To Paul: ghn. The p17. Think back here.

So what do we conclude?

We're trying to describe a cat by {g,h} + rels. Does it automatically provide an integral form of the category—
on additive category over the smallest ring $R$ of $C$ (or $\text{alg}$) where the cleft of the
relations live. For $\mathbf{rel}$ is defined over $\mathbf{Z}$ or $\mathbf{Z}(\mathbf{Q})$?

Categories have many different integral forms, it's nice to find one with meaning so that when you
specialize to $\mathbb{F}_p$ rather than $C$ you get something interesting—another story.
NOT include all irreps by in your groups! ⇒ all chaps broadly become
non, need eg $\frac{1}{\sqrt{6}}$K. But Rep is different in any finite characterisation:
$\frac{1}{\sqrt{6}}$ is in some chaps ⇒ $K \rightarrow \mathbb{Q} \rightarrow$ need or-by any relation!!

Suggests intractibility: - inegengence - inefficiency - uselessness (+ no specialization)

Get some try until simple if you place $L$ off-l, int and restrict to $F_d$. Then use chains to recover info about $L$. Note: $\text{Hom}(L, \mathbb{Z}) = \text{Hom}(L, \mathbb{Z}) \mathbb{Z}$

Sends great but... if you had a basis for $\mathbb{Z}$, still don't get a basis for $\text{Hom}(L, \mathbb{Z})$

Get cyclic relations. Also, I int relate $\mathbb{Z}$ to maps of $\mathbb{Z}$ into $\mathbb{Z}$ which are categories of chaps but not easy one. $\mathbb{Z}$ + Chaps = Inefficient presentation for Rep.

**For you have a s.s $\otimes$ cat?**

**Goal 1:** Find an additive $\otimes$ tensor + describe it by group relations w/ plan diagrams?

(link: Fund only correct to type $A_1$)

**Goal 2:** Find all the chaps -project to irreps inside reduced expression

**Goal 3:** (Dreamy) Find efficient presentation of the whole category.

**Rep:**

**Goal 1:** $\mathbb{Z}_2$ TL rank 2 Keras! $\approx$ type $A_1$ CKM else unknown!

**Goal 2:** $\mathbb{Z}_2$ JW rank 2 Kim,conjecture type $A_1$ we, conjecturally.

**Goal 3:** $\mathbb{Z}_2$ maybe? else - unknown.

Fact found: $\approx G_2$ symbols, etc

End talk-

End talk: What structure should this additive $\otimes$ cat have? "Shadow of semi-simplicity" How does this inform Goals 1+2? Elegy is easy + well-known but simpler is new.

Toolbox for "generally s.s. outranges & cats." Same idea of very different

detail worked $\rightarrow$ Hake category

End of Learn-1.
3: \( \frac{5}{2} \) \( \frac{3}{2} \) but a mystery, probably tractable.

some: a toolbox philosophy for studying zonal spectral geometry and their integral forms. Not just Rep \( g \), exact one toolbox for Siegel bundles.

Some ideas, but very different details.

Focus on Rep \( g \) to make most accessible. Give concrete example before abstract nonsense.

Let \( L_0 = \mathbb{C} \text{, } L_1 = V = \mathbb{C}^n \) with basis \( \{ e_1, e_2, \ldots, e_n \} \)

\( L_2 = \Lambda^2 V \) with basis \( \{ e_1 e_2, e_1 e_3, \ldots \} \)

\( L_k = \Lambda^k V \).

So \( L_0 = \Lambda^0 V \) is the \( \text{trivial } \) repr, and \( L_k = 0 \) for \( k > n \). (and \( k \leq 0 \) by convention)

All of them are reprs of \( g_{\mathbb{C}} \), and \( \{ L_k \}_{k \geq 0} \) are the \( \text{fundamental reprs of } g_{\mathbb{C}} \).

Generating a monoidal category \( \text{Rep}_{g_{\mathbb{C}}} \).

Now \( \Lambda^* V \) is an algebra (w/ gen-equiv mult.) and even a Frobenius algebra.

Have a mult-map \( L_j \otimes L_j \to \Lambda^j \) and its adjoint \( \Lambda^j \to L_j \otimes L_j \)

My favorite way to compute: For \( S \subseteq \{ 1, 2, \ldots, n \} \) let

\( e_S = e_{s_1} e_{s_2} \ldots e_{s_k} \) \( \text{L.k. } S = (s_1, \ldots, s_k) \) in order.

\( \{ e_S \} \subseteq \mathbb{C} \) is a basis for \( L_k \).

\( e_S \otimes e_{S'} = \begin{cases} 0 & \text{if } S \neq S' \\ \epsilon_S \epsilon_{S'} & \text{if } S = S' \end{cases} \)

where \( \epsilon_S = \frac{1}{|S|} \) \( \text{sgn}(S) \)

\( \Delta(e_T) = \sum (-1)^{|e_S|} e_S \otimes e_S' \)

\( \epsilon_T = \sum (-1)^{|e_S|} \epsilon_S \epsilon_S' \)

\( m(\Delta(e_T)) = \sum (-1)^{|e_S|} \epsilon_S \epsilon_S' e_T = (\epsilon_T) e_T \)

so \( m \circ \Delta = (\epsilon_T) \circ id \).
Time to draw it.

