→ study Rep of $\mathfrak{g}$ ⇐ G s.s. l.a.

If you took a class you probably think we know everything!

- classification of irreducible repns
- formula for their characters / "graded" dimensions
- It's $\otimes$. Decomposition formulas $L_\lambda \otimes L_\mu \cong \bigoplus L_\nu^{\oplus m_\nu}$ multiplicity, irreps

Objects, what about morphisms?

- It's a semisimple category – easy right? Schur's lemma: $\mathrm{Hom}(L_\lambda, L_\mu) = \begin{cases} \mathbb{C}\cdot\mathrm{id} & \lambda=\mu \\ 0 & \text{else} \end{cases}$

If $X = \bigoplus L_\lambda^{\oplus x_\lambda}$ $Y = \bigoplus L_\lambda^{\oplus y_\lambda}$ then $\mathrm{Hom}(X,Y) = \bigoplus_\lambda \mathrm{Hom}(L_\lambda^{\oplus x_\lambda}, L_\lambda^{\oplus y_\lambda}) \cong \mathrm{Hom}_\lambda(X,Y)$

this is canonical! Isotypic components always are. Noncanonically, $\mathrm{Hom}_\lambda(X,Y) \cong \mathrm{Mat}_{y_\lambda \times x_\lambda}(\mathbb{C})$

by choosing inclusion + projection maps $X \twoheadrightarrow L_\lambda \hookrightarrow Y$. Composition is matrix mult, for each λ.

BUT · it's still $\otimes$. $\otimes$ on morphisms? Matrix decomp does not lead to good description

Modern language for morphisms in a monoidal cat: planar diagrammatics. Before discussing the philosophy further I want to show you example of $\mathfrak{gl}_n$. Makes easier to motivate discussion. You're stuck here so don't need motivation yet...

To p2, Poll, $\mathfrak{gl}_n$. Then p7. Then back here.

So what do we conclude?

We're trying to describe a cat by gens + relns. Doing so automatically produces an integral form of the category – an additive category linear over the smallest subring $\mathbb{k}$ of $\mathbb{C}$ (or $\mathbb{Q}(q)$) where the coeffs of the relations live. Ex: Web^+ is defined over $\mathbb{Z}$ or $\mathbb{Z}[q,q^{-1}]$.

Categories have many different integral forms, it's nice to find one w/ meaning so that when you specialize to $\mathbb{F}_p$ rather than $\mathbb{C}$ you get something interesting – another story.

NOT include all irreps L_λ in your generators! $\Rightarrow$ all clasp formulas become nonsense, need e.g. $\frac{1}{[3]} \in K$. But Rep $\mathfrak{g}$ is different in any finite characteristic $\rightsquigarrow$ $\frac{1}{p}$ is in some clasp $\rightsquigarrow K \supset \mathbb{Q}$ $\rightsquigarrow$ need ∞-ly many relations!! (Q)

Suggests • intractability • inelegance • inefficiency • uselessness (+ no specialization)

Get something useful + simple if you place L_λ off-limits + restrict to Fund$\mathfrak{g}$.

Then use clasps to recover info about L_λ. Note: $\mathrm{Hom}(L_\lambda, -) = \mathrm{Hom}(L_{\underline{\lambda}}, -)e_\lambda$

Sounds great but... if you had a basis for webs, still don't get a basis for $\mathrm{Hom}(L_\lambda, -)$ b/c $-\circ e_\lambda$ creates relations. Also, $\exists$ interesting relations for maps e.g. $L_\lambda \otimes L_\mu \to L_\nu$ which are consequences of clasps but not easy ones. Webs + Clasps = Inefficient presentation for Rep$\mathfrak{g}$.

So you have a s.s. $\otimes$ cat?

Goal 1: Find an additive $\otimes$ subcat + describe it by gens + relations w/ planar diagrams. Tractable

(Rmk: Find $\mathfrak{g}$ only "correct" in type A.)

Goal 2: Find all the clasps. - projection to irreps inside "reduced expression".

Goal 3: (Dreamy) Find an efficient presentation of the whole category. ???

for Rep $\mathfrak{g}$ –

Goal 1: $\mathfrak{sl}_2$ TL rank 2 Kuperberg!! type A: CKM else - unknown!

Goal 2: $\mathfrak{sl}_2$ JW rank 2 Kim, incomplete type A: me, conjecturally.

Goal 3: $\mathfrak{sl}_2$ maybe? else - unknown.

6j symbols, etc

Future talks:

Goal Omega: What structure should this additive $\otimes$ cat have? "Shadows of semisimplicity"

How does this inform Goals 1+2? Everything is easy + well-known but synthesis is new.

Toolbox for "generically s.s. ~~enriched~~ $\otimes$ cats." Same ideas w/ very different details worked for Hecke category.

End of Lecture 1.

~~$\mathfrak{F}$: $\mathfrak{sl}_2$ ✓ but a mystery, probably intractable.~~ ②

~~series: a toolbox + philosophy for studying monoidal semisimple categories and their integral forms. Not just Rep $\mathfrak{g}$, exact same toolbox for Soergel bimodules. Same ideas, but very different details.~~

Focus on Rep $\mathfrak{gl}_n$ to make most accessible, give concrete example before abstract nonsense.

10:28

$\boxed{\mathfrak{g}=\mathfrak{gl}_n}$ Let $L_0 = \mathbb{C}_{triv}$, $L_1 = V = \mathbb{C}^n$ w/ basis $\{e_1, e_2, \dots, e_n\}$ 10:33

$L_2 = \Lambda^2 V$ w/ basis $\{e_1 \wedge e_2, e_1 \wedge e_3, \dots\}$ $\quad L_k = \Lambda^k V$.

So $L_n = \Lambda^n V$ is the determinant rep, and $L_k = 0$ for $k > n$. (and $k<0$ by convention)

All of them are reps of $\mathfrak{gl}_n$, and $\{L_k\}_{1\le k\le n}$ are the fundamental reps of $\mathfrak{gl}_n$, generating a monoidal category $\mathrm{Fund}\,\mathfrak{gl}_n \subseteq_{\text{full}} \mathrm{Rep}\,\mathfrak{g}$, Ob: $L_{k_1} \otimes L_{k_2} \otimes \dots$

Now $\Lambda^* V$ is an algebra (w/ $\mathfrak{gl}_n$-equiv. mult.) and even a Frobenius algebra.

Have a mult map $L_i \otimes L_j \xrightarrow{m} L_{i+j}$ and its adjoint $L_{i+j} \xrightarrow{\Delta} L_i \otimes L_j$

$v \otimes w \mapsto v \wedge w$

My favorite way to compute: For $S \subset \{1, 2, \dots, n\} = [n]$ let

$e_S = e_{s_1} \wedge e_{s_2} \wedge \dots \wedge e_{s_k} \in L_k$ when $|S| = k$, $S = \{s_1, \dots, s_k\}$ in order.

Then $\{e_S\}_{S \subset [n], |S|=k}$ is a basis for L_k.

$$e_S \wedge e_{S'} = \begin{cases} 0 & \text{if } S \not\perp S' \\ (-1)^{\ell(S,S')} e_{S \sqcup S'} \end{cases}$$

$(-q)^{\ell(S,S')} \rightarrow$

where $\ell(S,S') = \#\{(s,s') \in S \times S' \mid s' < s\}$

$$\Delta(e_T) = \sum_{\substack{S \sqcup S' = T \\ |S| = i \\ |S'| = j}} (-1)^{\ell(S,S')} e_S \otimes e_{S'}$$

$(-1)^{\ell(S,S')} q^{-\ell(S',S)}$

first computation: $m(\Delta(e_T)) = \sum_{S \sqcup S' = T} (-1)^{\ell(S,S')} (-1)^{\ell(S,S')} e_T = \binom{i+j}{i} e_T$

So $m \circ \Delta = \binom{i+j}{i} \circ \mathrm{id}, \dots$

10:41

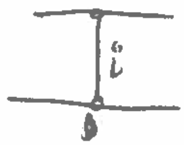
Time to draw it.

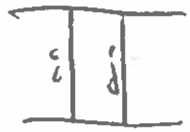
<u>Notation</u>: $\underline{i} = i_1, i_2, \ldots, i_d$ a sequence (think $i \leftrightarrow L_i$)

$L_{\underline{i}} = L_{i_1} \otimes L_{i_2} \otimes \cdots \otimes L_{i_d}$.

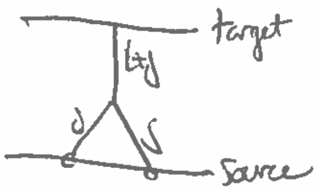
Draw $\underline{i}$ as 

, $\otimes$ is (horiz) concatenation.

Draw id_{L_i} as 

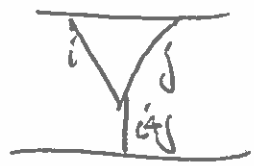
and $\mathrm{id}_{L_i \otimes L_j}$ as 

$\otimes$ is horiz concat

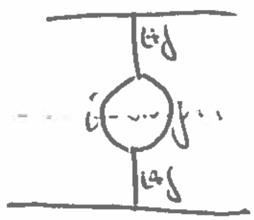
Draw M as 

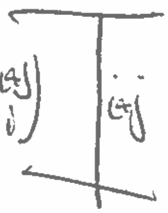
read bot $\to$ top

so composition is vertical stacking.

