \)

but the whole family $\xi \mathcal{u}_X$ for a common summand, i.e. a bunch of summands with fixed isomorphisms between them!
Def: For \( x \in \mathbb{N} \), a clasp is a family satisfying \( \bigotimes \) and picking out the summand \( Y \).

We'll denote \( \alpha \) with oval: \( \bigotimes \).

\[
\begin{align*}
\text{Ex:} & \quad \frac{1}{o} = \frac{1}{1} \quad P(o) = \{o\} \quad (o).
\end{align*}
\]

\[
\begin{align*}
\text{Ex:} & \quad P(\{1,2\}) = \{1,2\} \quad \bigotimes = \begin{array}{c}
1 \quad - \frac{1}{6}\end{array} \end{align*}
\]

\[
\begin{align*}
\text{Ex:} & \quad P(\{1,2\}) = \{1,2\} \quad \bigotimes = \begin{array}{c}
1 \quad - \frac{1}{6}\end{array} \end{align*}
\]

Exercise: Check \( \otimes \).

\[
\begin{align*}
\text{Moral 1:} & \quad \text{Kar is nice as a formal construction. But in practice, you need to find nontrivial claspers explicitly to use it.}
\end{align*}
\]

\[
\begin{align*}
\text{Moral 2:} & \quad \text{Claspers are nasty - linear combos of diagrams, and not easy to find!}
\end{align*}
\]
Moral 3: Clasps have denominators - defined in $\text{Webs}^+ \otimes \mathbb{Q}$ (but indep of $\mathfrak{n}$ really!)

Don't exist in $\text{Webs}^+ \otimes \mathbb{F}_p$ ... which is why rep theory in finite char is not semisimple and hard.

Moral 4: This is the reason to stick to $\text{Fnd}$ when projecting the category!

$\text{Fnd}$ is easy, but all reps are hard. If you wanted to project the category

with objects $L_{x_1} \otimes \cdots \otimes L_{x_n}$, then i.p would be morphisms and

i.p = clasps would be a relation! Category would be defined over $\mathbb{Q}$, not $\mathbb{Z}$, and would read only very nasty relations...

Why is $\frac{1}{2} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ correct?

Recent:

$L_{x_1} \otimes L_{x_2} \cong L_{x_1} \otimes L_{x_2}$, i.e., $V \otimes V \cong S^2 V \oplus S^2 V$.

So unique map up to scalar.

We can already compute $L_{x_1} \otimes L_{x_2} \xrightarrow{p} L_{x_2}$ for this other summand.

$p$ is multiple of $2 \chi_1$, $p = \frac{1}{2} \chi_1$.

$id_{L_2} = p \circ i = \frac{1}{2} \chi_1 \\ 1 \chi_1 = k_{L_1} (\chi_1) |_2 \\
2 k_{L_1} id_{L_2} \\
\Rightarrow k_{L_1} = \frac{1}{2}$.

Meanwhile, $e_{L_2} = i \circ p = k_{L_1} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \chi_2 \end{pmatrix}$. 

$\text{Schur} \Rightarrow \text{dim Hom}(L_{x_1} \otimes L_{x_2}) = 1$
Now \( \lambda_{4+1} = e_{L_{2,0}} + e_{L_2} \leftarrow \text{orthog. decompost.} \)

\[ \Rightarrow e_{L_{2,0}} = \Pi - e_{L_2} = 1 - \frac{1}{2} x^2. \]

For second example, \( L_1 \otimes L_2 = L_{1,1} \oplus L_{1,3}. \)

**Reason 2:** Suppose you've checked \( e^{E \mathfrak{sl}(L_i)} = e \) is an idempotent.

**What is \( \text{Im} e \)?** Some (not necessarily indecomposable) summand of \( L_i. \)

**How would you know it was \( L_1 \)?** Well, remember \( \otimes \)

\( \text{Im} e \equiv L_1 \iff \text{Im} e \neq 0 \) and \( \text{Hom}(\text{Im} e, L_1) = 0 \forall \mu < 1. \)

\( \Rightarrow e \neq 0 \) and \( \text{Hom}(L_1, L_1) e = 0 \forall \nu < 1 \)

