MAT 253 LECTURE NOTES by Ben Gun 

Boring stuff - day 2. Website: on board. Tea! Today: overview.

How does your computer know what $\sin(1.234)$ is? Does it have “sin” the function built in? (Yes/No) No, it approximates it, never knowing the exact answer.

Is there an exact answer (Yes/No) - Yes, but it's an irrational number w/o 20 digits so no human will ever know it beyond 5 digits or in any form that's not $\sin(1.234)$... not a problem.

How to approximate it? Well, we know $\sin(0) = 0$. So let's approximate w/ constant function $C$, since $1.234$ is not far from $0$.

$T_C(x) = C = \frac{1}{0} x + C$

Not exactly equal, but very close, must zoom in to see difference.

$T_L(x) = x$

Definitely not equal!

$T_0(1.234) = 0$

Not a very good approximation, but a constant function won't be great. What's the best constant function?

$f(x) = \sin(1.234)$ of course, but can't find that!

Calc II: The best linear approximation near $0$ is the tangent line.

$\sin'(0) = \cos(0) = 1 \Rightarrow$ slope

$\sin(0) = 0 \Rightarrow$ intercept

So $T_1(x) = 1 \cdot x + 0 = x$

$T_1(1.234) = 1.234$

Better idea: approximate linearly near $\frac{\pi}{2}$, where we also know $\sin(\frac{\pi}{2}) = 1$

$\sin'(\frac{\pi}{2}) = 0$

Let's postpone this idea.

$T_1$ is better than $T_0$, but still not great. What is the best quadratic polynomial approximation of $\sin x$ near $0$? $T_2(x) = ax^2 + bx + c$... which is best?

What makes $T_1(x)$ the best linear approx? $T_0(x) = \sin(x)$ (same intercept) $T_2(x) = T_1'(x) = \sin'(x)$ (same slope).

Let's choose $T_2(x)$ so that $T_2''(0) = \sin''(0)$ Note: $T_1''(0) = 0$, because linear.
Well, \( \sin''(0) = -\cos(0) = 0 \Rightarrow T_2 = T_1 \).

What is the best cubic poly approx of \( \sin x \) near 0? \( \sin''(0) = -\cos(0) = 0 \)

\[
T_3(x) = \frac{-1}{6} x^3 + x \\
T_3'(x) = \frac{-3}{6} x^2 + 1 \\
T_3''(x) = \frac{-6}{6} x + 0 \\
T_3'''(x) = \frac{-6}{6} = -1.
\]

\[
T_5(x) = \frac{1}{120} x^5 - \frac{1}{6} x^3 + x
\]

\[
T_3(1.234) \approx 1.208 \quad T_5(1.234) \approx 1.247.
\]

\[
T_7(1.234) \approx 1.238 \quad T_9(1.234) \approx 1.238185 \quad T_{11}(1.234) \approx 1.2381822.
\]

Then approximations are called the \( n \)th degree Taylor polynomial of \( \sin x \) centered at 0. They are functions and when you plug in 1.234 you get successive approximations of \( \sin(1.234) \).

Computers can evaluate polynomials only by adding and multiplying.

**Questions:**

1) You can compute \( T_n(x) \) as long as you can compute derivatives!

It's actually pretty easy.

2) Are the numbers \( T_3(1.234), T_4(1.234), T_5(1.234), T_6(1.234) \ldots \) getting closer to some number? Sure looks like it.

2') Is that number \( \sin(1.234) \)? My calculator thinks so. But this is what the calculator is doing - it's not giving me \( \sin(1.234) \), it is doing \( T_n(1.234) \) for some large \( n \).

Computers can add, multiply, but not do crazy things.

3) How big do I need to go? (Ask how many steps.) It depends how accurate you want to be. Calculator has 8 digits, google has 12.

(Ask: Until stabilize? Will it stabilize?)

3') and another way - can I know the error? \( |T_n(1.234) - \sin(1.234)| \)?

This result, the Taylor Inequality estimate, is the most important thing in this course!
Here's another function: $f(x) = \frac{1}{1-x}$ (for $x \not= 1$)

- $f(0) = 1$
- $f'(0) = 1$
- $f''(0) = 2$
- $f'''(0) = 6$

$f(a) = \frac{1}{1-a}$

When does Taylor approximation work? For this function, $T_n(x)$ will give a better and better approximation if $|x| < 1$.

Taylor polynomial question. What does it mean to get closer and closer to $f(x)$ by choosing a larger $n$? How can you tell if the approximation will diverge?