Δ as 

We just computed:

(1) $M \circ \Delta =$ 

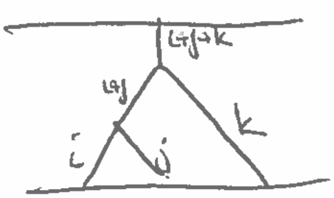
$= \binom{i+j}{i}$ 

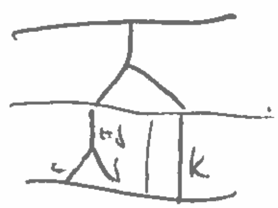
this is a linear combo of diagrams

If a diagram represents a morphism $\underline{i} \to \underline{j}$ (bot, top), then a linear combo w/ same bot + top represents a morphism too.

Call this the <u>bigon relation</u>.

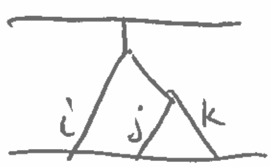
(2) Mult is <u>associative</u>.



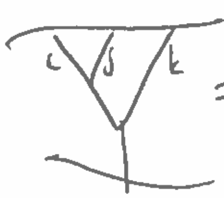
$=$ 

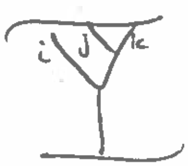
$M_{i+j,k} \circ (M_{i,j} \otimes \mathrm{id}_k)$

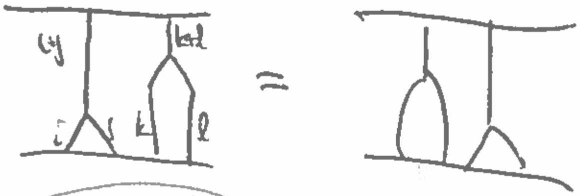
$\|$



(3) Δ is <u>coassoc</u>.



$=$ 

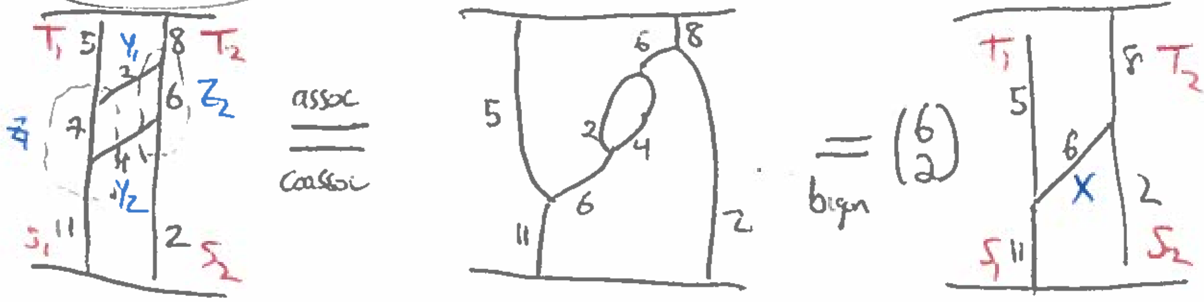


by laws of $\otimes$.

Rectilinear isotopy, property of all planar diagrammatics.

16.44 "Lean space"

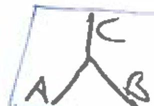
Ex:



The parallelogram squish or rung squish is a consequence of other relations.

Relating back to gl_n reps: Label all input & output strands w/ subsets of $[n]$.

The coeff of $\boxed{e_{T_1}\otimes e_{T_2} \text{ in the map applied to } e_{S_1}\otimes e_{S_2}}$ ← a matrix coeff is a signed count of the ways to label all the other strands compatibly. Sits



mean $A\sqcup B = C$.

generators have easy matrix.

RHS count: Clearly ~~X = ...~~ $X = T_2\setminus S_2 = S_1\setminus T_1$

So (w/o coeff) $\begin{cases} 1 & \text{if } S_1\setminus T_1 = T_2\setminus S_2,\ S_1\subset T_1,\ T_2\subset S_2 \\ 0 & \text{else} \end{cases}$

LHS count: letting $X = S_1\setminus T_1 = T_2\setminus S_2$ any division $X = Y_1\sqcup Y_2$ gives a valid diagram (+ determines Z_1, Z_2)

so get $\begin{cases} 0 & \text{else} \\ \binom{6}{2} & \text{if} \end{cases}$ (need to check some signs)

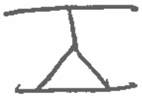
This is like a "state model" evaluation of webs.

10:50

go slower

Def: Let Webs_n^+ be the monoidal $\underline{\mathbb{Z}\text{-linear}}$ $\overset{\otimes}{\text{cat}}$ w/ presentation:

Ob: gen by $1 \le i \le n$, so $\mathrm{Ob} = \{\underline{i} = i_1 i_2 \ldots i_d\}$

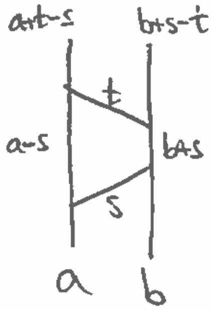
Mor: gen by  and 

Relns: Assoc, Coassoc, Bigon, (general relns for $\otimes$ presentations), and Square Flop

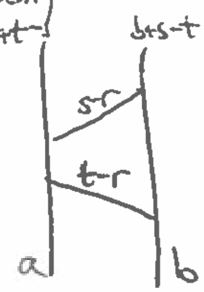
i.e. $\mathrm{Hom}(\underline{i}, \underline{j})$ = liner combo of diagrams w/ bottom $\underline{i}$, top $\underline{j}$.

Modulo relations applied locally to subdiagrams.

Square Flop:



$$= \sum_{r \ge 0} \begin{bmatrix} a+t-(b+s) \\ r \end{bmatrix}$$



Convention: some of these diagrams will be zero!

- If $a+t-r > n$
- If $b+r-t < 0$
- If $s-r<0$, $t-r<0$

$\boxed{\text{finite sum}}$ $\boxed{r \le \min(s,t)}$

Def: let $\mathrm{eval}: \mathrm{Webs}_n^+ \to \mathrm{Fund}\, gl_n$ be the obvious functor. (Check: Square flop)

Thm (CKM): $\mathrm{eval}_{\mathbb{C}}: \mathrm{Webs} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{Fund}$ is an equiv of cats.

i.e. • webs give all the maps, and • we found all the relations.

(We'll really unravel this theorem, just wait.)

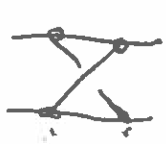
Rmks: ① This story has a q-deformation: $\mathrm{Webs}^+_{n,q}$ is $\mathbb{Z}[q,q^{-1}]$-linear

$\binom{a}{b}$ replaced by $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}[q,q^{-1}]$ quantum number? (look it up) $\boxed{\text{fix eval } \sout{\text{an exercise}}}$

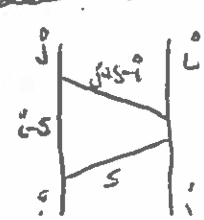
CKM: $\mathrm{eval}_{\mathbb{Q}(q)}: \mathrm{Webs}_n^+ \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to \mathrm{Fund}\, U_q(gl_n)$ is equiv of cat.

② My work gave meaning to Webs before specializing to $\mathbb{C}$ or $\mathbb{Q}(q)$, i.e. the integral form.

③ $L_i \otimes L_j \cong L_j \otimes L_i$ since symmetric (braided for $q \ne 1$) monoidal cat.



$$= \left(\sum_{s \ge 0} \right.$$



$$\left. (-q)^{j-s} \right) \left[(-1)^{ij} q^{-\frac{ij}{n}} \right]$$

(4) Dependence on n is minimal. $\exists$ Webs^+_∞ w/ $\mathrm{Webs}^+_\infty \to \mathrm{Webs}^+_n$ which sends $L_k \mapsto 0$ $\forall k>n$. Really most of what we do works in Webs^+_∞!

(5) Why $\mathrm{Webs}^{\oplus}$? Can also throw in L_k^* $\downarrow^k$ (draw L_k as $\uparrow_k$).

When working w/ $\mathfrak{sl}_n$, $L_k^* \cong L_{n-k}$ so have isom  (Note: relations have annoying sign issues depending on n!)

[Get Webs_n.] Webs^+ has all the complexity of behavior w/o a lot of technical nastiness.

11112

Clasps How to study L_λ using Webs? Not an object per se, but a direct summand of one!