Notation: \( i = i_1, i_2, \ldots, i_d \) a sequence (think \( i \to \text{Li} \)).

\[ L_{i_d} = L_{i_1} \otimes L_{i_2} \otimes \cdots \otimes L_{i_d}. \]

Draw \( i \) as \( \begin{array}{c} i_1 \\downarrow \\downarrow \\downarrow i_d \end{array} \), \( \otimes \) is (knot) concatenation.

Draw \( L_{i_d} \) as \( \begin{array}{c} i_1 \\downarrow \\downarrow \\downarrow i_d \end{array} \) and identity as \( \begin{array}{c} i_1 \\downarrow \\downarrow \\downarrow i_1 \end{array} \) \( \otimes \) is knot concatenation.

Draw \( M \) as \( \begin{array}{c} \text{target} \end{array} \) \( \begin{array}{c} \text{source} \end{array} \). \( \Delta \) as \( \begin{array}{c} \text{real} \quad \text{left} \quad \text{top} \end{array} \) \( \begin{array}{c} \text{bottom} \quad \text{top} \end{array} \) \( \text{source} \quad \text{bottom} \quad \text{top} \). So \( \Delta \) is virtual stacking.

We just computed:

\[ \text{MoA} = \begin{array}{c} \text{target} \end{array} \begin{array}{c} \text{source} \end{array} \begin{array}{c} i_1 \\downarrow \\downarrow \\downarrow i_d \end{array} \]

\[ \begin{array}{c} \text{This is a linear combo of diagrams} \\ \text{if a diagram represents a morphism} \\ \begin{array}{c} \text{then a linear combo of} \\ \text{bottom top} \\ \text{source bottom top represents a morphism too.} \end{array} \end{array} \]

\[ \Delta \quad \text{is associative.} \]

\[ \begin{array}{c} \text{M o (M o M)} \end{array} \]

\[ \begin{array}{c} \text{MoA} \end{array} \begin{array}{c} \text{(MoA)} \end{array} \begin{array}{c} \text{(MoA)} \end{array} \]

\[ \begin{array}{c} \Delta \quad \text{is coassociative} \end{array} \]

\[ \begin{array}{c} \text{MoA} \end{array} \begin{array}{c} \text{MoA} \end{array} \begin{array}{c} \text{MoA} \end{array} \]
The parallelogram squash or any squash is a covariant of other relations.

Relating back to glm reps: Label all input-output strands w/ subsets of \([\mathbb{A}]\).

The coeff of \(E_1 \otimes E_2\) in the map applied to \(E_{5,5} \otimes E_{5,5}\) is a signed count of the ways to label all the other strands compatibly. Sits in the matrix.

RHS count: Clearly \(X = T_1 \otimes T_2 = S_{5,5}\)

so \[\begin{cases} 0 \text{ if } S \otimes T_1 = T_1 \otimes S_2, & S \otimes T_1, S_2 \otimes S_2 \\ 1 \text{ else } & \end{cases}\]

LHS count: Letting \(X = S_1 \otimes T_1 = T_1 \otimes S_2\), any division \(X = y_1 \cdot y_2\) gives a valid chain (+elements \(B_1 B_2\)).

so get \[\begin{cases} (6) \text{ if } \text{(need to check some signs)} & \end{cases}\]

This is like a "state-model" evaluation of \(y_1 y_2\).

10:50

Go slower.
Def: Let $\text{Web}_n^+$ be the monoidal $\mathbb{Z}$-linear $\otimes$-cat $\mathcal{W}$ with presentation:

- Objects given by $1 \leq i \leq n$, so $\text{Ob} = \{e_i = i_{l-i+l}^j\}$
- Mor. gen by $\otimes$ and $\overline{1}$

Relns: Associated, Coassoc., Bigon, (general rels for $\otimes$ presentation), and Square Flap

The hom($\text{Web}_n^+$) = linear combo of diagrams w/ bottom $\overline{1}$, top $\overline{1}$.

Square Flap:

$$\text{Square Flap} = \sum_{r \geq 0} \left[ \text{cont} \right] - (\text{bis-t})$$

Def: Let eval: $\text{Web}_n^+ \rightarrow \text{Find qgl}$ be the domain functor. (Check: Square Flap)

Thm (CKM): eval: $\text{Web}_n^+ \otimes \mathbb{Z} \rightarrow \text{Find}$ is an equiv of cats.

- webs give all the maps, and we found all the relations.
- (Well really unravel this theorem, just wait.)

Ranks:

1. This story has a q-deformation: $\text{Web}_n^+\mathbb{Z}_q$ is $\mathbb{Z}_q$-linear.
2. replaced by $\left[ \delta^i_j \right] \in \mathbb{Z}_q$-linear (look it up)
3. Fix eval on exec

CKM: eval: $\text{Web}_n^+ \otimes \mathbb{Q}(q) \rightarrow \text{Find qgl}$ is equiv of cats.

2. My work gave meaning to webs before specializing to $\mathbb{C}$ or $\mathbb{Q}(q)$, i.e. the category $\mathcal{W}_n$.

3. $\text{Web}_n^+ \otimes \mathbb{C}$ since $\text{Web}_n^+ \otimes \mathbb{C}$ symmetric (braided $\otimes q=1$) monoidal cat.
4. Dependence on \( n \) is minimal. \( \exists \text{Webs}^+_{\infty} \) w/ \( \text{Webs}^+_{\infty} \rightarrow \text{Webs}^+_n \) which sends \( L_k \rightarrow 0 \) \( \forall k \geq n \). Really most of what we do works in \( \text{Webs}^+_{\infty} \)!