More fundamental question: What does it mean for a sequence of numbers (like $T_1(1.234), T_2(1.234), \ldots$) to have a limit (i.e., to get closer and closer to some number)? Do infinite sums make sense? How can you tell whether they will converge to a limit?

This is when we begin.

This class:

- sequences
- series
- power series
- Taylor series of functions
- applications of Taylor series
- all of these in various physics, computer science, etc. on exams.
Second half: really just MAT251 derivative, similar to stuff you've done before, fairly mechanical. BUT THE WHOLE POINT OF THE COURSE!!

First half: new, hard, conceptual. Good to take a break from calculus and do something different!

In MAT 251-2: here are some tools, deriv-natural. Hummm...saw, Now practice.

Now! here is a blue print. Figure out what tools you need. MUCH HARDER.

Men like a puzzle. GOOD LUCK.

\[ \sqrt{\text{MAT316-7: WHY the tools work! What's inside the box? What is } e \text{ a number?}} \]

Before: Give me the right answer.

Now: Give me an approximate answer. "Def: An approximation is a intelligent wrong answer!"

Takes smarts to be "correctly wrong, gotta know what you're dang.

How tall are you?
Boring stuff: Office hours, library, exams W4,8, quizzes weekly on H/W day. Expect HW harder than 251-2 because takes more time in class. Solution to one problem won't help you do the next. Answer HW problem in class on Monday.

**Definition:** A sequence is an infinite list of numbers.

\[ a = (a_1, a_2, a_3, a_4, ...) \] or \[ (a_n)_{n=1}^\infty \].

**Example 1:** \[ b = (1, 2, 3, 4, 5, ...) \] means \[ b_1 = 1, b_2 = 2, ..., b_n = n \].

**Example 2:** \[ c = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...) \] guess \( c_n = ? \)

\[ c_n = \frac{n}{n+1} \].

**Example 3:** \[ d = (1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, ...) \]

\[ a_n = \begin{cases} 50 & \text{if } n \text{ is even} \\ \frac{1}{2^k} & \text{if } n \text{ is odd} \end{cases} \]

\[ n = 2k-1 \]

**Example 4:** \[ p = (3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, ...) \]

\[ p_n = ? \]

\[ p_n = \text{the } n\text{th digit of } \pi \] (no formula).

**Example 5:** \[ q = (5, 17, -3.7, 0.2, 00000006, ...) \] need not be a pattern!!

Conv: When you see \( ... \) there is probably a pattern that is supposed to be obvious, like Ex 1-3. If not, it probably says so.

**Example 6:** \( a_1 = a_2 = 1 \), \[ a_n = a_{n-1} + a_{n-2} \] for \( n \geq 3 \). "Fibonacci sequence"

\[ a = (1, 1, 2, 3, 5, 8, ...) \]

Fibonacci sequence has recursive formula.

Rec formula is opposed to a closed formula which says what \( a_n \) is exactly without referring to other values.

Sequences are typically defined via:

- list of numbers with \( ... \) guess the pattern, or maybe it doesn't matter.
- closed formula
- recursive formula

but some sequences defy description.
Example 7: \( \vec{a} = (5, 8, 6, 6, 6, \ldots) \) \( a_n = \begin{cases} 5 & n = 1 \\ 8 & n = 2 \\ 6 & n \geq 3 \end{cases} \)

Next 8: \( \vec{a} = (5, 8, 6) \) must be infinite to be a sequence. Why this defn?

Because we care about behavior as \( n \to \infty \).

Ex: \( a_n = \frac{1}{2^n} \) \( \vec{a} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) \)

Plot sequence: \( \mathbb{R}^2 \)  1  2  3  4  5

Def: A sequence \( \vec{a} \) has a limit \( L \) if \( |a_n - L| \) eventually gets and stays arbitrarily close to \( L \). We say \( \vec{a} \) is convergent, and converges to \( L \). \( \lim_{n \to \infty} a_n = L \)

A sequence \( \vec{a} \) has a limit \( L \) if \( \lim_{n \to \infty} (a_n - L) = 0 \)

Ex: \( a_n = \frac{1}{(-2)^n} \)

\( \lim_{n \to \infty} a_n = 0 \)

Ex 2: \( a_n = \frac{n}{n+1} \)

\( \lim_{n \to \infty} a_n = 1 \)

