Math 253 (Calc III), Winter 2019
HW 9

1. Basic understanding problems: do exercises 0.2.4, 0.2.5, and 0.2.8 from the Diffy Qs online textbook.

   *Solutions:* 0.2.4: If $x(t) = e^{4t}$ then $x'(t) = 4e^{4t}$, $x''(t) = 16e^{4t}$, and $x'''(t) = 64e^{4t}$. Thus

   $$x''' - 12x'' + 48x' - 64x = e^{4t}(64 - 12 * 16 + 48 * 4 - 64) = e^{4t}(0) = 0$$

   as desired.

   0.2.5: If $x(t) = e^t$ then all its derivatives are also $e^t$, so

   $$x''' - 12x'' + 48x' - 64x = e^t(1 - 12 + 48 - 64) = e^t(-27) \neq 0.$$

   0.2.8: If $x(t) = Ce^{-2t}$ then $x'(t) = (-2)Ce^{2t}$ so

   $$x' + 2x = Ce^{-2t}(-2 + 2) = Ce^{-2t}(0) = 0,$$

   and $x$ is a solution to this differential equation. Also, $x(0) = C$, so if $x(0) = 100$ then $C = 100$. The solution to this initial value problem is $x(t) = 100e^{-2t}$.

2. Consider the differential equation $y'' - y' - y = 0$ with initial condition $y(-2) = 3$ and $y'(-2) = 2$.

   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.

   (b) Find a recursive formula for the general solution centered at $-2$.

   (c) Verify your computation of $T_4(x)$ using the recursive formula.

   *Solution:* a) Starting with

   $$y'' = y' + y$$

   we have

   $$y''(-2) = y(-2) + y'(-2) = 2 + 3 = 5.$$

   Taking the derivative we get

   $$y''' = y'' + y'$$

   so

   $$y'''(-2) = y''(-2) + y'(-2) = 5 + 2 = 7.$$

   Taking the derivative we get

   $$y'''' = y''' + y''$$

   so

   $$y''''(-2) = y'''(-2) + y''(-2) = 7 + 5 = 12.$$

   Thus the degree four Taylor polynomial (which should be centered at $-2$) is

   $$T_4(x) = \frac{3}{0!} + \frac{2}{1!}(x+2) + \frac{5}{2!}(x+2)^2 + \frac{7}{3!}(x+2)^3 + \frac{12}{4!}(x+2)^4 = 3 + 2(x+2) + \frac{5}{2}(x+2)^2 + \frac{7}{6}(x+2)^3 + \frac{1}{2}(x+2)^4.$$

b) If $y(x) = \sum a_n(x+2)^n$ then

$$y'(x) = \sum_{n=0} a_{n+1}(n+1)(x+2)^n$$

and

$$y''(x) = \sum_{n=0} a_{n+2}(n+2)(n+1)(x+2)^n.$$

(We've already done the reindexing for these formulas, see the assignment for a write-up of that.) Thus

$$y'' - y' - y = \sum_{n=0}(a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n)(x+2)^n = 0.$$

Setting each coefficient to zero we get

$$a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n = 0$$

or

$$a_{n+2} = \frac{(n+1)a_{n+1} + a_n}{(n+2)(n+1)}.$$

This formula holds for all $n \geq 0$.

The base case for the recursion is the initial conditions $y(-2) = a_0$ and $y'(-2) = a_1$.

c) Let $a_0 = 3$ and $a_1 = 2$. The recursive formula for $n = 0$ gives

$$a_2 = \frac{1a_1 + a_0}{(2)(1)} = \frac{5}{2}.$$

The recursive formula for $n = 1$ gives

$$a_3 = \frac{2a_2 + a_1}{(3)(2)} = \frac{5+2}{6} = \frac{7}{6}.$$

The recursive formula for $n = 2$ gives

$$a_4 = \frac{3a_3 + a_2}{(4)(3)} = \frac{\frac{21}{6} + \frac{5}{2}}{12} = \frac{\frac{36}{6}}{12} = \frac{6}{12} = \frac{1}{2}.$$

This matches our answer in part a).

3. Consider the differential equation $y'' + 3y' + y = 0$ with initial condition $y(0) = 1$ and $y'(0) = 1$.

   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.

   (b) Find a recursive formula for the general solution centered at $0$.

   (c) Verify your computation of $T_4(x)$ using the recursive formula.