5. Why \( \text{Webs}^+ \)? Can also throw in \( L_k^* \bigoplus L_k \) (draw \( L_k \approx 1_k \)).

When working w/ \( \text{Sh} \) \( L_k^* \approx L_{n-k} \) so have isom \( \bigoplus_{i=1}^{n-k} 1_{i} \) (annoying sign issues).

Note: relative homology depends on \( \tau \)!

Webs\(^+\) has all the complexity of behavior w/o a lot of technical nuisance.

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**Clasps**

How to study \( L \) using \( \text{Webs}^+? \) Not an object perse, but a direct summand of one!

Space \( \lambda = \sum q_i e_i \), \( q_i > 0 \). We'll write \( P(A) = \) "radical expression of \( \lambda \)

\[ = \bigoplus_i \text{with } A_i \text{ coping with } e_i \].

Ex: \( P(2\alpha_1 + 2\beta) = \{112, 121, 211\} \)

If \( (\alpha, \lambda) \) then \( L_i \bigoplus L^*_i \). So \( \exists \) canonical (degenerate) \( e_i \) (or \( e_i \) if \( i \))

Understand.

---

Projecting to \( L_1 \). We call \( e_i \) the clasp. Also call it the "top degenerate" - related to the fact (important soon!) that all other summands \( L_i \bigoplus L^*_i \) satisfy \( u \leq 1 \) and all alike.

In the eliminating order.

Again, \( L_1 \) is not an object, but the degenerate gives "all the info":

\( \text{Hom}(L_1, -) = \text{Hom}(L_2, -) e_1 \)

(All" in the abstract sense but maybe not explicit enough.

Ex: \( gl_2 \), clasps = Jones-Wenzl projectors.

\[
L_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}
\]

(5 terms w/ coeffs)

---

**Rank:**

Clasps live in \( \text{Webs} \otimes \mathbb{Q}(q) \), not in integral form. In specialization where \( [2] = 0 \),

Lam may not exist, or decamp. Maybe change. Shouldn't exist before many \( [2] \)...

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Why correct? \[ \text{Explanation 1: } L_1 \otimes L_2 = L_2 \otimes L_{a_1}, \quad L_2 \text{ is now decomposed.} \]

\[ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ up to scalar } \] can not be jointly but require \[ P^c = i \otimes L_2 \]

\[ i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ P^c = \text{ scalar } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \text{so scalar } = \frac{1}{\sqrt{2}} \]

The idealized projector to \( L_2 \) is the \( \epsilon_2 = \frac{1}{\sqrt{2}} [X_1, X_2] \)

The clap is the complementary idealized.

\[ \text{Explanation 2: (same ideas) Assume } \mathcal{G}_d(L \otimes L_2) \text{ is spanned by } 1, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for now.} \]

So \( e_{a_0} = a^1 + b^1 X \). By \( \text{Hom}(L_2, L_2)^3 \) so \[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ 0 = a^1 X + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b^1 \text{ so } a = -[2] b. \]

Also, \[ e^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ so } a = 1. \]

The key was the decomposition \( L \otimes L_2 = L_2 \oplus L_{a_0} \)

Claps are best computed using a chiral recursive formula, see lecture 3, Exercise.

\[ \text{PROBABLY END OF LECTURE 1.} \]

\[ \text{Back to new p.} \]
Recall: If $X = \bigoplus \mathbb{A}^n$ and $Y = \bigoplus \mathbb{A}^n$, then $\text{Hom}(X,Y) = \bigoplus \text{Hom}(\mathbb{A}^n, Y)$

$\cong \bigoplus \text{Mat}(\mathbb{A}^n, (Y))$ and $\dim \text{Hom} = \bigoplus \text{dim} \text{Hom} = \bigoplus \text{dim} \mathbb{A}^n \\ \bigoplus \text{Hom} (\mathbb{A}^n, Y) \otimes \text{Hom} (Y, \mathbb{A}^n)$

To get $\oplus$, choose basis of $\text{Hom}(\mathbb{A}^n, Y)$.

Key Idea: If $\oplus$ basis adapted to $\otimes$ structure, it is "canonical"!

To p8, plethysm, then p13.
How big is $\text{Hom}_{\mathbb{L}}(L_x, L_y)$? In the semiample setting?

Recall: If $X = \bigoplus L_i$ and $Y = \bigoplus L_i$ then $\text{Hom} = \bigoplus \text{Hom}$, representing the $L_i$ and then $\text{Hom}(X, Y) = \bigoplus \text{Hom}(L_i, L_j)$.

Let $\text{Hom}(x, y) = x^* \cdot y_1$. Choose a projection $X \rightarrow L_x$ and an inclusion $L_x \hookrightarrow Y$.

How do we find a basis for the projective space $\text{Hom}(X, L_y)$ when $X = L_x$?

**Plithysm** - a Greek word for mult/decomposing tensor product. Let's ask: what is $L_x \otimes L_i$?

**GL weights:** $\Lambda \in \mathbb{Z}^n \Rightarrow \lambda = (b_1, b_2, \ldots, b_n)$. $\text{wts}(\text{rep})$ is on $S_n$-multiset in $\Lambda$.