S'pose $\lambda = \sum_i a_i \varpi_i$, $a_i \geq 0$. We'll write $P(\lambda)$ = "reduced expressions of λ"

$= \{\underline{i}$ with a_i copies of $\varpi_i\}$. Ex: $P(2\varpi_1+\varpi_2) = \{112, 121, 211\}$

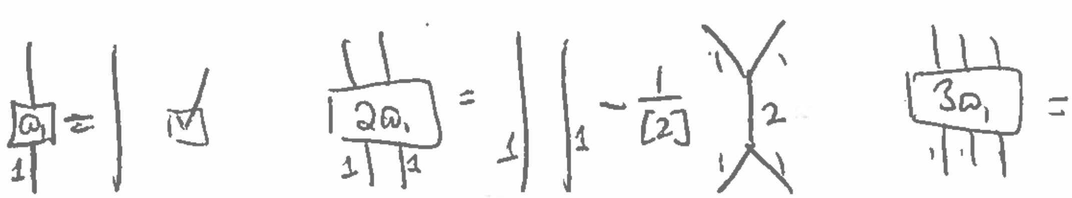
If $\underline{i} \in P(\lambda)$ then $L_\lambda \overset{\oplus}{\underset{1}{\subseteq}} L_{\underline{i}}$. Why? hw spaces. So $\exists$ canonical idempotent $e_{\underline{i}}$ (or e_λ if $\underline{i}$ understood) $\in \mathrm{End}(L_{\underline{i}})$

b/c mult. 1

projecting to L_λ. We call $e_{\underline{i}}$ the clasp. Also call it the "top idempotent" - related to the fact (important soon!) that all other summands $L_\mu \overset{\oplus}{\subseteq} L_{\underline{i}}$ satisfy $\mu < \lambda$ (and all other weights) in the dominance order.

Again, L_λ is not an object, but the idempotent gives "all" the info: $\mathrm{Hom}(L_\lambda,-) = \mathrm{Hom}(L_{\underline{i}},-)e_{\underline{i}}$ ("all" in the abstract sense but maybe not explicit enough.) in a minute

Ex: $\mathfrak{gl}_2$ clasps = Jones-Wenzl projectors



5 terms w/ coeffs $1, \frac{1}{[3]}, \frac{[2]}{[3]}$

(+6th term for $\mathfrak{gl}_n, n>2$)

Rmk: clasps live in $\mathrm{Webs} \otimes \mathbb{Q}(q)$, not in integral form. In specialization where $[2]=0$, $L_{2\varpi_1}$ may not exist + decomp. may change. Shouldn't even bother inverting $[2]$...

Why correct? Explanation 1: $L_1 \otimes L_1 \cong L_2 \oplus L_{2\varpi_1}$, L_2 is in our diagrammatics.

$p =$ up to scalar; can rescale jointly but require $pi = id_{L_2}$

$i =$

$pi = \text{scalar} \cdot \; = \text{scalar} \cdot [2] \,\big|_2$ so scalar $= \frac{1}{[2]}$

The idempotent projecting to L_2 is then $ip\,\text{scalar} = \frac{1}{[2]}$

The clasp is the complementary idempotent.

Explanation 2: (Same idea) Assume $End(L_1 \otimes L_1)$ is spanned by $||$, for now.

So $e_{2\varpi_1} = a\,|| + b$. But $Hom(L_{2\varpi_1}, L_2) = 0$ so $= 0$

$0 = a + [2]b$ so $a = -[2]b$

Also, $e^2 = a + b = ae$ so $a = 1$.

11:25

$\boxed{\text{The key was the decomposition } L_1 \otimes L_1 = L_2 \oplus L_{2\varpi_1}}$

Clasps are best computed using a double recursive formula, see lecture 3, Exercise.

PROBABLE END OF LECTURE 1.

Back to new p1

Lecture 2 start | Want to understand shadows of semisimplicity, ③

Work backwards – assume it works, see how things turn out to be. This lecture works from perspective of semisimplicity – we reason it next time. I.e. Rep gl_n not $\mathrm{Fund}\, gl_n$.

Recall: If $X = \bigoplus L_\lambda^{\oplus x_\lambda}$ $Y = \bigoplus L_\lambda^{\oplus y_\lambda}$ then $\mathrm{Hom}(X,Y) \overset{\text{canon}}{=} \bigoplus_\lambda \mathrm{Hom}_\lambda(X,Y)$

$\underset{\text{non-canon}}{\cong} \bigoplus_\lambda \mathrm{Mat}_{y_\lambda \times x_\lambda}(\mathbb{C})$ and $\dim \mathrm{Hom}_\lambda = x_\lambda \cdot y_\lambda$

|| canon

$\bigoplus_\lambda \mathrm{Hom}(L_\lambda, Y) \otimes \mathrm{Hom}(X, L_\lambda)$

To get isom, choose basis of $\mathrm{Hom}(X, L_\lambda)$ ← choice.

Qn: When $X = L_{\underline{i}}$, how to choose a basis?

Key Idea: $\exists!$ basis "adapted to $\otimes$ structure", its "canonical"!

To p8, Plethysm, then p13.

How big is $\mathrm{Hom}_{\mathbb{C}}(L_{\underline{i}}, L_{\underline{j}})$? ← in the semisimple setting!

Recall: If $X = \bigoplus L_\lambda^{\oplus x_\lambda}$ $Y = \bigoplus L_\lambda^{\oplus y_\lambda}$ then $\mathrm{Hom} = \bigoplus \mathrm{Hom}_\lambda$ ← maps factoring thru L_λ

and $\dim \mathrm{Hom}_\lambda(X,Y) = x_\lambda \cdot y_\lambda$: choose a projection $X \twoheadrightarrow L_\lambda$ and an inclusion $L_\lambda \hookrightarrow Y$.

How do we find a basis for the projection space $\mathrm{Hom}(X, L_\lambda)$ when $X = L_{\underline{i}}$?

Plethysm — a greek word for mult / (the art of) decomposing tensor products. Let's ask: what is $L_\lambda \otimes L_{\underline{i}}$?

gl_n weights: $\Lambda_{wt} = \mathbb{Z}^n \ni \lambda = (b_1, b_2, \dots, b_n)$. wts(rep) is an S_n-invt multiset in Λ_{wt}.

λ is polynomial if $b_i \geq 0\ \forall i$. Note: det weight is $(1,1,\dots,1) \leftarrow \mathrm{wts}(L_n)$

$\otimes$-ing w/ $\det^{\otimes k}$ will make any rep purely polynomial — may as well restrict to them.

Poly Rep gl_n

λ is dominant if $b_1 \geq b_2 \geq \dots \geq b_n$. Irreps $\Leftrightarrow$ Dom wts Λ^+_{wt}

$L_\lambda \longleftarrow \lambda$ (h.w.)

Ex: $\mathrm{wts}(V) = \left\{ \begin{array}{l} (10000) \leftarrow e_1 \text{ hw} \\ (01000) \leftarrow e_2 \\ \vdots \end{array} \right\}$ $\mathrm{wts}(\Lambda^2 V) = \left\{ \begin{array}{l} (11000) \leftarrow e_1 \wedge e_2 \text{ hw} \\ (10100) \leftarrow e_1 \wedge e_3 \\ \vdots \end{array} \right\}$

all multiplicity one! Key feature of type A!

Let $\varpi_k = (\underbrace{1111}_{k}000)$. Then $\Lambda^k V = L_{\varpi_k}$. If $\lambda = \sum a_i \varpi_i$

then $a_i = b_i - b_{i+1}$ $(b_{n+1} = 0)$ and λ dominant $\Leftrightarrow a_i \geq 0$ $\forall\, 1 \leq i \leq n-1$

λ dom't poly $\Leftrightarrow a_i \geq 0\ \forall 1 \leq i \leq n$.

$L_\lambda \overset{\oplus}{\subset} L_{\underline{i}}$ when $\underline{i} \in P(\lambda)$.

General $\otimes$ rule: $L_\lambda \otimes L_\mu = \bigoplus_{\nu \in \mathrm{wts}(L_\mu)}$ "$L_{\lambda+\nu}$"

If $\lambda+\nu$ dominant, $L_{\lambda+\nu}$ makes sense. If $\lambda+\nu$ not dominant, need to interpret correctly.

~~Really "$L_{\lambda+\nu}$" ...~~

I won't explain the rule entirely, but when $\lambda+\nu$ is on a wall, i.e.

$\lambda+\nu = \sum a_i \varpi_i$ with some $a_i = -1$, then "$L_{\lambda+\nu}$" $= 0$.

So what is $L_\lambda \otimes L_{\varpi_k}$?

Recall $a_i = b_i - b_{i+1}$

Ex: $k=3$ $(n\geq 7)$

$$\mathrm{wts}(L_{\varpi_k}) = \left\{\begin{array}{l} \overset{b_1 b_2 \cdots}{(11100\ldots)} \\ (11010\ldots) \\ (11001\ldots) \\ \vdots \\ (10110\ldots) \\ \vdots \\ (00101010\ldots) \end{array}\right\} = \left\{\begin{array}{l} \varpi_3 \\ \varpi_2 - \varpi_3 + \varpi_4 \\ \varpi_2 - \varpi_4 + \varpi_5 \\ \\ \varpi_1 - \varpi_2 + \varpi_4 \\ \\ \boxed{\varpi_0} - \varpi_2 + \varpi_3 - \varpi_4 + \varpi_5 - \varpi_6 + \varpi_7 \end{array}\right\}$$

zero, this helps.

Coeffs all in $\{-1,0,1\}$ so if $\lambda = \sum a_i \varpi_i$ $a_i \geq 0$ then

$\lambda + \nu = \sum a_i' \varpi_i$ $a_i' \geq -1$. $\lambda+\nu$ is either dominant or on the wall.

$\Longrightarrow$ Plethysm Rule: $L_\lambda \otimes L_{\varpi_k} \cong \bigoplus_{\substack{\nu \in \mathrm{wts}(L_{\varpi_k}) \\ \lambda+\nu \text{ dominant}}} L_{\lambda+\nu}$.