Ex: \( \vec{f} = (5, 8, 6, 6, 6, \ldots) \) \( \lim_{n \to ?} \vec{f} = 6 \).
\[ a_n = \begin{cases} 
5 & \text{if } n = 10^k \text{ for some } k \\
\frac{1}{a_n} & \text{otherwise}
\end{cases} \]

**Ex.** \( a = (6, 6, 0, 66, 0, 66, 0, \ldots) \) \( a_n = \begin{cases} 
50 & \text{if } n \text{ is a multiple of 3} \\
6 & \text{else}
\end{cases} \)

limit? No. Get close to 6 but don't stay close to 6.

\[ \text{less than .01 to 6? } \checkmark \times \checkmark \checkmark \checkmark \ldots \text{ never stays true.} \]

**Ex.** \( b = (1, 6, \frac{1}{2}, 6, \frac{1}{3}, 6, \frac{1}{4}, 6, \ldots) \) \( b_n = ? \) \( \text{limit?} \)

**Ex.** \( c = (1, 6, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots) \) \( c_n = \begin{cases} 
\frac{1}{a_n} & \text{if } n = 2k-1 \text{ odd} \\
6 & \text{if } n = 2k \text{ even}
\end{cases} \)

Plot:

Doesn't have to stay low the first time it goes low, but eventually has to stay low!

**Ex.** \( a_n = \begin{cases} 
\frac{1}{a_n} & \text{unless } n = 10^k \text{ for some } k \\
5 & n = 10^k \text{ for some } k
\end{cases} \)

looks like it goes to 0 except at \( n = 1, 10, 100, 1000, 10000, \ldots \)

No limits.
Given a sequence, does it converge or diverge?
- if it converges, can you find the limit?
We’ll give you a toolbox, but figuring out what tool to use can be tricky. Not the only method.

**Tool 1:** Algebra with limits. \( \lim_{n \to \infty} 5 + \frac{1}{n} = 5 \)
- If \( c_n = a_n + b_n \), then \( \lim c_n = \lim a_n + \lim b_n \) if both limits exist.
  \( \lim (5 + \frac{1}{n}) = \lim 5 + \lim \frac{1}{n} = 5 + 0 = 5 \)
  the sequence \((5, 5, 5, 5, \ldots)\)
- \( c_n = \lambda a_n \) then \( \lim c_n = \lim \lambda a_n \).
- \( \lim (a_n b_n) = \lim a_n \cdot \lim b_n \)
  [Text continuation and examples]

**Note:** This only works if the limits exist! \( \lim \frac{5t+2}{12t+3} \neq \frac{\lim 5t+2}{\lim 12t+3} \)

**Ex:** \( \lim (n + \frac{1}{n}) = ? \) Ask. DNE \( = \lim n + \lim \frac{1}{n} \)

- Case: If asked, then \( \lim n = \lim (n + \frac{1}{n}) - \lim \frac{1}{n} \) would also exist.
  If \( a_n \) converges and \( b_n \) diverges, then \( a_n + b_n \) **diverges**.

**Tool 2:** Extension to a function.

We learned in Calc I how to take \( \lim f(t) \) for a function \( f \).

If \( a_n \) extends to \( f \), i.e., \( f \) is a function so that \( f(n) = a_n \) for all \( n \),
and if \( \lim f(t) = L \), then \( \lim a_n = L \).
Ex: \( a_n = \frac{n}{n+1} \), \( P(t) = \frac{t}{e^t+1} \)
\[
\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{t}{t+1} = \lim_{t \to \infty} \frac{1}{1} = 1 = \lim_{n \to \infty} a_n.
\]

This is the main use of L'Hôpital's rule. It allows one to use calculus to evaluate limits.

Ex: \( \lim_{n \to \infty} \frac{5n^2+3n}{12n^2-5} = \frac{5}{12} \).

For ratios of polynomials,
\[
\lim_{n \to \infty} \frac{5n^2+3n}{12n^2-5} = \frac{5 \cdot \text{DN6}}{12 \cdot \text{DN6}} = \frac{5}{12}.
\]

Reminders:
- For ratios of polynomials, \( \lim_{n \to \infty} \frac{c^n}{d^n} = \begin{cases} \frac{c}{d} & \text{if } m = m' \\ 0 & \text{if } m < m' \\ \text{DN6 & if } m > m' \end{cases} \),

- For exponential functions, \( \lim_{t \to \infty} e^{rt} = \begin{cases} \text{DN6} & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ \text{DN6} & \text{if } r > 0 \end{cases} \),

For sequences, this will become: \( \lim_{n \to \infty} x^n = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{DN6} & \text{if } |r| > 1 \end{cases} \),

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for All values of r, except r = 0.
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Ex: \( \lim_{n \to \infty} \frac{n^2-n}{n^2} = \lim_{n \to \infty} \frac{t^2}{e^t} = \lim_{t \to \infty} \frac{2t}{e^t} = \lim_{t \to \infty} \frac{2}{e^t} = 0 \)

Ex: \( \lim_{n \to \infty} \frac{10000}{(1.1)^n} = 0 \)  

"Exponential grow faster than polynomials eventually."