- $\lambda$ is polynomial if $b_i \geq 0 \; \forall i$. Note: det weight is $(1, 1, \ldots, 1) \in \text{wts}(L_n)$.
- $\otimes$-v $\lambda$ det will make any rep purely polynomial - may as well retract to then.

$\lambda$ is dominant if $b_1 > b_2 > \cdots > b_n$. $\text{Irrep} \leftrightarrow \text{Dom} \in \Lambda^+$

Let $L^{-} \rightarrow \lambda$ (hw).

**Example:**

$\text{wts}(V) = \left\{ \begin{array}{l} (10000) \in \mathbb{Z} \times \mathbb{W} \hspace{1cm} \text{hw} \end{array} \right\}$

$\text{wts}(A^V) = \left\{ \begin{array}{l} (1000) \in \mathbb{Z} \times \mathbb{W} \\ (1010) \in \mathbb{Z} \times \mathbb{W} \end{array} \right\}$

all multiplicity one! Only feature of type A!

Let $\Theta = \left( \begin{array}{c} 1 \cdots 1 \end{array} \right)$. Then $\Lambda^V = L_{\Theta}$. If $\lambda = \sum b_i \cdot \Theta_i$ then $\lambda$ is dominant if $b_i > 0$ and $\lambda$ is dominant if $b_i > 0$ for all $i$.

$\lambda$ is dominant if $a_i > 0$. $\text{L}_1 \otimes \text{L}_{a_1}$ where $a_i > 0$.

**General $\otimes$ rule:** $\text{L}_1 \otimes \text{L}_{a_1} = \bigoplus \text{wts}(L_a)$

If $a_i > 0$, $\text{L}_1 \otimes \text{L}_{a_i}$ makes sense. If $a_i < 0$, not defined, need to interpret correctly.

Really $L_1 \otimes L_{a_1}$ where $a_i > 0$. I won't explain the rule entirely, but when $L_1 \otimes L_{a_1}$ is on a wall, i.e.$L_1 \otimes L_{a} = \sum a_i \cdot \Theta_i$ with some $a_i = -1$, then $L_1 \otimes L_{a_1} = 0$. 

So what is $L_a \otimes L_b$?

Recall $a_i = b_i - 5w_i$

$\begin{pmatrix}
\begin{array}{c}
0 \\
11100 \\
11010 \\
11001 \\
10110 \\
00101010 \\
\end{array}
\end{pmatrix} = \begin{pmatrix}
0 \\
\omega_3 \\
\omega_2 - \omega_3 + \omega_4 \\
\omega_2 - \omega_4 + \omega_5 \\
\omega_1 - \omega_2 + \omega_4 \\
\omega_0 - \omega_2 + \omega_3 - \omega_4 + \omega_5 - \omega_6 + \omega_7 \\
\end{pmatrix}$

Coeffs all in $\mathbb{F}_{1,0,1,0}$ so if $\lambda = \sum a_i \omega_i \quad a_i \geq 0$ then

$\lambda + \nu = \sum a_i' \omega_i' \quad a_i' \geq 1$. $\lambda + \nu$ is ever dominated or on the wall.

$\Rightarrow$ Plethysm Rule: $L_a \otimes L_b \cong \bigoplus_{\nu \in \text{wall}(\lambda \otimes \omega_b)} L_{\lambda + \nu}$.

$\lambda + \nu$ dominates.

Ex: Can choose $\nu = (11010\ldots)$ if $a_3 \geq 1$

$\nu = (0010101\ldots)$ if $a_2, a_4, a_6 \geq 1$

If $\lambda$ large enough, get all $(\nu)$ weights.

Def: Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ a seq. of dual whts. A plethysm path for $i$ is

a seq. $T = (\lambda_1, \ldots, \lambda_i)$ of dom. whts s.t. all $L_\lambda \subset L_{\lambda_{i-1}} \otimes L_{\lambda_i}$ ($\lambda_0 = 0$)

i.e. $L_i = L_{\lambda_{i-1}} \otimes L_{\lambda_i} \otimes L_{\lambda_{i+1}} \otimes \ldots$

$E(\nu, \lambda) = 0$ plethysm path ends in $\lambda = \lambda_0$

(also called Miller-Littlewood paths)

(Deborahs)
Then \( L_x = \bigoplus L_y \) where \( n_x = \# E(x,y) \).

**Ex:** \( x = (2,2,1) \) Plothyin paths are:

- \( 0 \rightarrow 2 \rightarrow q = 2 + (1000) \rightarrow 2q + \overline{q} \) topological
- \( 2 \rightarrow q + \overline{q} = 2 + (1010) \rightarrow 2q + \overline{q} \) and multiplied 2
- \( \overline{q} = 2 + (0011) \rightarrow q + \overline{q} \)

For each \( T \in E(x,y) \) can choose projection \( d_T : L_x \rightarrow L_y \) compatible w/ path, up to scalar. Here's the diagramatic schematic. As in previous notes, simply use clayps freely or write \( \Lambda \) just to mean the object \( L_x \).

To build it needs to know:

\[ p : L_x \otimes L_y \rightarrow L_{x+y} \]

for each.

Call \( \delta \) the cliped light ladder. It is unique up to scalar b/c each \( p \) is.

**Remark:** This idea works in any \( \text{ss cat} \), but it is greatly simplified by the fact that plothyin is fundamentally \text{mutation-free} so \( p \) is ! up to scale.