Ex: Can choose $\nu = (11010\ldots)$ if $a_3 \geq 1$

$\nu = (0010101)$ if $a_2, a_4, a_6 \geq 1$

If λ large enough, get all $\binom{n}{k}$ weights.

Def: Let $\underline{i} = (i_1, \ldots, i_d)$ a seq. of fund wts. A plethysm path for $\underline{i}$ is a seq $T = (\lambda_1, \ldots, \lambda_d)$ of dom. wts s.t. $L_{\lambda_j} \overset{\oplus}{\subset} L_{\lambda_{j-1}} \otimes L_{i_j}$ $(\lambda_0 = 0)$

i.e. $L_{\underline{i}} = \underbrace{L_{i_1}}_{\substack{\| \\ L_{\lambda_1}}} \otimes L_{i_2} \otimes L_{i_3} \otimes \cdots$

$\underbrace{\qquad\qquad}_{\cup\oplus}$

$L_{\lambda_2} \otimes L_{i_3} \otimes L_{i_4} \otimes \cdots$

$\cup\oplus$

$L_{\lambda_3} \otimes L_{i_4} \otimes \cdots$

$\ddots$ $\cup\oplus$

$E(\underline{i}, \lambda) = \{\text{plethysm paths ending in } \lambda_d = \lambda\}$

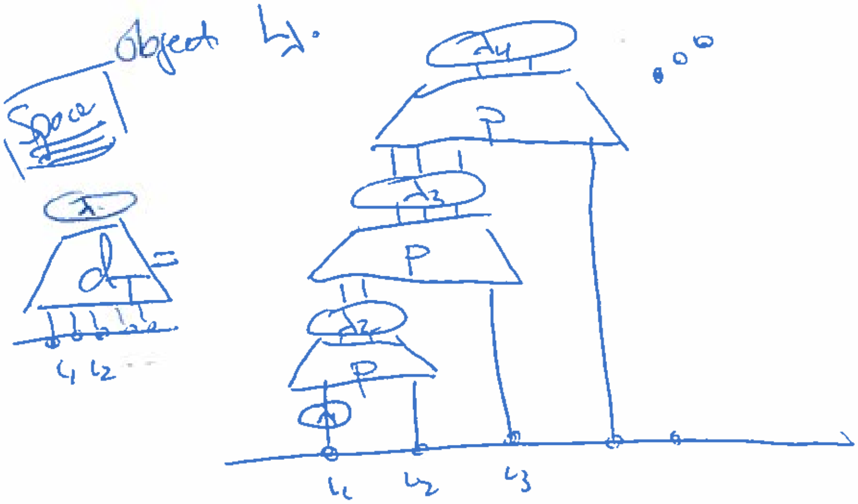
(also called ~~Minuscule~~ Littelmann paths)

Then $L_{\underline{\iota}} = \bigoplus L_\lambda^{\oplus n_\lambda}$ where $n_\lambda = \# E(\underline{\iota}, \lambda)$.

Ex: $\underline{\iota} = (2,2,1)$ Plethysm paths are

$$\begin{array}{l} 0 \rightsquigarrow \varpi_2 \rightsquigarrow 2\varpi_2 = \varpi_2 + (1100) \rightsquigarrow 2\varpi_2 + \varpi_1 \quad \underline{\text{top summand}} \\ \varpi_2 \rightsquigarrow \varpi_1 + \varpi_3 = \varpi_2 + (1010) \rightsquigarrow \varpi_2 + \varpi_3 \leftarrow \text{multiplicity } 2 \\ \varpi_2 \rightsquigarrow \varpi_4 = \varpi_2 + (0011) \rightsquigarrow \varpi_1 + \varpi_4, \ \varpi_5 \\ \text{also } 2\varpi_1 + \varpi_3 \end{array}$$

For each $T \in E(\underline{\iota}, \lambda)$ can choose projection $d_T : L_{\underline{\iota}} \twoheadrightarrow L_\lambda$ compatible w/ path, ! up to scalar. Here's the diagrammatic schematic. As in preface, we're thinking semisimply — use clasps freely, or write $\boxed{\lambda}$ just to mean the object L_λ.



To build it need to know $p : L_\lambda \otimes L_k \twoheadrightarrow L_{\lambda+\nu}$ for each potential summand

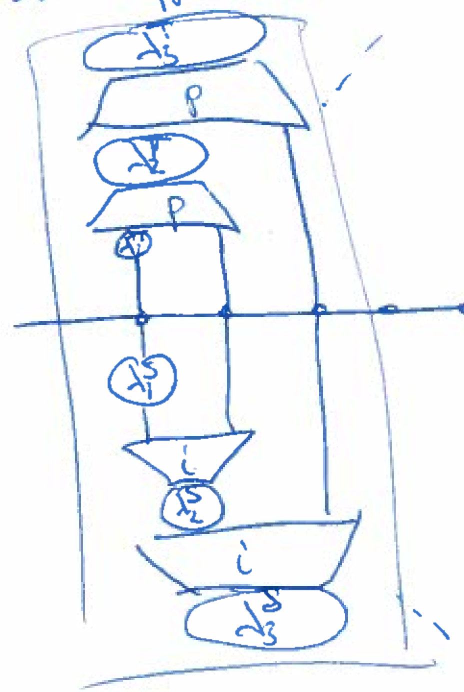
Call d_T the clasped light ladder. It is unique up to scalar b/c each p is.

Rmk: This idea works in any $\otimes$ ss cat, but it is greatly simplified by the fact that plethysm w/ fundamentals is multiplicity-free so p is ! up to scalar. Otherwise must choose basis for projections, do linear algebra ← yuck!

Can do the same for inclusions too. For reasons of sanity, let $M(\lambda, \underline{\iota}) = E(\underline{\iota}, \lambda)$ but parametrizing inclusions. Instead let $\mathbb{D}$ be the bijection b/w sets.

For $S \in M(\lambda, \underline{\iota})$ define d_S w/ the same picture upside-down, w/

What happens when we compose $d_T \circ d_S$?



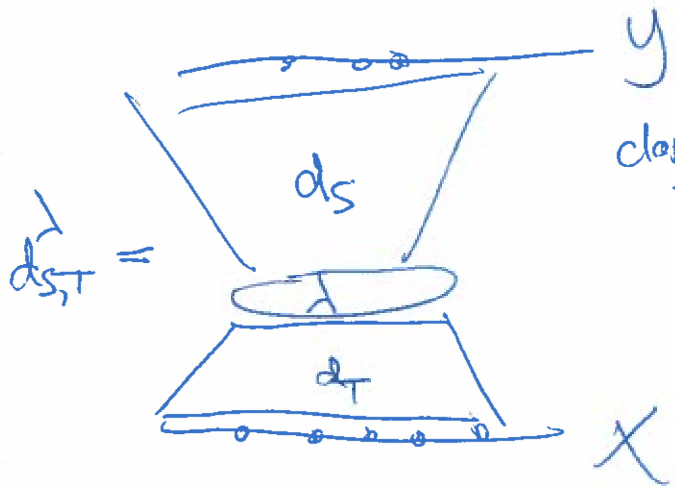
If $\lambda_3^S \neq \lambda_3^T$ then morphism in box is zero since $\mathrm{Hom}(L_{\lambda_3^S}, L_{\lambda_3^T}) = 0$.

$\Rightarrow d_T \circ d_S = 0$ if $D(S) \neq T$.

If $D(S) = T$, $d_T \circ d_S = id_{L_\lambda}$.

Then $\{d_S\}$ $\{d_T\}$ are dual bases for $\mathrm{Hom}(X, L_\lambda)$ and $\mathrm{Hom}(L_\lambda, X)$ when $X = L_{\underline{i}}$!

$\mathrm{Hom}(X,Y)$ has basis $d^\lambda_{S,T}$ for $S \in M(\lambda, Y)$ $T \in E(X, \lambda)$



clasped double ladder.

Moreover, $\underset{E(Y,\mu)}{\underset{\cap}{d^\mu_{U,V}}} \; \underset{M(\lambda,Y)}{\underset{\cap}{d^\lambda_{S,T}}} = \begin{cases} 0 & \text{if } \lambda \neq \mu \text{ or } S \neq D(V) \\ d^\lambda_{U,T} & \text{if } \lambda = \mu, \; S = D(V) \end{cases}$

It's our matrix basis for $\mathrm{Hom}_\lambda(X,Y)$!!

This is completely abstract nonsense. Does it help? You bet, stay tuned.

Still need to be specific about $p: L_\lambda \otimes L_k \to L_{\lambda+\nu}$ $\nu \in \mathrm{wts}(L_k)$.

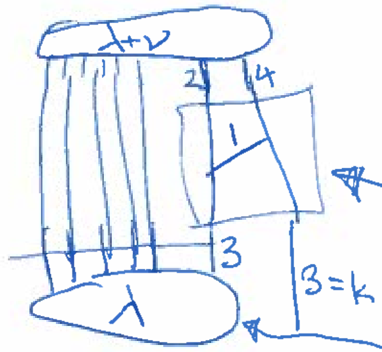
Details specific to setting finally.

Thm (E): p can be described as an <u>elementary light ladder</u> ELL

lose this from λ (which has at least one ϖ_3 or $\lambda+\nu$ not dominant)

<u>Ex 1:</u> $k=3$ $\nu = (11010\ldots) = \varpi_2 - \varpi_3 + \varpi_4$

gain this.