Big Important! Factorials: \( (1, 2, 6, 24, 120, 720, \ldots) \)

What is \( n! \)? \( 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \)

But is there a continuous function \( f(t) \) with \( f(n) = n! \) for each \( n \)?

What is \( \frac{1}{2}! \) ?

Gamma function, \( \frac{1}{2}! = \sqrt{\pi} / 4 \).

Is \( \frac{1}{2}! \) = \( 1 \) by convention.

Yes! In fact, there's even a smooth one, called the Gamma function, \( \frac{1}{2}! = \sqrt{\pi} / 4 \).

Now \( \lim_{n \to \infty} \frac{1}{n} = 0 \), even \( \lim_{n \to \infty} \frac{10n^5}{n^3} \leq 0 \), "factorials grow faster than exponentials, eventually."

but must use different tools to see it.

**Why Factorials?**

What is \( f(n) \) for \( \begin{align*} f(0) &= x \quad f' = 1 \ \\ f(1) &= x^2 \quad f'^{'} = 2 \ \\ f(2) &= 3x^2 \quad f'' = 6x \quad f''^{''} = 6 \ \\ f(3) &= 4x^3 \quad f^{'''} = 4.3x^2 \quad f^{''''} = 432x \ \\ f(4) &= x^4 \quad f^{'''''} = 432x \quad f^{''''''} = 4.3! \end{align*} \)

they show up naturally when taking derivatives of polynomials. Super important for Taylor series, so they'll be everywhere in this class.

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**Tool 3:** Evaluation under a continuous function.

**If** \( f \) is continuous and \( b_n \to L \) then \( f(b_n) \to f(L) \) (Aside! This is actually a definition of continuity!)

\[ \lim_{n \to \infty} \sin \left( \frac{1}{n} \right) = \sin \left( \lim_{n \to \infty} \frac{1}{n} \right) = \sin(0) = 0 \]

\[ \lim_{n \to \infty} \ln \left( \frac{2n+1}{3n+2} \right) = \ln \left( \lim_{n \to \infty} \frac{2n+1}{3n+2} \right) = \ln \left( \frac{2}{3} \right) \]

\[ \lim_{n \to \infty} \frac{e^{\left( \frac{n+1}{n} \right)}}{\sin \left( \frac{2n+1}{n+1} \right)} = \frac{\lim_{n \to \infty} e^{\left( \frac{n+1}{n} \right)}}{\lim_{n \to \infty} \sin \left( \frac{2n+1}{n+1} \right)} = \frac{e^0}{\sin(3)} = \frac{1}{\sin(3)} \]

Note: If \( b_n \) diverges, it is still possible for \( f(b_n) \) to converge.

Ex: \( b_n = 2n \). diverges

\( \sin(b_n) = 0 \) for all \( n \), converges.

\( \lim_{n \to \infty} \left( \frac{\sqrt{9n^2+3n}}{4n^2-n+2} + \sin \left( \frac{n+1}{2} \right) \right) = ? \)
**Tool 4: Monotonicity**

**Def:** A sequence $\alpha$ is increasing if $\alpha_{n+1} \geq \alpha_n$ for all $n \geq 1$; decreasing if $\alpha_{n+1} \leq \alpha_n$.

A sequence $\alpha$ is bounded above by $M$ if $\alpha_n \leq M$ for all $n$. It is bounded below if $\alpha_n \geq m$.

**Ex:** $\alpha = \left( \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots \right)$, $\alpha_n = \frac{n}{n+1}$. Is it monotone? Bounded? Above? Below?

- Bounded above by 1 since $\frac{n}{n+1} < 1 \iff n < n+1$.
- Bounded below by $\frac{1}{2}$ since $\alpha_{n+1} \geq \alpha_n \iff \alpha_{n+1} - \alpha_n \geq 0$.

\[
\alpha_{n+1} - \alpha_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0.
\]

**Ex:** $\alpha = (1, \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots)$, $\alpha_n = \left( \frac{2}{n+1} \right)^{n-1}$.