\( \Rightarrow e \neq 0 \) and \( \text{Hom}(L_1, L_j) e = 0 \forall j \in P(\mu) \forall \nu < 1. \)

**Exercise:** \( e \neq 0 \) and \( \text{Hom}(L_j, L_1) e = 0 \).

**Ex:** What is \( \otimes 2 \omega_1 \)? Just \( \omega_2. \)

**In \( \Lambda^4, \text{dim} = 21 \)**

\( \text{Hom}(L_1 \otimes L_1, L_2) \) spanned by \( \chi_1 \)

(its 1D by Schur lemma)

\[ \chi_1^2 = \chi_1 - \frac{1}{2} \chi_1^3 = \chi - \chi^2 = 0. \]

The claim is orthogonal to lower terms. **How to check?** Need to know

a basis of maps to lower term \( \Rightarrow \) soon.
Exercise: Show that $\varphi \in \text{End}(L_i)$ is equivalent to $\text{id}$ modulo $\text{Hom}_W(L_i, L_i)$. Use this to deduce that $x \varphi x$ is unipotent (given orthogonally to lower terms). Is $x \varphi x$ isomorphic to lower terms?

Exercise: Compute $\varphi^2$. You can use $L_i \otimes L_i \cong L_{2g} \oplus L_{2g+2}$.

are that $\text{End}(L_i \otimes L_i)$ has basis $\{x^1, x^2, \ldots, x^{2g} \}$.

Exercise: To compute the clamp, you need only compute a single $\varphi^2$ because

\[ \text{Thm: } \begin{array}{cc}
\frac{b-a}{b} \times \frac{b}{b} = \frac{b-a}{b-a}
\end{array} \]

and so, for $a < b$.

To prove, look at
\[ \begin{array}{cc}
\frac{a}{b} \times \frac{b}{a} = (\text{Square Fpp}) \frac{1}{a} + \text{Hom}_{\omega_a \omega_b} \end{array} \]

and deduce the result.
Exercice: Compute \( L^2 \otimes L^2 = L_{22} \oplus L_{2+3} \oplus L_{24} \).

To really work with webs we need a basis for \( \text{Hom} \) spaces.

Recall: \( \text{Hom}(L_i, L_j) = \bigoplus \text{Hom}_k(L_i, L_j) \) (span of maps \( L_i \to L_j \to L_j \))

and \( \text{Hom}(L_i, L_j) \otimes \text{Hom}(L_i, L_k) \xrightarrow{\text{comp}} \text{Hom}(L_i, L_j) \) by Schur's lemma.

\[ \begin{array}{c}
\text{Hom}(L_i, L_j) \\
L \rightarrow L_i \\
L \rightarrow L_j \\
L \rightarrow L_j
\end{array} \]

\[ \begin{array}{c}
\text{Hom}(L_i, L_j) \\
L \rightarrow L_i \\
L \rightarrow L_j
\end{array} \]

Qn: How to find a basis for \( \text{Hom}(L_i, L_k) \)? Don't call it too high!!

Motto: We hate linear algebra!

Glorious Fact: The real reason we care! Find \( \lambda \) multiplicity-free branching

\[ \forall \lambda \in \Delta^+ \quad L^\lambda \otimes L^\lambda = \bigoplus L^\nu \quad \text{and} \quad C^\lambda \in \{0,1\} \]

\[ \forall i \geq 0 \]

So \( \text{Hom}(L^i \otimes L_j, L^j) \) is at most 1D. Basis is unique up to rescaling!!

No linear alg.

We'll bootstrap into a canonical (up to rescaling) basis of \( \text{Hom}(L_i, L_j) \) adapted to monoidal structure.

First, let's just look at \( L^i \otimes L^i \).
For pedagogical reasons I will develop the theory of bases assuming we can compute all the chaps! Which seems ridiculous...

But later I'll argue you can remove all the chaps and still get a basis.