(3) $\underline{j}$

(4) $j \in P(\lambda+\nu)$ ending in 24

(5) WLOG $\underline{i} \setminus 3 = \underline{j} \setminus 24$ so just use identity map

(6) elementary light ladder, first appears in minimal version of plethysm, when $\lambda = \varpi_3$ itself!

$\underline{i}$

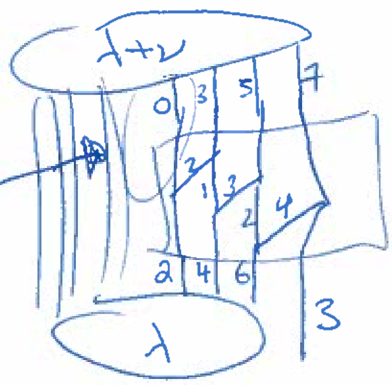
$3 = k$

(1) some $\underline{i} \in P(\lambda)$ ending in 3 $\quad e_L$ (2)

or maybe just the inclusion.

(~~plethysm patterns for~~ ~~...~~ really matter!)

<u>Ex 2:</u> $k=3$ $\nu = (0010101) = \boxed{\varpi_0} - \varpi_2 + \varpi_3 - \varpi_4 + \varpi_5 - \varpi_6 + \varpi_7$



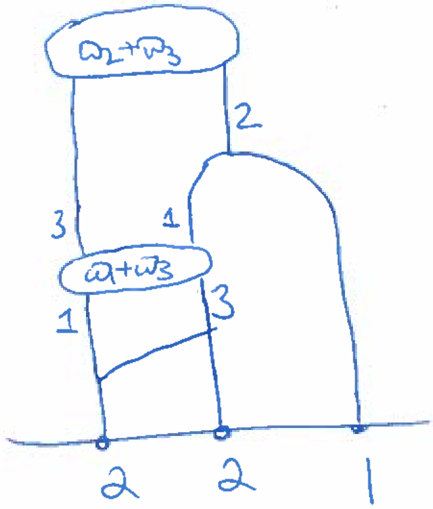
make it dotted

this kind of web, built from rungs $\vdash\!\dashv$ or $\dashv\!\vdash$ is called a <u>ladder</u>

the <u>elementary light ladder</u> E_ν, first appears when $\lambda = \varpi_2 + \varpi_4 + \varpi_6$

Big idea: The form of p does not depend on λ, only on the minimal λ where plethysm appears – ∞-ly many $L_\lambda \otimes L_k \to L_{\lambda+\nu}$ but finitely many <u>plethysm patterns</u>, one for each $\nu \in \mathrm{wts}(L_k)$

Ex: $\underline{i}=(2,2,1)$ $\quad T=(\omega_2,\ \underbrace{\omega_1+\omega_3}_{=\,\omega_2+(\omega_1-\omega_2+\omega_3)},\ \overbrace{\omega_2+\omega_3}^{=\,\text{prev}+(\omega_2-\omega_1)})$

$d_T =$ 

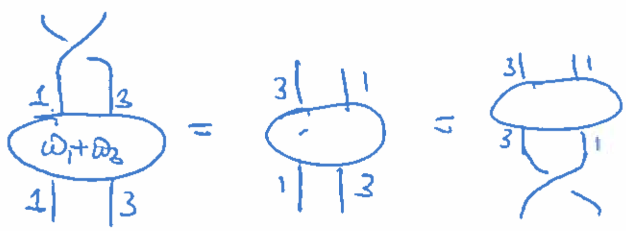
Rmk: To define this need to know

← not an idempotent e_{31} or e_{13}

but $L_1\otimes L_3 \twoheadrightarrow L_{\omega_1+\omega_3} \hookrightarrow L_3\otimes L_1$

Still called the clasp

need to compute these too.

Next key idea: 

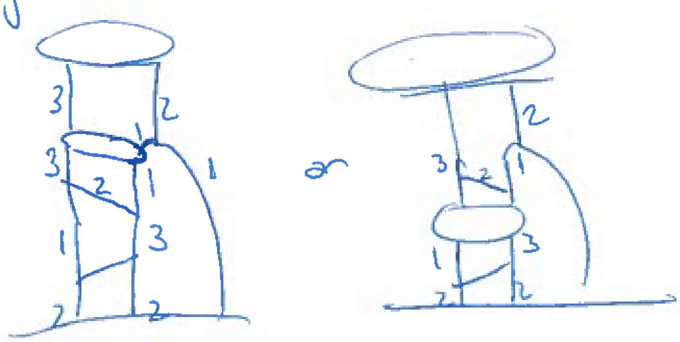
Makes sense. But braiding is crazy + complicated sum

Thankfully, 

b/c other terms in sum are orthogonal to the clasp.

Neutral ladder

Only need to compute one idempotent $e_{\underline{i}}$ for $\underline{i}\in P(\lambda)$ and get the rest by applying neutral ladders to reorder indices

$d_T =$ 

So modulo clasp-finding, have completely explicit basis for Hom spaces!!

<u>Factorization Philosophy</u>: Let $\mathcal{C}$ be s.s. cat, $\mathrm{Irr}\,\mathcal{C} = \Lambda$, satisfying Schur's lemma. (14)

$$L_\lambda \longleftrightarrow \lambda$$

$$X = \bigoplus L_\lambda^{\oplus x_\lambda} \qquad Y = \bigoplus L_\lambda^{\oplus y_\lambda} \qquad \mathrm{Hom}(X,Y) = \bigoplus \mathrm{Hom}_\lambda(X,Y)$$

where $\mathrm{Hom}_\lambda(X,Y) = \mathrm{Hom}(L_\lambda, Y) \otimes \mathrm{Hom}(X, L_\lambda)$ $\quad \dim = x_\lambda \cdot y_\lambda$.

$$g \circ f \longleftarrow g \quad f$$

a morphism which truly factors thru L_λ.

<u>Note</u>: Not everything in Hom_λ truly factors, just as not every element of a tensor product is a pure tensor. By <u>factory thru L_λ</u>, we mean morphisms in the span of truly factory morphisms.

<u>Rmk</u>: For any object Z, $I_Z = \{\text{morphisms factory thru } Z\}$ is an ideal in $\mathcal{C}$, i.e. closed under pre- and post-composition w/ any morphism. $\quad I_{M \oplus N} = I_M + I_N$

What if (as in Fund/Webs) the irrep L_λ is inaccessible. ← As it shall be, remember lecture 1. Instead assume

↳ i.e. not in subcategory.

<u>Object ~~unitriangularity~~ triangularity</u>: $\exists$ poset structure on Λ, and accessible objects B_λ

s.t. $B_\lambda \cong L_\lambda \oplus \bigoplus_{\mu < \lambda} L_\mu^{\oplus \bullet}$. Ex: Choose some $\mathrm{DSP}(\lambda)$, set $B_\lambda = L_{\underline{w}}$.

No immediate relationship between I_{B_λ} and I_{L_λ}, but morphisms factory thru

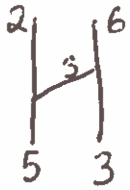
B_μ for $\mu \le \lambda$ $\iff$ factors thru L_μ for $\mu \le \lambda$. I.e.

$$I_{\le \lambda} = I_{\bigoplus_{\mu \le \lambda} B_\mu} = I_{\bigoplus_{\mu \le \lambda} L_\mu}.$$

One still has a <u>filtration</u> of $\mathcal{C}$ by ideals, not a splitting $\mathrm{Hom}(X,Y) = \bigoplus \mathrm{Hom}_\lambda(X,Y)$. We can recover Hom_λ $\mathrm{Hom}_{\le \lambda}(X,Y)$ only in the associated graded. When λ is fixed, $I_{<\lambda}$ is called <u>lower terms</u>.

$$\mathrm{Hom}(X, B_\lambda) / I_{<\lambda} \text{ is a model for } \mathrm{Hom}(X, L_\lambda) \quad \text{(they're isom v.s.)}$$

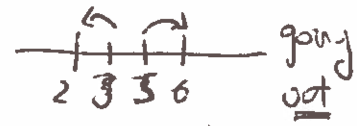
Ex: $\lambda = \sum_i a_i \varpi_i$ then if $i \in P(\lambda)$, $L_\lambda \overset{\oplus}{\subset}_1 L_{\underline{i}}$ and all other summands μ are $\lhd \lambda$ in the dominance order. How to remember dominance order?



$\varpi_5+\varpi_3 = (22211\,000)$ higher in dominance

$\downarrow$ subtract $(00100{-}100)$

$\varpi_6+\varpi_2 = (221111\,00)$ lower in dominance



going out is lower.

lowest.

$B_{\varpi_5+\varpi_3} = L_5 \otimes L_3 = L_{\varpi_5+\varpi_3} \oplus L_{\varpi_6+\varpi_2} \oplus L_{\varpi_7+\varpi_1} \oplus L_{\varpi_8}$

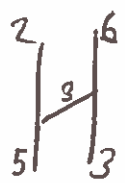
$B_{\varpi_6+\varpi_2} = L_6 \otimes L_2 = L_{\varpi_6+\varpi_2} \oplus L_{\varpi_7+\varpi_1} \oplus L_{\varpi_8}$.

~~CRUCIAL ARROW HERE~~

Back to describing $Hom(X,Y)$. $\bigoplus Hom(B_\lambda, Y) \otimes Hom(X, B_\lambda)$ is too big.