- Bounded above by 1, bounded below by $\frac{1}{1000}$ (or $\frac{1}{2}$).

- Neither increasing nor decreasing.

**Thm:** If $\alpha$ is increasing and bounded above by $M$, then it has a limit $L$ and $L \leq M$.

If $\alpha$ is decreasing and bounded below by $m$, then $L \geq m$.

This method tells you to converge, but not what the limit is. (Aside: the smallest upper bound is the true limit but hard to find.)
Tool 5: Squeezing

Lemma: Let \( a_j \leq b_j \leq c_j \) be sequences with \( a_n \leq b_n \leq c_n \) for all \( n \) and suppose \( \lim a_n = \lim c_n = L \). Then \( \lim b_n = L \).

Picture:

Eventually, \( a_n \) and \( c_n \) are close to \( L \), thus so does \( b_n \).

Ex: \( b_n = \frac{\sin n}{n} \to 0 \). Why? \(-1 \leq \sin n \leq 1\) so \( \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \).\[\lim n = 0 \to 0 \]

Ex: \( \frac{n!}{n^n} \to 0 \). Why? \( \frac{n!}{n^n} = \frac{1}{n} \left( \frac{2}{n} \right) \left( \frac{3}{n} \right) \ldots \left( \frac{n}{n} \right) \) \[\leq \frac{1}{n} \to 0 \]

Ex: \( \frac{100^n}{n!} \to 0 \). \[\frac{100^n}{n!} = \frac{100}{n} \frac{100}{n-1} \ldots \frac{100}{2} \frac{100}{1} \]

\[\leq \left( \frac{100}{10^7} \right)^n \to 0 \quad \text{if } n > 10^7 \]

Can use squeeze for lots of things but takes insight and creativity! + practice.


\[
\begin{aligned}
\text{Ex:} \quad 0 \leq \frac{2^n}{3^n + n^2} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \rightarrow 0 \quad \text{so} \quad \frac{2^n}{3^n + n^2} \rightarrow 0 \\
\text{by} \quad 3^n + n^2 \rightarrow 3^n, \quad r = \frac{2}{3} < 1 \\
\text{Can use squeeze to isolate leading terms of numerator + denominator.}
\end{aligned}
\]

\[
\begin{aligned}
\text{Ex:} \quad 0 \leq \frac{2^n + 15n}{3^n} \\
2^n + 15n < 2^n, \quad \text{ehm.} \\
\text{let} \quad 15n < 2^n \quad \text{for n by egraph} \\
\text{so} \quad 2^n + 15n < 2^n + 2^n = 2 \cdot 2^n \quad \text{for n by egraph} \\
\frac{2^n + 15n}{3^n} \leq 2 \cdot \left(\frac{2}{3}\right)^n \rightarrow 0. \quad \boxed{\text{Good egraph}}.
\end{aligned}
\]

\[
(\text{Aside: All rules apply if you can use them for n big enough, since covariance + limit only depend on the } \infty \text{ tail of the sequence).}
\]

\[
\begin{aligned}
\text{Ex:} \quad 0 \leq \frac{2^n}{3^n - n^2} \leq \frac{2^n}{\frac{1}{2}(3^n)} = 2 \cdot \left(\frac{2}{3}\right)^n \rightarrow 0 \\
3^n - n^2 > \frac{1}{2}(3^n) \quad \text{for n large enough}.
\end{aligned}
\]

\[
\text{PRACTICE:} \quad \text{It's hard.}
\]

People need lots of help w/ algebra of inequalities!

\[
\begin{array}{c}
\text{If } a \leq b \Rightarrow ac \leq bc \text{ if } c > 0 \\
\text{If } ac \leq bc \text{ if } c < 0 \\
\frac{1}{a} \geq \frac{1}{b} \text{ if } ab > 0 \quad \text{strictly}
\end{array}
\]

\[
\text{Be careful!}
\]
Two big consequences of squeeze theorem:

1) Abs. Value Test: If $|a_n| \to 0$ then $a_n \to 0$.

Ex: $\left(\frac{-1}{2}\right)^n \to 0$ by $(\frac{1}{2})^n \to 0$.

Why? For any $x, \# x \leq |x|$ so $-|a_n| \leq a_n \leq |a_n| \implies -0 = 0 \to 0$.