*Solution:* This one is very similar to the previous one. I'll be a little less wordy.

a) We have

$$y''(0) = -3y'(0) - y(0) = -4,$$
$$y'''(0) = -3y''(0) - y'(0) = 11,$$
$$y''''(0) = -3y'''(0) - y''(0) = -29.$$

So

$$T_4(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{-4}{2!}x^2 + \frac{11}{3!}x^3 + \frac{-29}{4!}x^4 = 1 + x - 2x^2 + \frac{11}{6}x^3 - \frac{29}{24}x^4.$$

b) b) If $y(x) = \sum a_n x^n$ then

$$y'(x) = \sum_{n=0} a_{n+1}(n+1)x^n$$

and

$$y''(x) = \sum_{n=0} a_{n+2}(n+2)(n+1)x^n.$$

Thus $y'' + 3y' + y = \sum_{n=0}((n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + a_n)x^n = 0$. Setting each coefficient to zero, we get

$$a_{n+2} = \frac{-3(n+1)a_{n+1} - a_n}{(n+1)(n+2)}$$

for all $n \geq 0$.

The base of the recursion is the initial conditions $y(0) = a_0$ and $y'(0) = a_1$.

c) Let $a_0 = 1$ and $a_1 = 1$. The recursive formula for $n = 0$ gives

$$a_2 = \frac{-3a_1 - a_0}{(2)(1)} = -2.$$

The recursive formula for $n = 1$ gives

$$a_3 = \frac{-3(2)a_2 - a_1}{(3)(2)} = \frac{12 - 1}{6} = \frac{11}{6}.$$

The recursive formula for $n = 2$ gives

$$a_4 = \frac{-3(3)a_3 - a_2}{(4)(3)} = \frac{\frac{-33}{2} + 2}{12} = \frac{-29}{24}.$$

This matches.

4. Consider the differential equation $y' - y = \frac{1}{1-x}$ with initial condition $y(0) = 4$.

   (a) Compute the degree three Taylor polynomial $T_3(x)$ for a solution to this initial value problem.

   (b) Find a recursive formula for the general solution centered at $0$.

(c) Verify your computation of $T_3(x)$ using the recursive formula.

*Solution:* a) $y' = y + \frac{1}{1-x}$ so $y'(0) = 4 + \frac{1}{(1-0)} = 5$. Taking the derivative we get

$$y'' = y' + \frac{1}{(1-x)^2}$$

so

$$y''(0) = 5 + \frac{1}{1^2} = 6.$$

Taking the derivative we get

$$y''' = y'' + \frac{2}{(1-x)^3}$$

so

$$y'''(0) = 6 + \frac{2}{1^3} = 8.$$

Thus $T_3(x) = 4 + 5x + \frac{6}{2}x^2 + \frac{8}{6}x^3$.

b) If $y(x) = \sum a_n x^n$ then

$$y'(x) = \sum_{n=0} a_{n+1}(n+1)x^n.$$

Also,

$$\frac{1}{1-x} = \sum_{n=0} x^n.$$

Thus

$$y' - y - \frac{1}{1-x} = \sum_{n=0}((n+1)a_{n+1} - a_n - 1)x^n = 0.$$

Setting each coefficient to zero we get

$$a_{n+1} = \frac{a_n + 1}{n+1}$$

for all $n \geq 0$.

The base of the recursion is the initial condition $y(0) = a_0$.

c) Let $a_0 = 4$. The recursive formula for $n = 0$ gives

$$a_1 = \frac{4+1}{1} = 5.$$

For $n = 1$ it gives

$$a_2 = \frac{5+1}{2} = 3.$$

For $n = 2$ it gives

$$a_3 = \frac{3+1}{3} = \frac{4}{3}.$$

This matches.

5. Consider the differential equation $y'' + x^2 y = 0$ with initial condition $y(0) = 4$ and $y'(0) = -1$.

(a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem. (Be careful! What is the derivative of $x^2y$?)

(b) Find a recursive formula for the general solution centered at $0$.

(c) Use this recursive formula to find $T_8(x)$. (If you did things correctly to this point, this is not as bad as it sounds.)