Spose you have a map $E(X,\lambda) \hookrightarrow Hom(X, B_\lambda)$ s.t. $\{G_T\}$ descends
$T \longmapsto G_T$
to a basis of $Hom(X,B_\lambda)/I_{\lhd\lambda}$, Similarly $M(\lambda, Y) \hookrightarrow Hom(B_\lambda, Y)$.
$S \longmapsto C_S$

Then $C^\lambda_{S,T} := C_S \circ G_T \in Hom(X,Y)$, $\{C^\lambda_{S,T}\}_{S \in M(\lambda,Y),\, T \in E(X,\lambda)}$ descends to a basis of $Hom_\lambda(X,Y)$ modulo $I_{\lhd\lambda}$. So $\{C^\lambda_{S,T}\}_{\lambda,S,T}$ is a basis for $Hom(X,Y)$!

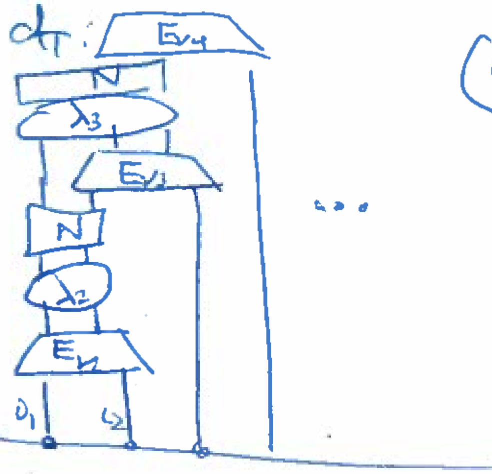
Ex:  is a complicated map, having nontrivial coeffs in all 3 common summands.

BUT it is nonzero modulo $I_{<\varpi_2+\varpi_6}$ so it descends to a basis of $Hom(53,26)/I_{<\varpi_2+\varpi_6} \cong Hom(L_5 \otimes L_3, L_{\varpi_6+\varpi_2})$

~~How to find the sets E, M and the morphisms C_S, G_T? This uses the monoidal structure, plethysm paths. Another module.~~

So who needs clasps?? To new pdf

In our example, how to construct G_T. Recall d_T:



light ladders

where N neutral

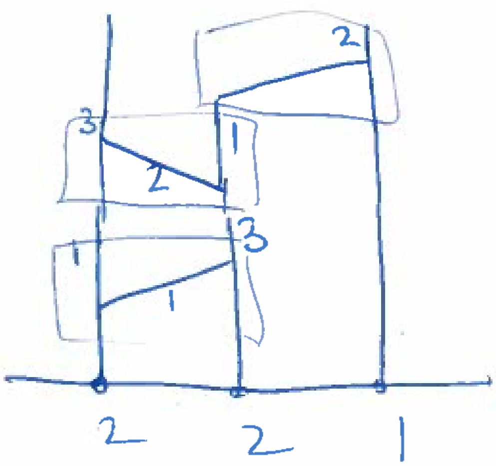
λ clasp

$E_\downarrow$ elementary light ladder.

What happens when we leave out the clasps?

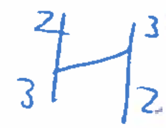
Ex: $\underline{i} = 221$ $\quad T = (\varpi_2, \varpi_1+\varpi_3, \varpi_2+\varpi_3)$

$G_T =$



get a map $L_{221} \longrightarrow L_{32}$

(if $B_{\varpi_2+\varpi_3} = L_{23}$, apply one more neutral ladder



)

Using that $L_{\underline{i}} = L_\lambda \oplus \left(\bigoplus_{\mu<\lambda} L_\mu\right)$ for $\underline{i} \in P(\lambda)$, deduce that

$$G_T \equiv d_T + \sum_{S<T} \text{coeff}\cdot d_S + I_{<\lambda}$$

$\leftarrow$ path dominance order, $\lambda_j^S \leq \lambda_j^T \iff S \leq T$.

So $\{G_T\}$ descends to a basis. (not quite $\{d_T\}$ but unitriangular c.o.b.)

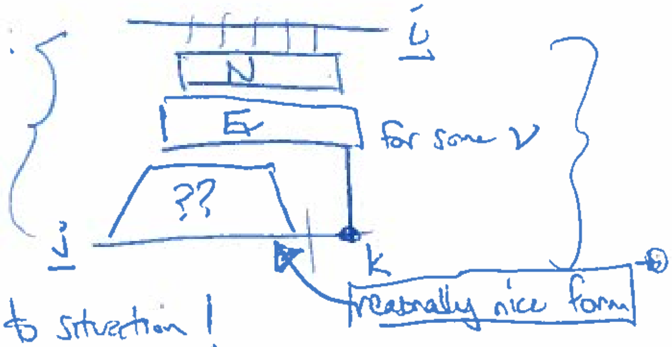
Hooray, completely explicit basis. $\{C^\lambda_{S,T}\}$ double ladders

Thm (E) Double ladders form a basis for Webs^+ over $\mathbb{Z}$!!!

On the proof: Suppose you've defined $\{c^\lambda_{S,T}\}$ but don't know it spans. Can let $J_{<\lambda} = \mathrm{Span}\{c^\mu_{ST}\}_{\mu<\lambda}$

Prop: For $\underline{i} \in P(\lambda)$, $\mathrm{Hom}(L_{\underline{\lambda}}, L_{\underline{i}})$ is spanned by the following modulo $J_{<\lambda}$:

(or really $I_{<\lambda}$!!)



Pf of prop is very specific to situation!

But pf of thm from prop is abstract inductive argument. Pf of lin. indep. of c^λ_{ST} also abstract using upper tri to d^λ_{ST}.

Consequences: When is your presentation correct? Need:

- Enough generating morphisms to construct:
  - projections for each plethysm pattern (eg. elem LL)
  - Neutral maps b/w any two red exp for λ (eg neutral ladders) ⊛
- Enough relations to prove:
  - Prop above
  - neutral ladders are isom modulo $J_{<\lambda}$
  - ⊛ one more result saying morphisms are somewhat manageable — "Morse theory" don't get too "wide"

Then can repeat proof + win!

They go down in B.O. on Λ, then back up, no need to get higher!!

To p16

Crucial Aside: Why put L_λ off-limits? We're trying to describe by gens + rels. Doing so automatically produces an integral form of the category — a category linear over the smallest subring $\mathbb{K}$ of $\mathbb{C}$ (or $\mathbb{Q}(q)$) where the coeffs of the relations reside. Ex: Webs is defined over $\mathbb{Z}$ / $\mathbb{Z}[q,q^{-1}]$.

Categories have many integral forms, over different rings sometimes, and it is nice to find one with independent meaning, but that's another story.

Why not have $L_\lambda \in Ob$? projection $L_\lambda \otimes L_\mu \to L_\chi$ in gen. morphisms? B/c then relations will be nasty! If you force semisimple behavior then you NEED $\mathbb{K}$ very large ($\supset \mathbb{Q}$) because Rep gl_n is different in finite characteristic. Good luck finding that relation with $\frac{37}{101}$ ~~[illegible]~~ a coefficient. ← NOT FINITELY PRESENTED!! Suggests intractability.

Instead you want a minimal presentation w/ simple relations + most room for specialization, so you can use it to study char p!! That's better, but it places L_λ off-limits.

↳ which isn't so bad, as we'll see.

What structure does the basis $\{c^\lambda_{S,T}\}$ have?

Let $a \in \mathrm{Hom}(Y,Z)$. $Z \xleftarrow{a} Y \xleftarrow{c_S} B_\lambda \xleftarrow{c_T} X$. Modulo $I_{<\lambda}$, ac_S is a linear combo of c_U, $U \in M(\lambda, Z)$

$$ac_S \equiv \sum_U l(a,S,U)\, c_U .$$

$\equiv$ ← means "modulo $I_{<\lambda}$"

Then $\circledast$ $$a c^\lambda_{S,T} \equiv \sum l(a,S,U)\, c^\lambda_{U,T} \quad \text{modulo } I_{<\lambda}$$

these coeffs are independent of T! this stuff isn't.

$\circledast$ is called the "cellular formula" and it implies that $\{c^\lambda_{S,T}\}$ is a cellular basis.

[Modulo one other property...]

Similarly, $c^\lambda_{S,T}\, a \equiv \sum c_{S,V}\, r(a,T,V) + I_{<\lambda}$.

Cool implication: $c^\lambda_{S,T} \circ c^\lambda_{U,V} \equiv \sum_{\substack{\uparrow \\ \text{indep of } V}} \circ\, c^\lambda_{-,V} \equiv \sum_{\substack{\uparrow \\ \text{indep of } S}} \circ\, c^\lambda_{S,-}$ so $\equiv \varphi(T,U)\, c^\lambda_{S,V}$

That is

$$Z \xleftarrow{S} B_\lambda \xleftarrow{T} Y \xleftarrow{U} B_\lambda \xleftarrow{V} X$$

There exists $\varphi(T,U)$ s.t.

$$C_T C_U = \varphi(T,U) \cdot id_{B_\lambda} + I_{<\lambda}$$

since $\mathrm{End}(B_\lambda)/I_{<\lambda}$ is spanned by id_{B_λ}.