So just pretend right now that chaps aren't awful linear combos but are instead nice happy black boxes... en... ovuls.

\[ \text{Hom} \left( L^1 \otimes L^5 \otimes L^n, L^2 \otimes L_2 \right) = \text{span of "diagrams" of the form} \]

Pick \( \sigma_6 \in \mathbb{P}(\mu) \) but choice is irrelevant.
Steinberg \( \otimes \) formula: \[ L_{\lambda} \otimes \mu = \bigoplus_{\nu \in \text{Wts}(\lambda)} (\text{dim } L_{\lambda} \otimes L_{\nu}) \cdot L_{\nu} \]

**Example:** \( n = 2 \), \( \lambda = (2,0) \) \( \otimes \mu = (1,1) \), \( \nu = (0,2) \)

So \[ L_{(2,0)} \otimes L_{(1,1)} = L_{(3,0)} \oplus L_{(1,1)} \oplus L_{(0,2)} \]

**Problem:** \( \lambda + \nu \) might not be dominant! What is \( L_{\lambda + \nu} \) then?

In general, some "signed" copy which might cancel out another summand!!

**But** - recall \( \lambda = \sum q_i \omega_i \) is dominant \( \iff q_i \geq 0 \). \( \forall i \)

If \( q_i = -1 \) for some \( i \), we say \( \lambda \) is "on the wall."

If \( \lambda \) is on the wall, then \( \"L_\lambda \" = 0 \).

**Example:** \( L_{(1,0)} \otimes L_{(1,0)} = L_{(2,0)} \oplus L_{(1,1)} \oplus L_{(0,2)} \)

\[ \lambda = (a_1, a_2, \ldots, a_n) \]

\( n = 7 \), \( \nu = (0,1,1,0,0,1,0) \)

\( \lambda + \nu = (a_1 + 1, a_2 + 1, a_3 + 1, a_4, a_5 + 1, a_6 + 1, a_7) \) where \( a_i \geq 0 \)

If \( a_i = a_2 \) then \( a_i \leq a_2 \) \( \iff \) \( q_{i+1} \) is dominant!

If \( a_i > a_2 \) then \( a_i \geq a_2 + 1 \)

Similarly, if \( a_5 = a_6 \) then on the wall, else \( a_5 \geq a_6 + 1 \). So \( \lambda + \nu \) is dominant or on the wall.
Another way: \[ y = -w_1 + w_2 - w_5 + w_6 \]

\[ x = \text{Re } \omega \]

So \[ f(x) = (q-1)w_1 + (q+1)w_2 + (q^3+1)w_3 + w_4 + (q^3+q^2+1)w_5 + (q^2+q+1)w_6 \]

cells \[ \geq -1 \] and if \( \geq 0 \) then dominant.

When not dominant? When \( q = 0 \) (i.e., \( a_1 = a_2 \)) or \( a_3 < 0 \) (i.e., \( a_5 = a_6 \)).

Now, \( \text{wts}(L_k) = \{ (0010101101111...) \} \) for all \( \alpha \).

\[ h_K \text{ being } \alpha \text{ mult } 1. \]

**Example:**

\( L_5 \otimes L_1 = L_{(11110000)} + (10000) \otimes L_{(11110000)} + (01000) \otimes ... \)

\[ = L_{(21110000)} \otimes L_{(21110000)} \otimes ... \otimes L_{(11110000)} \otimes L_{(11101000)} \]

\[ = L_{w_5 + w_6}. \quad \text{Proj maps up to scalar: } \frac{1}{1} \quad \text{and} \quad 3 \]

**Example:**

\( L_5 \otimes L_2 = L_{(22111000)} \otimes L_{(22111000)} \otimes L_{(111111000)} \)

\[ = L_{w_2 + w_6} \otimes L_{w_2 + w_6} \otimes L_{w_2}. \]

Proj maps up to scalar: \( \frac{1}{1} \quad 6 \quad 4 \quad 1 \)

\[ \text{There are nonzero maps} \quad \text{lying in } \mathbb{Z}^D \]

\( \text{hom spaces,} \)

\( \text{show sum of} \quad \text{map is nonzero on} \)

\( \text{this vector.} \)
Ex: When does \( L_3 \otimes L_3 \) have summand \( L_{j+n} \)? When \( \lambda \in \mathcal{P}(\frac{3}{2}) \).

So \( \exists \lambda \in \mathcal{P}(\alpha) \) of the form \( (k, 1, 5) \).

Let \( L_3 \otimes L_3 \rightarrow L_{j+n} \).