*Solution:* a) Using $y'' = -x^2y$ we get

$$y''(0) = -0^2 4 = 0.$$

Taking the derivative we get

$$y''' = -2xy - x^2y'$$

so

$$y'''(0) = -2 \cdot 0 \cdot 4 - 0^2(-1) = 0.$$

Taking the derivative we get

$$y'''' = -2y - 2xy' - 2xy' - x^2y'' = -2y - 4xy' - x^2y''$$

so

$$y''''(0) = -2(4) - 0 - 0 = -8.$$

Thus

$$T_4(x) = 4 - x + 0x^2 + 0x^3 + \frac{-8}{24}x^4 = 4 - x - \frac{1}{3}x^4.$$

b) If $y(x) = \sum a_n x^n$ then

$$y'(x) = \sum_{n=0} a_{n+1}(n+1)x^n$$

and

$$y''(x) = \sum_{n=0} a_{n+2}(n+2)(n+1)x^n.$$

Also

$$x^2y = \sum_{n=2} a_{n-2}x^n,$$

a sum which starts at $n = 2$ (we already did the reindexing). Thus

$$y''+x^2y = \sum_{n=0} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2} a_{n-2}x^n = 2a_2x^0 + 6a_3x^1 + \sum_{n=2}((n+1)(n+2)a_{n+2} + a_{n-2})x^n = 0.$$

(We separate the first two terms and combined the rest of the sum starting at $n = 2$.) Setting each coefficient to zero we get

$$2a_2 = 0, \quad 6a_3 = 0,$$

and

$$a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}$$

for all $n \geq 2$.

The base of our recursion is the initial condition $y(0) = a_0$ and $y'(0) = a_1$, together with the special values $a_2 = 0$ and $a_3 = 0$. The recursive formula is $a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}$ for all $n \geq 2$.

c) Let $a_0 = 4$ and $a_1 = -1$. We know that $a_2 = a_3 = 0$. From the recursive formula for $n = 2$ we get

$$a_4 = \frac{-a_0}{(4)(3)} = \frac{-1}{3}.$$

From the recursive formula for $n = 3$ we get

$$a_5 = \frac{-a_1}{(5)(4)} = \frac{1}{20}.$$

From the recursive formula for $n = 4$ we get

$$a_6 = \frac{-a_2}{(6)(5)} = 0,$$

and similarly for $n = 5$ we get $a_7 = 0$. Then for $n = 6$ we get

$$a_8 = \frac{-a_4}{(8)(7)} = \frac{\frac{1}{3}}{8 \cdot 7} = \frac{1}{3 \cdot 8 \cdot 7}.$$

Now for the optional extra exercises.

1. Repeat the standard problem for $y' - (2+3x)y = 0$, centered at $0$, with $y(0) = 3$.

   *Solution:* a)
$$y'(0) = 2y(0) = 6$$
$$y'' = (2+3x)y' + 3y$$
$$y''(0) = 2y'(0) + 3y(0) = 21$$
$$y''' = (2+3x)y'' + 3y' + 3y' = (2+3x)y'' + 6y'$$
$$y'''(0) = 2y''(0) + 6y'(0) = 42 + 36 = 78.$$

   So
$$T_3(x) = 3 + 6x + \frac{21}{2}x^2 + \frac{78}{6}x^3.$$

   I didn't bother to do $T_4$ here.

   b) $y(x) = \sum_0 a_n x^n$ so
$$y'(x) = \sum_{n=0} a_{n+1}(n+1)x^n$$

   and
$$(2+3x)y = \sum_0 2a_n x^n + \sum_0 3a_n x^{n+1} = \sum_0 2a_n x^n + \sum_1 3a_{n-1}x^n = 2a_0x^0 + \sum_1 (2a_n + 3a_{n-1})x^n.$$

   Thus
$$y' - (2+3x)y = (a_1 - 2a_0)x^0 + \sum_1 ((n+1)a_{n+1} - 2a_n - 3a_{n-1})x^n = 0.$$

   Hence
$$a_1 = 2a_0$$

   and
$$a_{n+1} = \frac{2a_n + 3a_{n-1}}{n+1}$$

   for $n \geq 1$.

   The base case is $a_0 = y(0)$ and $a_1 = 2a_0$.

   c) If $a_0 = 3$ then $a_1 = 6$ and
$$a_2 = \frac{2a_1 + 3a_0}{2} = \frac{21}{2}$$

   and
$$a_3 = \frac{2a_2 + 3a_1}{3} = \frac{21 + 18}{3} = 13.$$

2. Consider the differential equation $y'' + x^2y = 0$ with initial condition $y(1) = 4$ and $y'(1) = -1$.

   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.