Otherwise one must choose basis for projections, do linear algebra — yuck!

Can do the same for inclusions too. For reasons of sanity let \( \mathbf{MA}, i \) = \( E(x,y) \) but *parameter* inclusions. Instead let \( \mathcal{D} \) be the objects. Write retr. for \( \text{S r MA, i} \) define \( A_x \) as in the same picture upside-down, w/
What happens when we compute \( d_T \circ d_J \)?

If \( X_3 \neq X \), then the morphism in box is zero since \( \text{Hom}(L^s_{X_3}, L^T_{X_3}) = 0 \).

\[ d_T \circ d_J = 0 \quad \text{if} \quad D(S) \neq T. \]

If \( D(S) = T \), \( d_T \circ d_J = \text{id}_{L^s_X} \).

Then \( \{d_J, d_T\} \) are dual bases for \( \text{Hom}(X, L^s_X) \) and \( \text{Hom}(L^s_X, X) \) when \( X = L^s_X \).

\( \text{Hom}(X, Y) \) has basis \( d_{S,T}^1 \) for \( \text{Sel}(X, Y) \to E(X, X) \).

Moreover, \( d_{U,V}^1 d_{S,T}^1 = E(y, y)(M, Y) \):

\[
\begin{cases}
0 & \text{if } x \neq y \text{ or } S \neq ID(V) \\
\{d_{U,T}^1 \text{ if } x = y, S = ID(V) 
\end{cases}
\]

It's our matrix basis for \( \text{Hom}_X(X, Y) \)!!

This is completely abstract nonsense. Does it help? You bet, stay tuned.
Ex: \( \mathbf{v} = (2, 2, 1) \) \implies T = \begin{pmatrix} 2 \theta_2, \theta_1 + \theta_2, \theta_2 + \theta_3 \\ \theta_1 + (\theta_2 - \theta_2 + \theta_3) \end{pmatrix}

\[ \begin{array}{c}
\theta_1 = 3 \\
\theta_2 = 1 \\
\theta_3 = 2
\end{array} \]

**Properties:** To define this, need to know
- Trivially identical
- \( e_3 \) or \( e_1 \)
- But not the others

Still called the **clasp**
- Need to compute these too.

**Next key idea:**
\[
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

**Thankfully**

- The other terms in sum are orthogonal to the clasp.

**Neutral ladder**

Only need to compute one: identical \( e_3 \) for \( \phi(P(A)) \) and get the rest by applying neutral ladder to reorder indices.

\[ \begin{array}{c}
\theta_1 = 3 \\
\theta_2 = 1 \\
\theta_3 = 2
\end{array} \]

So note clasp finding, have completely explicit basis for Hom spaces.
Let $\mathcal{C}$ be a cocomplete, locally presentable category $\mathcal{L}$, satisfying Brown's lemma.

\[ X = \bigoplus L_i \quad Y = \bigoplus L_i^\otimes \quad \text{Hom}(X,Y) = \bigoplus \text{Hom}_i(X,Y) \]

where \[ \text{Hom}_i(X,Y) = \text{Hom}(L_i,Y) \otimes \text{Hom}(X,L_i) \quad \text{dim} = x_i \cdot y_i \]

\[ x \quad \quad f \quad \quad \text{a morphism which satisfies the } L_i \text{ condition} \]

**Note:** Not everything in Hom$_i$ truly factors, just as not every element of a tensor product is a pure tensor. By *factor* $L_i$, we mean morphisms in the span of truly factored morphisms.

**Rank:** For any object $Z$, $I_Z = \{ \text{morphisms factoring through } Z \}$ is an ideal in $\mathcal{C}$, i.e., closed under pre- and post-composition of any morphism $i_{i_0}$. $I_{	ext{mean}} = I_{\text{mean}}^{\text{infty}}$ as it should be present beneath $\text{L}_i$. What if (as in Fuld/Weib) the morph $L_i$ is inaccessible. Instead assume $I_{\text{mean}} = I_{\text{mean}}^{\text{infty}}$. Object $\text{triangularity} = \exists$ point structure on $\Lambda$, and accessible objects $B_{\mu}$ st. $B_{\mu} = \bigoplus L_i \oplus \bigoplus L_i^\otimes$. (Existence of $\text{GP}(\text{L})$, set $B_{\mu} = L_{\mu}$.)

No immediate relationship between $I_{B_{\mu}}$ and $I_{Y_i}$, but morphisms factoring through $B_{\mu}$ for $\mu \leq \Lambda$ are exactly those $L_i \in \mu \leq i$. I.e.

\[ I_{\text{GP}} = I_{\text{GP}_{\text{mean}}} = I_{\text{GP}_{\text{L}}}. \] One still has a filtration of $\mathcal{C}$ by ideals, not a splitting $\text{Hom}(X,Y) = \bigoplus \text{Hom}_i(X,Y)$. We can recover $\text{Hom}$ $\text{Hom}_i(X,Y)$ only in the associated graded. When $\Lambda$ is fixed, $I_{\Lambda}$ is called lower term.

\[ I_{\text{Hom}(X,B_{\mu})}/I_{\Lambda} \] is modeled by $\text{Hom}(X,B_{\mu})$ (there is an isomorphism).
Ex. $\lambda = \sum a_{ij} t_i e_j$ then if $\lambda \in \text{PA}$, $L_A \otimes L_B$ and all the summands $\mu$ on $\lambda$ in the dominion order. How to recover dominion order?