Thus $\varphi_{Y,\lambda}(-,-) : E(Y,\lambda) \times M(\lambda, Y) \longrightarrow \mathbb{K}$ is called the cellular pairing (form) and its properties (e.g. rank, etc) really determine the structure of the category.

Exercise: Rank $\varphi_{Y,\lambda}$ = mult of L_λ in Y.

~~Rmk: There is one more feature in the literature on cellular algebras/category which is really inessential: an anti-autoequivalence i of $\mathcal{C}$ sending $C^\lambda_{S,T} \mapsto C^\lambda_{i(T),i(S)}$ via some identification of $M(\lambda,Y) \overset{i}{\leftrightarrow} E(Y,\lambda)$.~~

Rmk: There is one more feature in the standard defn of cellular categories which is really not essential. Suppose $\mathcal{C}$ has a contravariant autoequiv. $\mathbb{D}$, fixing objects.

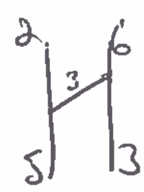
Since $\mathrm{Hom}(X, B_\lambda) \overset{\mathbb{D}}{\cong} \mathrm{Hom}(B_\lambda, X)$ expect $M(\lambda, X) \overset{\mathbb{D}}{\longleftrightarrow} E(X,\lambda)$, so let's assert that then $\mathbb{D}(C^\lambda_{S,T}) = C^\lambda_{\mathbb{D}(T),\mathbb{D}(S)}$.

For any ss cat can construct such iota NONCANONICALLY, by choosing bases for projections (i.e. the matrix description) and taking matrix transpose.

Sometimes iota comes naturally from some duality functor, as it does for Webs_n. I won't bother to explain why this

$\mathbb{D}$(diagram) = diagram flipped upside down

is natural.

Eg.  $= C_T$ for $T \in E(L_5 \otimes L_3,\ \varpi_2 + \varpi_6)$

 $= C_{\mathbb{D}(T)}$.

Altogether we get the following structure. Ref: Elias–Lauda

Def: An *object-adapted cellular category* is • A $\Bbbk$-linear category $\mathcal{C}$

- A poset Λ, and an object B_λ for each $\lambda \in \Lambda$.
- For each $X \in Ob(\mathcal{C})$, $\lambda \in \Lambda$, sets $M(\lambda, X) \xleftrightarrow{i} E(X, \lambda)$ in bijection
- Maps $c: M(\lambda, X) \to Hom(B_\lambda, X)$ $\quad E(X,\lambda) \to Hom(X, B_\lambda)$

  $S \mapsto c_S$

Now write $c^\lambda_{S,T} = c_S \circ c_T$ and define $I_{<\lambda}$ as the span of $c^\mu_{S,T}$ for $\mu < \lambda$, etcetera.

We require: ① $\{c^\lambda_{S,T}\}_{\substack{\lambda \in \Lambda \\ S \in M(\lambda, Y) \\ T \in E(X,\lambda)}}$ is a basis for $Hom(X,Y)$

② $i(c_S) = c_{i(S)}$ extends to an anti-auto-equiv

③ $M(\lambda, B_\lambda) = E(B_\lambda, \lambda) = *$ and $c_* = id$

④ $I_{\leq\lambda}$ is an ideal OR ⊛ holds OR ...

$\Rightarrow$ $I_{\leq \lambda}$ is the ideal of morphisms factoring thru B_λ.

Remarks:

- A unitriangular subcategory of a semisimple category is automatically an OACC.
- Integral forms may not be, but they are so long as ~~[illegible]~~ the maps c_S can be constructed integrally, and $c^\lambda_{S,T}$ form an integral basis.
- If you're trying to prove your diagrams are an integral form, find this structure first!!
- Get many things for free – Groth. group, trace decat, etcetera.
- Almost every cellular algebra is just an OACC in disguise. Factorization makes properties like ⊛ much more natural. (One "non-example" – $\mathbb{H}(S_n)$)

From the other module on plethysm paths, we see that it may be useful to consider not just one B_λ for each $\lambda \in \Lambda$, but a whole family of them!

Def: Multiobject adapted cellular category has • A set $P(\lambda)$ of objects for $\lambda \in \Lambda$

• A map C: which sends $S \in M(\lambda, X)$ to a map $B_\lambda \xrightarrow{C_S} X$ for some $B_\lambda \in \Lambda$ (different for each S)

• Neutral maps $B_\lambda^{(i)} \xrightarrow{{}_j\varphi_i} B_\lambda^{(j)}$ for each $B_\lambda^{(i)}, B_\lambda^{(j)} \in P(\lambda)$

such that ${}_k\varphi_j \, {}_j\varphi_i \equiv {}_k\varphi_i$ modulo $I_{<\lambda}$, and ${}_i\varphi_i = \mathrm{id}$

Now we set $C^\lambda_{S,T} = Y \xleftarrow{G} B_\lambda^{(2)} \xleftarrow{\varphi} B_\lambda^{(1)} \xleftarrow{C_S} X$.

This allows one the convenience of defining C_S to any object (simpler) rather than having to make an unnatural choice of reduced expression.

THAT'S A MODULE!

Clasps

Def: Fix $\lambda \in \Lambda$ w/ reduced expression $P(\lambda)$. The clasp, if it exists, is a family of morphisms ${}_j\varphi_i : i \to j$ for $i, j \in P(\lambda)$ satisfying

① ${}_k\varphi_j \circ {}_j\varphi_i = {}_k\varphi_k$

② ${}_i\varphi_i = id_i$ modulo $I_{<\lambda}$

③ ${}_j\varphi_i\, a = b\, {}_j\varphi_i = 0$ for any $a, b \in I_{<\lambda}$

(This way of formulating it obviates the need for discussing the object L_λ.)

Exercise: The clasp is unique if it exists.

Finding a closed formula for the clasp as a linear combo of webs seems out of reach. It was done by Morrison for gl_2 and it is complicated! But inductive formulas are philosophically important and practical too. Triple Clasp expansion –

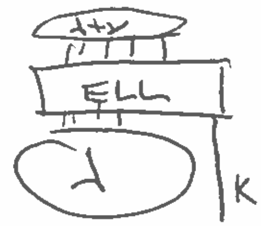
Suppose we have computed all the clasps less than $\lambda + \varpi_k$, including λ. 

ovals

We know $L_\lambda \otimes L_{\varpi_k} = \bigoplus L_{\lambda+\nu}$ for $\nu \in Wts(L_{\varpi_k})$ w/ $\lambda+\nu$ dominant.

One summand is $L_{\lambda+\varpi_k}$, we want this idempotent so we want to subtract off all the others!

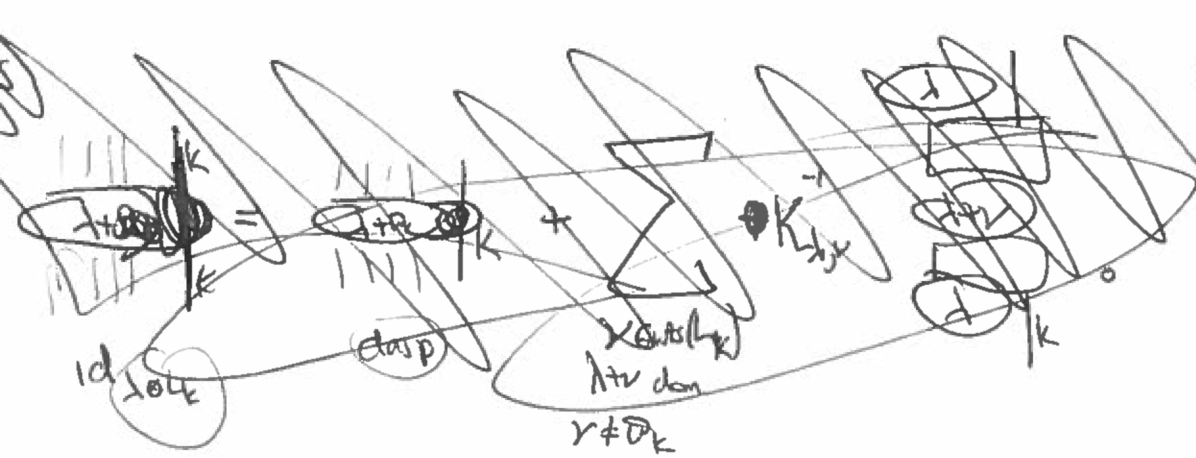
$\dim Hom(L_\lambda \otimes L_k, L_{\lambda+\nu}) = 1$ and we have a basis for it:



So the idempotent is $K^{-1}_{\lambda,\nu}$ 

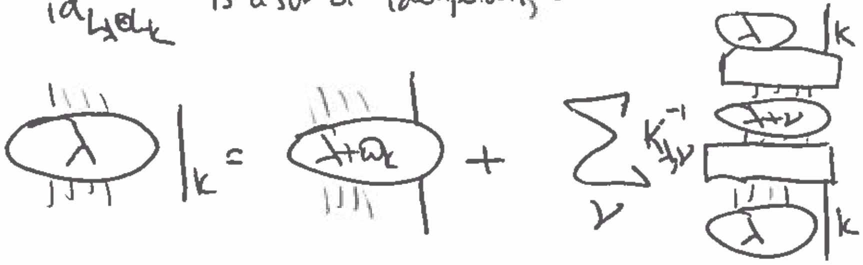
for some scalar $K^{-1}_{\lambda,\nu}$. ← reason for inverse coming soon

~~Thus~~



Great, completely formal argument

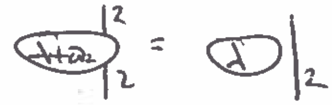
Thus $id_{L_\lambda \otimes L_k}$ is a sum of idempotents, or

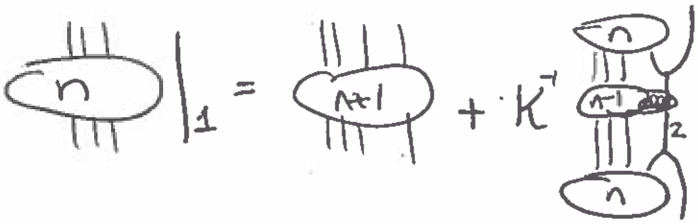


A formal argument gives this recursive formula!