(b) Suppose you try to find a recursive formula for the general solution centered at 1. What makes this problem subtle, and different from the previous problem? (If you don't get it, keep reading.)

(c) Find $b_0, b_1, b_2$ such that

$$x^2 = b_0 + b_1(x-1) + b_2(x-1)^2.$$

(d) Find the recursive formula for the general solution to the differential equation

$$y'' + b_0 y + b_1(x-1)y + b_2(x-1)^2 y$$

centered at 1. (Is this doable? Is this the same differential equation as before?)

*Partial Solution:* a) you can do this. $y''(1) = -1^2 y(1) = -4$. and so forth.

b) Well, if we write $y(x) = \sum a_n(x-1)^n$ then what is the formula for $x^2y$? It's not $\sum a_n x^2(x-1)^n$, that's not a power series.

c) Let $g(x) = x^2$. Then $g(1) = 1$, $g'(1) = 2$, $g''(1) = 2$, and all further derivatives are zero. So

$$g(x) = \frac{1}{0!} + \frac{2}{1!}(x-1) + \frac{2}{2!}(x-1)^2 = 1 + 2(x-1) + (x-1)^2.$$

d) So we should rewrite the differential equation as $y'' + (1 + 2(x-1) + (x-1)^2)y = 0$. Now we can write each term as a power series, for example

$$(x-1)^2 y = \sum_0 a_n(x-1)^{n+2} = \sum_2 a_{n-2}(x-1)^n.$$

I'll leave the rest as a further exercise. But it is doable, and it is the same differential equation as before, just written in a way which is more convenient for centering at 1.

3. Repeat the standard problem for $y' - (2+3x)y = 0$, centered at 1, with $y(1) = 3$.

   *Sketch:* The interesting part is that we should rewrite $2 + 3x$ as a power series centered at 1. We have $2 + 3x = 3(x-1) + 5$, so the ODE is better written as

$$y' - (3(x-1) + 5)y = 0.$$

Now when $y(x) = \sum a_n(x-1)^n$ we have

$$y'(x) = \sum_{n=0} a_{n+1}(n+1)(x-1)^n$$

and

$$(3(x-1)+5)y = \sum_0 5a_n(x-1)^n + 3\sum_1 a_{n-1}(x-1)^n = 5a_0(x-1)^0 + \sum_1 (5a_n + 3a_{n-1})(x-1)^n.$$

From this we get the formula

$$y' - (2+3x)y = (a_1 - 5a_0)(x-1)^0 + \sum_1 ((n+1)a_{n+1} - 5a_n - 3a_{n-1})(x-1)^n.$$

So the base case is $a_0 = y(1)$ and $a_1 = 5a_0$, and the recursive formula is

$$a_{n+1} = \frac{5a_n + 3a_{n-1}}{n+1}$$

for $n \geq 1$.

4. Consider the differential equation $y'' - y' = 0$, with general initial condition $y(0) = a_0$ and $y'(0) = a_1$.

   (a) Find a recursive formula for the general solution centered at $0$.

   (b) What can you say about the specific solution with $y(0) = 6$ and $y'(0) = 0$? Have you seen this function before?

   (c) What can you say about the specific solution with $y(0) = 6$ and $y'(0) = 6$? Have you seen this function before?

   *Solution* a) The recursive formula ends up being

$$a_{n+2} = \frac{(n+1)a_{n+1}}{(n+2)(n+1)} = \frac{a_{n+1}}{n+2}$$

   for all $n \geq 0$. The base case is $a_0 = y(0)$ and $a_1 = y'(0)$.

   b) When $a_0 = 6$ and $a_1 = 0$ we get $a_2 = \frac{a_1}{2} = 0$ and $a_3 = \frac{a_2}{3} = 0$ and $a_4 = \frac{a_4}{4} = 0$ and so forth. So $a_n = 0$ for all $n \geq 1$, and the solution is just $y(x) = 6$, a constant function.

   c) When $a_0 = 6$ and $a_1 = 6$ we get $a_2 = \frac{6}{2}$ and $a_3 = \frac{6}{3 \cdot 2}$ and $a_4 = \frac{6}{4 \cdot 3 \cdot 2}$ and so forth. So $a_n = \frac{6}{n!}$ for all $n \geq 0$, and the solution is just $y(x) = 6e^x$.

   Aside: In fact, the general solution is $C_1 + C_2 e^x$ for any numbers $C_1$ and $C_2$. You learn why in MAT256.