$H \uparrow^6 $
$5 \quad 3$
$a_5 + a_3 = (1111000)$
$\downarrow \text{Subtract (00100-100)}$
$a_6 + a_2 = (02111100)$

$B_{a_5+a_3} = L_5 \otimes L_3 = L_5 \otimes L_3 = L_{5+3} \otimes L_{5+3}$
$B_{a_6+a_2} = L_6 \otimes L_2 = L_6 \otimes L_2 = L_{6+2} \otimes L_{6+2}$

Crucial: When $G \leq G'$

Back to describing $\text{Hom}(X, Y)$. $\oplus \text{Hom}(B_3, Y) \otimes \text{Hom}(X, B_3)$ is too big.

Spice you have a map $E(X, Y)$ such that $\text{Hom}(X, B_3)$ s.t. $E(X, Y)$ descends to a base of $\text{Hom}(X, B_3)/I_X Y$. Similarly:

$M(X, Y) \otimes \text{Hom}(B_3, Y)$.

Then $C_{ST}^X \otimes C_{ST}^Y \in \text{Hom}(X, Y)$, $\{ C_{ST}^X \}$ descends to a base of $\text{Hom}_A(X, Y)$ modulo $I_X Y$. So $\{ C_{ST}^X \}_{ST}$ is a base for $\text{Hom}(X, Y)$!

Ex: $H \uparrow^6$
$5 \quad 2 \quad 3$

is a complicated map, being contravariant in all 3 common summands.

But it is contra-variant modulo $I_{a_5+a_3}$ so it descends to a base of $\text{Hom}(53, 26)/I_{a_5+a_3} \cong \text{Hom}(L_5 \otimes L_3, L_6 \otimes L_2)$

How to find the set $E \otimes M$ and the morphism $C_{ST}^X$? This uses the monad structure, plenty in paths. Another module.

So who needs claspor?? To meet
In our example, how to construct $G_T$. Recall $d_T : E \to E$.

Where $N$ neutral

light ladders

$X$ clap

$E_{x}$ elevating light ladder.

What happens when we lean out the claps?

$E_x^1 \quad l = 221 \quad T = (\omega_2, \omega_1, \omega_3, \omega_2 + \omega_3)$

$C_T = \sum_{S \in \mathcal{T}} d_T S + I_{x, l}$

Using that $L_x = L_{\alpha} \oplus (\oplus \mathcal{L}_y)$ for $x \in \mathcal{P}(W)$, deduce that

So $\{G_T^x \}$ descends to a basis (not quite $E_T$ but untriangular cols.)

Hooray, completely explicit basis, $\{G_T^x\}$ double ladder

$\text{Im}(E) \quad \text{Double ladder from a basis for } \text{Webs}^+ \quad \text{over } \mathbb{Z}$
On the proof: Suppose you're defined \( E_{ST} \) but don't have it span. Can let \( J_{ST} = \text{Span} \{ E_{ST} \} \).

Prop: For \( t \in \mathbb{P} \), \( \text{Hom}(L, L_t) \) is spanned by the following module \( J_{ST} \):

\[
\begin{array}{c}
\text{N} \\
\text{E} \\
? \\
k \\
\end{array}
\]

For some \( \nu \). 

Pf of pop is very specific to situation!

But pf of thin from pop is abstract relative argument. Pf of line length of \( \mathcal{G} \) also abstract very uppertri to \( \mathcal{G} \).

Consequence: When is your presentation correct? Need:

- Enough generating morphisms to construct:
  - projections, for each
    - platinum pattern (e.g., elem. line)
  - Neutral maps by any two red exp (e.g., neutral ladders)

- Enough relations to prove:
  - Prop above
  - Neutral ladders are isomy modules \( J_{ST} \)

- One more result saying morphisms are somewhat arrageable.

Then can repeat proof + win!

To p16

They go down in \( \mathbb{P} \) or \( \Lambda \), 
Then back up so need to get higher!!
Crucial Aside: Why put $L_\pm$ off-limits? We're trying to describe by generators.

Dary so automatically produces an integral form of the category—a category
liner on the subring $K$ of $\mathbb{C}$ (or $\mathbb{Q}_p$) where the coeff of the relations
reside. Each $W_b$ is defined on $\mathbb{Z}/\mathbb{Z}[g, q^{-1}]$.

Category, how may integral forms, are different rings sometimes, and it is nice to find one with
independent meaning, but that's another story.

Why not have $L_\pm \oplus 0_b$? Projection $L_{\pm} \oplus 0_b \rightarrow L_{\pm}$ in gen morphism? Bk then
relation will be nasty! If you force simple behavior then you need
$K$ vary large ($\mathbb{Q}$) because Rap gap is difficult in flat characteristic. Good luck
finding that relation with $\frac{37}{101}$ not a coefficient. Suggests intractibility.

Instead you want a minimal presentation with simple relation + must room for specialization,
so you can want to study the $p$'s. That's better, but it places $L_\pm$ off-limits,
which isn't so bad, or well see.

What structure does the basis $\{ C_{5T} \}$ have?