Note: \( (k, 3, 6) \in \mathcal{P}(\alpha+n) \)

For all \( x \in L_3 \), the projection maps to \( L_{j+n} \).

\[ L_3 \otimes L_3 \rightarrow L_{j+n} \]

\[ k = 1, 5 \text{ and } 3, 6. \]

Color solve problem for minimal example \( J = 0 \wedge 5 \wedge 5 \wedge 5 \).

This is why the problem is tractable!

Any \( x \), but only one minimal example!

To prove it works, just need to find a vector \( x \in L_3 \) such that:

\[ L_3 \otimes V \rightarrow \text{vanish out of } V_4 \]

Hint: \( x \in [V_j, V] \).

Exercise:

You've just experienced the phenomenon of branching patterns! The pattern \( +v \) matched a class of \( \lambda \) and the projection maps for this pattern were all contracting en masse from the minimal match.
Exercise: Find all elementary light ladders for $\text{wts}(L_2)$ when $n=4$.
Exercise: Find all elementary light ladders for $\text{wts}(L_3)$ when $n=6$.

Exercise:

Example:

$\mathcal{L} = (1100100) = w_2 + w_5 + w_6 \in \text{wts}(L_3)$

Minimal match: $1 = w_5 \land 1 + 2 = w_2 + w_6$

\[
\begin{array}{c|cc}
0 & 2 & 1 \\
\hline
1 & 3 & 6 \\
0 & 3 & 5 \\
\end{array}
\]

General projection

Ok, so we know how to project from $L \otimes L_i$ to any of its summands. Now what?

Let's decompose $L_2 \otimes L_2 \otimes L_2$ into its direct summands and find projection maps

$n \gg 0$

\[1 \otimes L_2 \otimes L_2 \otimes L_2 \] do it one $\otimes$ factor at a time

$1 \otimes L_2 \rightarrow L_2 = L_0$ only summand \[1 = L_0\]

Other direct summands $\otimes L_2 \ \text{wts}(L_2)$

$L_2 \otimes L_2 \sim L_2 \oplus L_4 \oplus L_5$ 

\[
\begin{array}{c|cc}
2 & 1 & 3 \\
\hline
4 & 1 & 4 \\
\end{array}
\]

Proj

\[
\begin{array}{c|cc}
0 & 2 & 1 \\
\hline
0 & 2 & 1 \\
0 & 2 & 1 \\
\end{array}
\]
Now $\otimes$ w/ $L_4$. Let's analyze $L_{1+\mu_2} \otimes L_4$. 

\[ L_{1+\mu_2} \otimes L_4 = L_{(31\pm\infty)} \otimes L_{(31\infty)} \]

\[ = L_{(31\infty)} \otimes L_{(22\infty)} \]

So they lead to:

![Diagram showing tree structures]

We switched the expression in $PL_{1+\mu_2}$.

This flexibility really helps!!

Hooray, clasps $>$ homotopies.

If we analyze $L_{2\mu_2} \otimes L_4$ and $L_{\mu_4} \otimes L_4$ similarly, we contract projections from $L_2 \otimes L_2 \otimes L_4$ to all summands.

Branching graph:

1 \rightarrow \omega_2 \rightarrow \omega_4, \omega_4w_2, \omega_4w_4 \rightarrow \omega_5w_2, \omega_5w_4, \omega_5w_5, \omega_5w_6, \omega_5w_7, \omega_5w_8, \omega_5w_9, \omega_5w_{10}.

Each path gave a summand of $L_1 \otimes L_4$ with a projection map.

Finding this the intermediate summands!
So, to summarize: to each path \( T \) in the branching graph of \( L_\mathfrak{g} \) we have a sum \( L_\mathfrak{g} \oplus L_\mathfrak{k} \) and a projection \( P_\mathfrak{f} : L_\mathfrak{g} \to L_\mathfrak{k} \).

\[ T = (1_b, 2, 3, \ldots, 1_d)^N = 1 \]

\( P_\mathfrak{f} \)

(Where each \( P_i \) is \( \mathfrak{U} \).)

Flipping upside-down is a symmetry of \( \mathfrak{U} \) giving \( t : L_\mathfrak{g} \to L_\mathfrak{k} \).