Ex: gl_2, write (n) for $n\omega_1 + x\omega_2$ where x determined from context —

ω_2 is determinant rep and



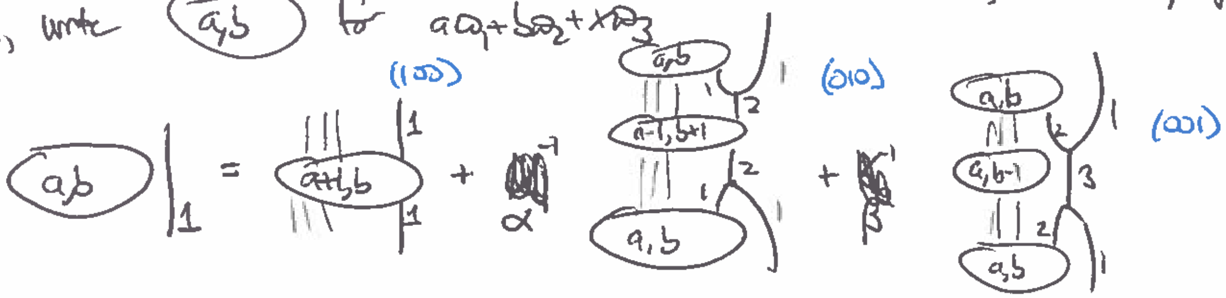


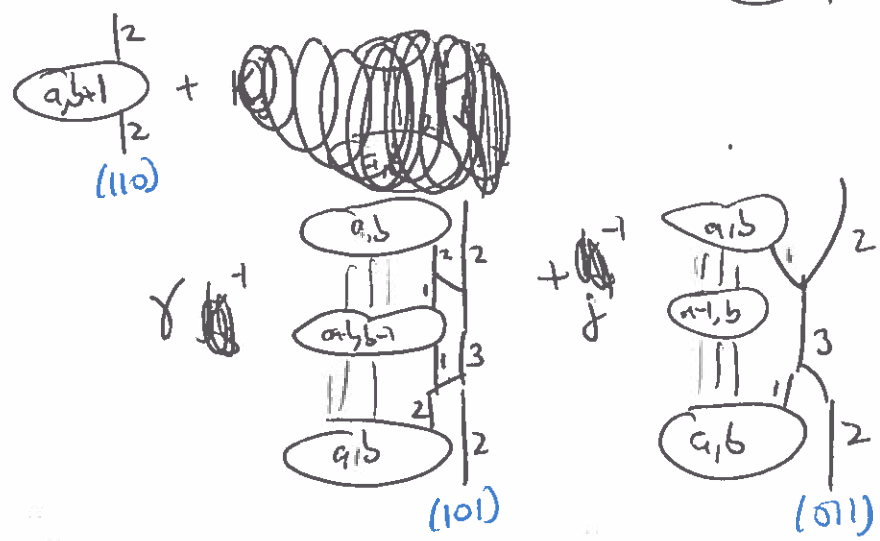
and $K^{-1} = \frac{[n]}{[n+1]}$

Note: 3 clasps have names.

term exists only when $n>0$! But when $n=0$ $K^{-1}=0$ anyway!

Ex: gl_3, write (a,b) for $a\omega_1+b\omega_2+x\omega_3$





$\alpha K^{-1} = \frac{[a]}{[a+1]}$

$\delta K^{-1} = \frac{[b]}{[b+1]}$

$\beta K^{-1} = \frac{[b][a+b+1]}{[b+1][a+b+2]}$

$\gamma K^{-1} = \frac{[a][a+b+1]}{[a+1][a+b+2]}$

How to find coeffs? ☺

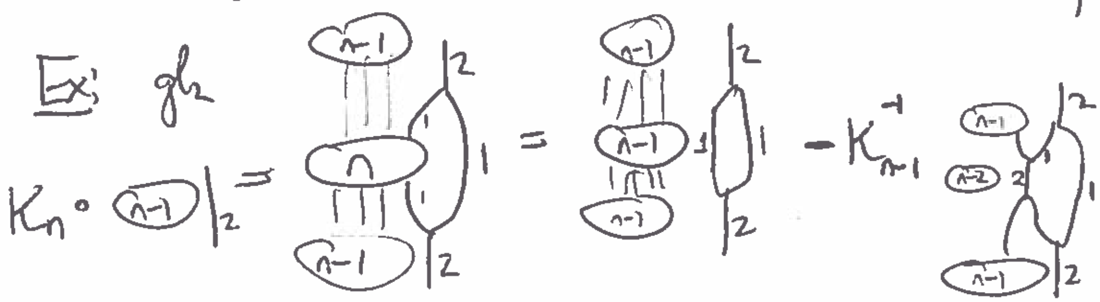
Well, it's supposed to be $ip = e$, with $pi = id_{L_{\lambda+\nu}}$. So want

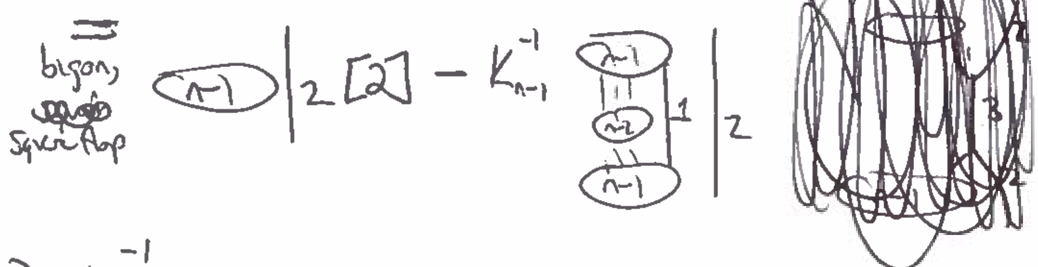
, i.e. $K_{\lambda,\nu}$ is coeff of id, i.e.

$K_{\lambda,\nu}$ = cellular pairing of E with itself (a 1×1 matrix) hooray for multiplicity freeness.

We can just compute this

<u>Ex</u>: gl_2





So $K_n = [2] - K_{n-1}^{-1}$.

As noted $K_0^{-1} = 0$ since this term does not exist. Can solve, get $K_n = \frac{[n+1]}{[n]}$

Using the formal recursion for clasp, get recursion for coeffs in that formal recursion.

<u>Ex</u>: gl_3 get $\alpha_{a,b} = [2] - \alpha_{a-1,b+1}^{-1}$ and $\alpha_{0,b}^{-1} = 0$

$\rightsquigarrow \alpha_{a,b} = \frac{[a+1]}{[a]}$

get $\beta_{a,b} = [3] - \frac{\alpha_{a+1,b-2}}{\gamma_{a,b-1}} - \frac{1}{\delta_{a,b-1}}$ and similar for δ...

quite complicated !! ${}^{\circ\circ}$

<u>Rmk</u>: (Reverse) dominance order on ν controls which coeffs can appear in recursive formula for which others.

Again, morally its clear what to do! Compute various $K_{\lambda,\nu}$ by finding recursion (C4)

relation + solving it. Both finding + solving take real work!! Life is much easier thanks

to simple plethysm rules

<u>Conjectural Solution:</u>

$$K_{\lambda,\nu} = \prod_{\substack{\text{pos roots } \alpha \\ \text{inverted in } \nu}} \frac{[\langle \lambda+\rho, \alpha\rangle]}{[\langle \lambda+\rho, \alpha\rangle - 1]}$$

$$\langle \lambda+\rho, \alpha\rangle - 1 = \langle \lambda+\nu+\rho, \alpha\rangle$$

<u>Ex1</u> $\nu = (0101100)$

then relevant α are $\varepsilon_1-\varepsilon_2,\ \varepsilon_1-\varepsilon_4,\ \varepsilon_1-\varepsilon_5,\ \varepsilon_3-\varepsilon_4,\ \varepsilon_3-\varepsilon_5$

so if $\lambda = a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n$ then $\lambda+\rho = (a_1+1)\omega_1 + (a_2+1)\omega_2 + \ldots$

$$K_{\lambda,\nu} = \frac{[a_1+1][a_1+a_2+a_3+3][a_1+a_2+a_3+a_4+4][a_3+1][a_3+a_4+2]}{[a_1][a_1+a_2+a_3+2][a_1+a_2+a_3+a_4+3][a_3][a_3+a_4+1]}$$

Please help me prove it / understand it!! Just guesswork!

Proven for $n \le 4$ and most ν for $n=5$