Math 253 (Calc III), Winter 2019
HW 9

I will post answers to these homework problems after the due date.

I've boiled down the problems I like to ask about ODEs on tests to the following three types.

a) Given an ODE with initial value, compute the Taylor polynomial T_N for some small value of N. This involves knowing where the center should be, and computing the first few derivatives at the center. (We learned this on Friday of W9.)

b) Given an ODE with a choice of center, find the recursive formula for the general solution. This involves identifying which coefficients of the power series are part of the initial conditions and become parameters, identifying which coefficients have special values, and finding the general recursive formula for the n-th coefficient for large enough n. (We learned this on Monday of W10.)

c) Same as b), and then I follow up by asking you to use the recursive formula to compute T_N for some initial value and some small value of N. (This is just checking that you can follow the recursive formula you constructed in an actual example.) Note: This is the same "problem" as a), but the method is different, and in particular computing derivatives directly is not the way to go, as it might be harder.

Here is a sample problem with a very wordy solution so you can see all three of these in action with some commentary. Though I have assigned all three parts in each problem in the homework, an exam problem might have just (a), or just (b) and (c), to be shorter.

*) Consider the differential equation $y' - xy = 0$ with initial condition $y(0) = 2$.

1. Compute directly the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.

2. Find a recursive formula for the general solution with center 0.

3. Verify your computation of $T_4(x)$ using the recursive formula.

Solution: a) The center for my Taylor polynomial should be at 0, that's where the information is.

We know that $y(0) = 2$ and $y' = xy$. Thus $y'(0) = 0 \cdot 2 = 0$. (I plugged in 0 for x and 2 for $y(0)$.) Taking the derivative of both sides of

$$y' = xy$$

we get that

$$y'' = xy' + y$$

(to get this, I used the product rule when taking the derivative of xy, which is secretly $xy(x)$). Thus $y''(0) = 0 \cdot 0 + 2 = 2$. Taking the derivative again we get that

$$y''' = xy'' + y' + y' = xy'' + 2y'.$$

Thus $y'''(0) = 0 + 0 = 0$. Taking the derivative one final time, we get

$$y'''' = xy''' + y'' + 2y'' = xy''' + 3y''.$$

Thus $y''''(0) = 0 + 3 \cdot 2 = 6$.

Using this data, and the formula $a_n = \frac{y^{(n)}(0)}{n!}$, we get

$$T_4(x) = 2 + 0x + \frac{2}{2}x^2 + 0x^3 + \frac{6}{4!}x^4 = 2 + x^2 + \frac{1}{4}x^4.$$

b) If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then our general initial condition is $y(0) = a_0$, and we use a_0 as a parameter.

If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then using standard manipulations and reindexing we get

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n,$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n.$$

(Note, we reindexed so we could add/subtract these two series.)

We want to set $y' - xy = 0$. Note that the power series for $xy(x)$ starts at $n = 1$, so the coefficient of x^0 is zero, and doesn't follow the rest of the formula. Thus the coefficient of x^0 in $y' - xy$ is $a_1 - 0 = a_1$, and since $y' - xy$ is zero, we get $a_1 = 0$. This is a coefficient with a "special value".

For $n \geq 1$, the coefficient of x^n in $y' - xy$ is

$$a_{n+1}(n+1) - a_{n-1}.$$

This is supposed to be zero, so $a_{n+1}(n+1) - a_{n-1} = 0$ or in other words

$$a_{n+1} = \frac{a_{n-1}}{n+1}.$$

This is true for all $n \geq 1$. This is the recursive formula for large enough n. Optional: you can reindex this recursion so that it says

$$a_n = \frac{a_{n-2}}{n}$$

for $n \geq 2$.

Thus the general solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is defined by the recursive formula: a_0 is a parameter, $a_1 = 0$, and $a_{n+1} = \frac{a_{n-1}}{n+1}$ for all $n \geq 1$.

c) Now we plug in $a_0 = 2$ to the previous recursion. We have $a_0 = 2$, $a_1 = 0$, $a_2 = \frac{a_0}{2} = 1$, $a_3 = \frac{a_1}{3} = 0$, and $a_4 = \frac{a_2}{4} = \frac{1}{4}$. It matches!

End of solution.

Now for the assignment. Below, $y(x)$ will always be a function of the variable x, and x_0 will denote the center of the power series.

1. Basic understanding problems: do exercises 0.2.4, 0.2.5, and 0.2.8 from the Diffy Qs online textbook.
2. Consider the differential equation $y'' - y' - y = 0$ with initial condition $y(-2) = 3$ and $y'(-2) = 2$.
   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.
   (b) Find a recursive formula for the general solution centered at -2.
   (c) Verify your computation of $T_4(x)$ using the recursive formula.
3. Consider the differential equation $y'' + 3y' + y = 0$ with initial condition $y(0) = 1$ and $y'(0) = 1$.
   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.
   (b) Find a recursive formula for the general solution centered at 0.
   (c) Verify your computation of $T_4(x)$ using the recursive formula.
4. Consider the differential equation $y' - y = \frac{1}{1-x}$ with initial condition $y(0) = 4$.
   (a) Compute the degree three Taylor polynomial $T_3(x)$ for a solution to this initial value problem.
   (b) Find a recursive formula for the general solution centered at 0.
   (c) Verify your computation of $T_3(x)$ using the recursive formula.
5. Consider the differential equation $y'' + x^2 y = 0$ with initial condition $y(0) = 4$ and $y'(0) = -1$.
   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem. (Be careful! What is the derivative of $x^2 y$?)
   (b) Find a recursive formula for the general solution centered at 0.
   (c) Use this recursive formula to find $T_8(x)$. (If you did things correctly to this point, this is not as bad as it sounds.)

Extra practice problems. You don't need to turn these in.

1. Repeat the standard problem for $y' - (2 + 3x)y = 0$, centered at 0, with $y(0) = 3$.

2. Consider the differential equation $y'' + x^2 y = 0$ with initial condition $y(1) = 4$ and $y'(1) = -1$.

   (a) Compute the degree four Taylor polynomial $T_4(x)$ for a solution to this initial value problem.

   (b) Suppose you try to find a recursive formula for the general solution centered at 1. What makes this problem subtle, and different from the previous problem? (If you don't get it, keep reading.)

   (c) Find b_0, b_1, b_2 such that

   $$x^2 = b_0 + b_1(x-1) + b_2(x-1)^2.$$

   (d) Find the recursive formula for the general solution to the differential equation

   $$y'' + b_0 y + b_1(x-1)y + b_2(x-1)^2 y$$

   centered at 1. (Is this doable? Is this the same differential equation as before?)

3. Repeat the standard problem for $y' - (2 + 3x)y = 0$, centered at 1, with $y(1) = 3$.

4. Consider the differential equation $y'' - y' = 0$, with general initial condition $y(0) = a_0$ and $y'(0) = a_1$.

   (a) Find a recursive formula for the general solution centered at 0.

   (b) What can you say about the specific solution with $y(0) = 6$ and $y'(0) = 0$? Have you seen this function before?

   (c) What can you say about the specific solution with $y(0) = 6$ and $y'(0) = 6$? Have you seen this function before?