**Im:** If \( T = T' \) then \( P_\mathfrak{f} T = 0 \).

\[ \text{Proof:} \]

Look at first place they differ, \( k \neq k' \).

This kind of structure makes sense for many other semisimple monoidal cats! We'll summarize our assumptions later.
Conclusion: A basis for $\text{Hom}(L_i, L_j)$ is

$$\{ T \mid \text{a branching path for } i \text{ only in } T \}$$

$$\text{do } \iota_{L_i} \text{ at } T'$$

and this is an arith.-wellestern basis!! (by previous theorem)

The problem: All those awful clamps. This basis is defined over $\mathbb{Q}$ not $\mathbb{Z}$ and every element is a linear combination of diagrams, impossible to compute...

The fix: What if we just ignore the clamps?

replace

$$\text{replace with}$$

$$\text{Call it } LL_T$$

instead of $\Pi_T$

$$\Pi_T \text{ instead of } 4.$$
What are we doing?

Replacing $T$ with $id_i$ or $\frac{a}{b}$ with $\frac{a}{a}$

They agree modulo $Hom(L_i)$.

By a terribly convoluted induction argument:

\[ \{ T^i \}_{i=0}^\infty \] is a basis of $Hom(L_i, L_i)$

with upper triangle coars. matrix to $\begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}$

But $\{ T^i \}_{i=0}^\infty$ is just a diagonal, not a linear combo of diagrams!

(Not canonical though, make some choice. Eg.

Look ma, no clamps!

Don't get an A-W basis, but don't expect one to be easy!!

Moral: In webs, we don't have a direct handle on $L_i$, just maps on clamps.

Can't access $Hom_i$. But we can access $Hom_{i1}$ because it agrees with maps on factors that $L_i$ is $P^1(1)$ for each.

Don't have clamps, but have a "heuristic ladder" which agrees with $Hom_{i1}$.

Good enough to crank the abstract machinery!
What really makes this tick: (Assumptions) For semisimple $\mathfrak{g}$ on $E$:

1) Monoidal untrangulants: $\forall \chi \in \text{Im } E \exists$ nonempty set $\mathcal{P}(\chi)$ of objects $\chi \otimes \mathcal{L}_i$ and all other summands $\chi \otimes \mathcal{L}_i$ satisfy $\mu < \lambda$ (for some $\mu$ on $\text{Im } E$).

This fails for other popular approaches to rep theory, e.g., the full subcat $\mathcal{N} \otimes \mathcal{L}$, which does contain every irreps as a summand.

M.U. allows us to study filtration $\text{Hom}_{\mathfrak{g}}(\chi, \text{End } E)$ on every object in $\mathfrak{g}$ and this to understand $\text{Hom}_{\mathfrak{g}}$ as $\text{Hom}_{\mathfrak{g}}/\text{Hom}_{\mathfrak{g}}^*$. Here $\text{Hom}_{\mathfrak{g}}^* = \{\text{Span of maps factoring thru } \mathcal{L}_i \text{ for } \mu \in \mathcal{P}(\mu)\}$ for $\mu < \lambda$.

2) Multiplicatively free branching: this is OPTIONAL.

It removes linear algebra from the picture, but would be outside type A.

E.g., find reps might have 2D weight space... a little linear, a lot.

3) line vectors: A technical tool for proving correctness of elementary projections, a nice testing spot - but there are alternatives.

4) Branching patterns: Surprisingly ubiquitous! Finite work $\Rightarrow$ infinite pleasure.
Summary:

• Within the integral from $W^+_\mathbb{Z}$ to $W^+_\mathbb{O}(Q)$ for each $\lambda$ and $i$ and sum over $Ly \otimes Li$ one has

\[ \int_{\lambda} \mathbb{P}(\mu) \]

which descends in $W^+_\mathbb{O}(Q)$ (when $\operatorname{cl}$ ap is existent) to

• For $L, L' \in \mathcal{P}(\mathbb{A})$ have neutral map $N_L$: $\mathbb{Q} \to \mathbb{A}$

• Using this, for each branching path $T$ subordinate to $f$ one has

\[ (\lambda_1, \lambda_2, \ldots, \lambda_d) \]

Write $T$ to $E(f, \lambda_d)$

• Similarly if $k \in \mathcal{P}(\mathbb{A})$ to $f$.