Let $o : \text{Horn}(Y, Z) : \mathbb{Z} \to \alpha \to \frac{C}{X} \leftarrow \beta \leftarrow \gamma$

such that $\alpha, \beta, \gamma$ is a linear combo of $C_0, U_{5T}(Y, Z)$

$C_{5T} \in \mathbb{Z} \to \alpha \to \frac{C}{X} \leftarrow \beta \leftarrow \gamma$

Then $\otimes_{B_{5T}} C_{5T} \cong \sum B(\gamma, \nu) C_{5T}$ mod $\mathbb{I}_{3T}$

This coeff are independent of $T$! (this stuff isn't)

$\otimes$ is called the "cellular formula" and it implies that $\{ C_{5T} \}$ is a cellular basis.

$\boxed{\text{[Modulo other property?]} \} \}

Similarly, $c_{5T} \alpha \equiv \sum_{\nu} C_{5T} \eta(\gamma, \nu) + \mathbb{I}_{3T}.$

Cool implication: $C_{5T} \circ C_{5T} \equiv \sum_{\nu} C_{5T} \eta(\gamma, \nu) \equiv \sum_{\nu} \gamma_{\nu} \circ C_{5T}$

So $\equiv \eta(T, \nu) C_{5T}$.
That is
\[ Z \leftarrow B \leftarrow \gamma \leftarrow B \leftarrow X \]

Then exist \( \gamma' \) s.t.
\[ C_{\gamma'} = (\gamma'_{\gamma}
\cdot \text{Id}_{B} + I_{\mathcal{E}} \text{.} \]

Since \( \text{End}(B) / I_{\mathcal{E}} \) is spanned by \( \text{Id}_{B} \).

Thus \( \gamma'(-, -) : B(y, x) \times B(x, y) \to \mathcal{E} \) is called the cellular form and its properties (e.g. rank, etc.) really determine the structure of the category.

**Exercise:** Rank \( \gamma_{y_{1}} = \text{null of } \gamma_{y_{1}} \) in \( Y \).

**Remark:** There is one more feature in the standard def of cellular categories which is really not essential. Suppose \( C \) has a couniversal autoequival, \( D \), fixy object.

Since \( \text{Hom}(X, B) \overset{D}{\Rightarrow} \text{Hom}(B, X) \), expect \( \text{Mor}(X, Y) \overset{D}{\Rightarrow} \text{Mor}(X, Y) \), so let us suggest

then \( \text{Then } D(C_{\gamma_{y_{1}}}^{x}) = C_{\gamma_{y_{1}}}^{x} \).

For any ss cat can contract such into NONSANDIRKALLY, by choosing bases for projection (ie. the matrix description) and taking matrix transpose.

Sometimes data comes naturally from some duality functor, as it does for \( \text{Web}_{n} \).

\( D(\text{diagram}) = \text{diagram flipped upside down.} \) I want to explain why this is natural.

Eg. \( \frac{3}{6} \times \frac{1}{3} = C_{\gamma} \) for \( \text{Mor}(L_{5}^{2}L_{3}, \Delta_{4}^{2n}) \)

\( \frac{3}{6} \times \frac{1}{3} = C_{\gamma} \)
 Altogether we get the following structure. Ref: Elie-Lada

**Def**: An object-oriented cellular category is:

- A $k$-linear category $C$
- A poset $\Lambda$ and an object $B_\lambda$ for each $\lambda \in \Lambda$.
- For each $X \in \mathcal{D}(k)$ in $\Lambda$, set $M(A,X) \rightarrow E(X,Y)$ in bijection.
- Maps $c: MA \rightarrow \text{Hom}(B_\lambda, X) \rightarrow E(X,Y) \rightarrow \text{Hom}(X, B_\lambda)$

Now write $C_{\text{eff}}^J = C^0_{\text{eff}} G^J$ and define $I_{\text{eff}}$ as the spin of $C_{\text{eff}}^J$ for $J$ and $\sigma_J$.

We require:

1. \{$c_{\lambda\mu}$\}$\lambda \in \Lambda$ is a basis $\text{Hom}(X,Y)$
   
2. $\text{End}(c) = C(\sigma) \text{ extends to an anti-auto-equiv}$

3. $MA \rightarrow E(B_\lambda, X) = \#$ and $C_{\#} = \text{id}$

4. **$I_{\text{eff}}$** is an ideal OR $G$ holds OR ... \(\Rightarrow I_{\text{eff}}\) is the ideal of morphism fixing the $B_\lambda$.

**Remark**: A unital categorical subcategory of a semisimple category is automatically an OACC.

Integral forms may not be, but they are so by as the maps $G$ can be constructed integrally and $C_{\text{eff}}^J$ from an integral basis.

- If you're trying to prove your diagrams or integral forms, find the structure first!
- Get any things for free - Grothendieck, group, trace, defect, etc.
- Almost every cellular algebra is just an OACC in disguise. Factorization makes popular like much more natural. (One example - $\text{IH}(S_n)$)
From the other module on plethysm paths, we see that it may be useful to consider not just one $B_s$ for each $s \in \Lambda$, but a whole family of them!

**Def.** Multispectral adapted cellular category has:

1. A set $P(\Lambda)$ of objects for $\Lambda$.
2. A map $C_i$ which sends $S_{iM}(X)$ to a map $B_{s_i} \xrightarrow{C_s} X$ for some $B_{s_i} \in \Lambda$ (different for each $S$).
3. Neutral maps $B_{s_i} \xrightarrow{\Phi} B_{s_j}$ for each $B_{s_i}, B_{s_j} \in P(\Lambda)$ such that $\Phi \circ \Phi = \Phi$ and $\Phi \circ \text{id} = \text{id}$.