• If $T \in E(f, \lambda_d)$ then $S \in E(f', \lambda)$

\[ L_{\lambda} \rightarrow L = L_{\lambda} \]

and $[L_{\lambda}]$ is basic for $\operatorname{Hom}(f', f)$ up to rescaling.

Now $W^+_\mathbb{Z}$ is a multi-objects adapted cellular category: lots of free machinery to study its rep theory.

\[ \boxed{120} \]
This helps you present your category too!

Object generators: enough to get maximal untriglability, but keep the branching graph simple with bar multiplicities.

Morphism generation: enough to have all $P_f$ and all $N$.

Relations: Enough to make $\{L_{5,T}\}$ span!

Go bad up $\Rightarrow$ ![Diagram](good and bad diagrams)

Any ![Diagram](up and down arrows) shall go to $\Rightarrow$ ![Diagram](up and down arrows),

**Ex:** $\partial y_j < \partial_i + y_j$ so $\partial y_j$ is down, $y_j$ is up

$$0 = (\cdot)$$ replace with $\square$; Yay.

**Ex:**

$$
\begin{bmatrix}
3 & 5 \\
1 & 7
\end{bmatrix}
$$

is up, $\begin{bmatrix}
1 \\
2
\end{bmatrix}$ is down (think $\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix}$)

$$
\begin{bmatrix}
1 \\
2
\end{bmatrix}
$$

$D_{1,2} = c_1 \begin{bmatrix}
1 \\
2
\end{bmatrix} + c_2 \begin{bmatrix}
1 \\
16
\end{bmatrix}$; Yay.
Additional Consideration: \[ \frac{\text{Hom}(i, i')}{\text{Hom}_X(i, i')} \] is 1D.

So neutral maps live in 1D space \textit{w.l.o.g.}

\[
\begin{bmatrix}
N & N \\
\hline
N & U \\
\end{bmatrix}
= \begin{bmatrix}
N \\
\hline
U \\
\end{bmatrix} + \begin{bmatrix}
D \\
\hline
D \\
\end{bmatrix}
\]

**Ex:**

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
2 & 4 & 6 & 1 & 1 & 1 & 1 & \ldots \\
6 & 4 & 2 & & & & & \\
2 & 6 & 2 & 6 & & & & \\
\end{array}
\]

There are a few more ingredients but there is almost a general proof of correctness of predation requiring relations of a certain form.
Clasps: Def: Fix $\lambda \in A \setminus \text{redial expression } PA$. The clasp (if it exists) is a family of morphisms $\psi_k: i \to j$ for $i,j \in \text{PA}$ satisfying

1. $\psi_k \circ \psi_k = \psi_k$
2. $\psi_k = \text{id}_i$ modulo $I \lambda$
3. $\psi_k \circ a = b \circ \psi_k = 0$ for any $a,b \in I \lambda$

(This way of formulating it obviates the need for discussing the object $L_1$.)

Exercise: The clasp is unique if it exists.

Finding a closed formula for the clasp as a linear combo of webs seems out of reach. It was done by Morrison for $\gamma_2$ and it is complicated! But inductive formulas are philosophically important and practical too. Triple Clasp Expansion -

Suppose we have computed all the clasps less than $\lambda + \infty k$, including $\lambda$. We know $L_{\infty k} = \bigoplus L_i$ for $i \in \text{Ind}(L_{\infty k})$ with domain $k$.

One summand is $L_i + \infty k$, we want this idempotent so we want to subtract $k \circ \infty k$ from all the other!

$\text{Hom}(L_i + \infty k, L_0 k) = 1$ and we have a basis for it.

So the idempotent is $\frac{1}{k} I_k$

For some $k \in I_k$.

Reason for more coming soon.