Now we set

$$C_{s_i, s_j} = \text{?} \xleftarrow{\Phi} B_{s_i} \xrightarrow{C_{s_j}} X.$$ 

This allows us to remove the arrows from $C$s to any object (several) rather than having to make a canonical choice of reduced expression.

**THAT'S A MODULE**
Clasp

**Def:** Fix $x \in \Lambda$ with reduced expression PA. The clasp, if it exists, is a family of morphisms $\xi_k : i \rightarrow j$ for $i, j \in PA$ satisfying

1. $\xi_k \circ \xi_k = \xi_k$
2. $\xi_k = id_i \mod \text{Im } \lambda$
3. $\xi_k \circ a = b \circ \xi_k = 0$ for any $a, b \in \text{Im } \lambda$

(This way of formulating it obviates the need for choosing the object $L_k$.)

**Exercise:** The clasp is unique if it exists.

Finding a closed formula for the clasp as a linear combo of webs seems out of reach. It was done by Morrison for $\chi_m$ and it is complicated! But inductive families are philosophically important and practical too.

**Triple Clasp Expansion**

Suppose we have computed all the claspers less than $\lambda + \otimes_k$, including $\lambda$. We know $L_{\lambda} \otimes L_{\lambda} = \bigoplus L_{\lambda + k}$ for $\forall \nu \in \text{Wd}(L_{\lambda})$ with relevant.

One question is $L_{\lambda + k}$, we want this identity, so we want to submit all the others.

In $\text{Hom}(L_{\lambda} \otimes L_{\lambda}, L_{\lambda + k}) = 1$ and we have a basis for it:

So the identity is $K^{-1}_{\lambda + k}$ for some unknown.

Reason for more coming soon.
Thus \( \text{id}_{\text{L}e_{\text{L}}} \) is a sum of idempotents, or

\[
\lambda \quad k = \quad \sum_{\nu} \lambda_{\nu}^{k} + \sum_{\nu} \lambda_{\nu}^{k}
\]

A formal argument gives the reverse formula.

**Example 1:** \( g_{2} \), write \( \Phi \) for \( m+1 \) when \( x \) is determined for context.

\[ \Phi_{2} \] is determined rep and

\[ \Phi_{2} \] is determined rep and

\[ \Phi_{2} = \Phi \]

**Note:** 3 clays have name.

**Example 2:** \( g_{3} \), write \( a,b \) for \( a+b \) when \( x \) is determined for context.

\[ \Phi_{3} \] is determined rep and

\[ \Phi_{3} = \Phi \]

**Note:** 3 clays have name.

**Example 3:** \( g_{4} \), write \( a,b \) for \( a+b \) when \( x \) is determined for context.

\[ \Phi_{4} \] is determined rep and

\[ \Phi_{4} = \Phi \]

**Note:** 3 clays have name.

How to find coeffs?
Well, it's supposed to be $p = e$, with $p_i = id_{e_{i+1}}$. So want

$$K_{i,i}^{-1} = \ldots$$

We can now compute this

$$\mathbb{E}^{g_2} \quad g_2$$

$$K_n \circ \Lambda^{-1} = \ldots$$

$$= \ldots$$

$$= \Lambda^{-1}$$

$$= 2$$

$$= K_{n-1}^{-1}$$

$$= K_n - K_{n-1}$$

$$= \Lambda^{-1}$$

So $$K_n = E_2 - K_{n-1}.$$ 

As noted, $K_0 = 0$ since this term does not exist. Can solve, get $K_n = \frac{E_{n+1}}{E_n}$

Using the formal recursion to drop, get recursion for coefficients in that formal recursion.

$$\mathbb{E}^{g_1} \quad g_1$$

get $a_{1,1} = [2] - a_{1,0}^{-1}$ and $a_{1,1}^{-1} = 0$

$$\rightarrow a_{1,1} = \frac{E_4}{E_3}$$

get $b_{1,1} = [3] - \frac{a_{1,1}^{-1} - 2}{a_{1,0}^{-1}} - \frac{1}{a_{1,0}^{-1}}$ and similar for $S$...

Note: (Revers) domain order on $\Lambda$ controls which coeffs can appear in recursion formulas for which others.

Quite complicated!!
Again, morally it's clear what to do! Compute map $K_{ij}$ by finding relations, solving it. Both finding relations take real work!! Life is hard, save that to simple, platitude rules.

**Conjectural Solution:**

$$K_{ij} = \prod \left[ \frac{\langle 4| \alpha \rangle}{\langle 4| \alpha \rangle} \right]$$

- $\nu = (0101100)$
- Then relevant $\alpha$s are $E_1 - E_2, E_3 - E_4, E_5 - E_6, E_5 - E_7, E_5 - E_8$
- So $\lambda = a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n$ thus $\lambda + i\beta = (a_1\omega_1 + (a_2 + 1)\omega_2 + \ldots$

$$K_{ij} = \frac{(a_1 + 1)(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})}{(a_1)(a_2 + a_3)(a_4 + a_5 + a_6)(a_7)(a_8 + a_9)}$$

Please help me prove it/understand it!! Just guess work!

Prove for $n \leq 4$ and most $\nu$ for $n = 5$. 