[Sketches of diagrams and equations related to the text are present.]
Thus, \( \text{id}_{\Lambda_k} \) is a sum of idempotents, or

\[
\Lambda_k = \begin{array}{c}
\, \\
\end{array} + \sum_k \Lambda_k^{-1}
\]

A formal argument gives this recursive formula.

\[ \begin{aligned}
\text{Ex: } \varphi_1, & \quad \text{write } n \text{ for } n \varphi_1 + x \varphi_2 \text{ when } x \text{ depend on context.}
\end{aligned} \]

\( x \varphi_2 \) is determined rep and

\[ \begin{aligned}
\left| \begin{array}{c}
\mathcal{A}_2
\end{array} \right|_2 &= 0
\end{aligned} \]

Note: 3 claps have name.

\[ \begin{aligned}
\text{Ex: } \varphi_2, & \quad \text{write } n \text{ for } a \varphi_2 + b \varphi_2 + c \varphi_2
\end{aligned} \]

\[ \begin{aligned}
\left| \begin{array}{c}
\mathcal{A}_2
\end{array} \right|_2 &= 0
\end{aligned} \]

But when \( n = 0 \), \( \mathcal{A}_2 = 0 \) anyway!

\[ \begin{aligned}
\alpha_{ab} &= \frac{[a]}{[a_1]}
\end{aligned} \]

\[ \begin{aligned}
\beta_{ab} &= \frac{[b]}{[a]}
\end{aligned} \]

\[ \begin{aligned}
\gamma_{ab} &= \frac{[c]}{[a_1][a_2^2]}
\end{aligned} \]

How to find coeff?
Well, it's supposed to be $i \rho = e$, with $\rho = i \delta_{i+j,\nu}$. So want

$$K_{\nu}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., $K_{\nu}$ is of $i \delta_{i+j,\nu}$, i.e.

$K_{\nu} = \text{cellular family of } E$ with itself (a $1 \times 1$ matrix).

We can now compute this.

$$K_n \circ \alpha^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - K_{n-1}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - K_{n-1}^{-1} \text{ [2]} = 2 \text{ [2]} - K_{n-1}^{-1} \text{ [2]} = 2 \text{ [2]} - K_{n-1}^{-1} \text{ [2]}$$

So $K_n = [2] - K_{n-1}^{-1}$.

As noted, $K_0 = 0$ since the term does not exist. Can solve, get $K_n = \frac{[n+1]}{[n]}$

Using the formal recursion for $\alpha$, get recursion for coeff in that formal recursion.

Get $\gamma_{ab}$ get $\alpha_{a,b} = [2] - \alpha_{a+1,b+1}^{-1}$ and $\alpha_{a,0}^{-1} = 0$

$\Rightarrow \alpha_{a,b} = \frac{[a]}{[b]}$

Get $\beta_{a,b} = [3] - \frac{\alpha_{a+1,b+2}}{\delta_{a,b-1}} = \frac{1}{\Theta S_{a,b-1}}$ and similar for $S$...

Quite complicated!!

Note: (Reverse) domain order on $\nu$ controls which coeffs can appear in recursion

forms for which others.
Morally it's clear what to do! Compute various $K_{l,m}$ by finding various formulas and solving them. Finding and solving each takes a lot of work, but it's still tractable thanks to branching patterns.

**Theorem (E conjecture 16) (Martin-Sparrow 122)**

$$K_{l,m} = \prod_{\text{pos root } \omega \in \mathcal{V}} \frac{<\lambda + \rho, \alpha>}{<\lambda, \alpha> - 1}$$

**Example:**

$\gamma = (0101100)$

So if $\lambda = \gamma_1 \omega_1 + \ldots + \gamma_n \omega_n$ then

$<\lambda + \rho, \xi - \xi_2> = g_i + 1$

$<\lambda + \rho, \xi - \xi_4> = (g_i + 1) + (c_4 + 1) + (c_3 + 1)$

$<\gamma, \xi - \xi_2> = -1$  
$<\gamma, \xi - \xi_4> = -1$  etc

$K_{l,m} = \frac{(c_1 + 1)(c_1 + c_2 + c_3 + 1)(c_4)(c_3 + c_4 + 1)}{(c_1)(c_1 + c_2 + c_3 + 2)(c_4 + c_2 + c_3 + 3)(c_3)(c_3 + c_4 + 1)}$

After reconfiguring M-S relate to work of Tatsuy '90, but I'm still not sure there's a good combinatorial or philosophical explanation.