2) (Not in book but awesome) Limit Ratio Test

If $\lim \frac{a_{n+1}}{a_n}$ exists then $\lim a_n = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{DNE if } |r| > 1 \\ \text{DNE if } r = 1 \text{ or } \text{or } r = -1 \end{cases}$

Ex: $2^n \cdot 3^{n+1} = \left( \frac{2}{3} \right)^n \lim \text{DNE}$.

Remember: $\lim a_n = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{DNE if } |r| > 1 \\ \text{DNE if } r = 1 \text{ or } \text{or } r = -1 \end{cases}$

Ex: $\lim \frac{2^{n+1}}{3^{n+1}} = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \lim \text{DNE}$.

Ex: $\lim \frac{a^{n+5}}{3^{n+7}}$.

So $\lim \frac{a^{n+1}}{a_n} = \frac{a^{n+5}}{3^{n+7}} \implies \frac{a^{n+1}}{3^{n+5}} \to \frac{2}{3} = \frac{2}{3} = r$.  

Conclude squeeze theorem or compare ratios.
Ex. \( \frac{1000^n}{n!} = a_n \)

\[ \frac{a_{n+1}}{a_n} = \frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^n}{n!}} = \frac{1000}{n+1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

For \( \lim a_n = 0 \).

Ex. \( a_n = n \)

\[ \frac{a_{n+1}}{a_n} = \frac{n+1}{n} \rightarrow 1 = r \quad \text{ratio test inconclusive.} \]

Ex. \( a_n = \frac{1}{n} \)

\[ \frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 = r \]

One more: Notice that \( e^x = \lim (1 + \frac{x}{n})^n = (\frac{n+X}{n})^n \)

Ex. \( \lim (\frac{n-1}{n})^n = e^{-1} \).

Ex. \( \lim (\frac{a_n+1}{n})^n = \lim 2^n \cdot (\frac{n+\frac{1}{2}}{n})^n = \lim 2^n \cdot \lim (\frac{n+\frac{1}{2}}{n})^n = \lim 2^n \cdot e^{\frac{1}{2}} \)

So \( \lim (\frac{a_n+1}{n})^n \quad \text{DNE} \).
Actual def'n of limit (see Appendix D)

I gave intuitive def'n of limit: \((a_n) \to L\) if the sequence gets arbitrarily close to \(L\). Let's make it into math.

distance to \(L\) is \(|a_n - L|\). Arbitrarily close: for any degree of accuracy \(\varepsilon\), we can get closer than \(\varepsilon\) for sufficiently small \(n\).

Rigorously get \(\varepsilon\)-\(N\) steps: there is some time \(N\) st \(\varepsilon\) for all later times \(n > N\).

we have \(|a_n - L| < \varepsilon\).

Def: \(\lim_{n \to \infty} a_n = L\) if for all "accuracies" \(\varepsilon > 0\) there is some "recipe" \(N\) such that \(|a_n - L| < \varepsilon\) for all \(n > N\).

Ex: \(\lim_{n \to \infty} \frac{n-1}{n} = 1\)? Well, \(|1 - a_n| = \left|\frac{1}{n}\right| = \frac{1}{n}\).

When \(\varepsilon_{001} = \frac{1}{100}\) choose \(N \geq 100\) for \(|1 - a_n| < \varepsilon = \frac{1}{100}\).

since \(\frac{1}{n} < \frac{1}{100} \iff n \geq 100\), \(N = 100\) works but \(N = 1000\) or higher.

When \(\varepsilon_{0001} = \frac{1}{10000}\) choose \(N = 10000\) or higher.

In general, \(\frac{1}{n} < \varepsilon \iff n \geq \frac{1}{\varepsilon}\) so can choose \(N = \frac{1}{\varepsilon}\).

This definition says: if you can solve for \(N\) given \(\varepsilon\) then you can confirm that \(a_n \to L\).

Ex: \(\lim_{n \to \infty} \frac{1}{3^n} = 0\) \(
0 - \frac{1}{3^n} = \left|\frac{1}{3^n}\right| = \frac{1}{3^n}, \quad \frac{1}{3^n} < \varepsilon \iff 3^n > \frac{1}{\varepsilon}
\)

so \(N = \lceil \log_3 \left(\frac{1}{\varepsilon}\right) \rceil\) works

\(\varepsilon_{001} \Rightarrow N = \lceil \log_3 (100) \rceil = 5\)
\(\varepsilon_{0001} \Rightarrow N = \lceil \log_3 (1000) \rceil = 9\) or 9?
This seems abstract but is really what you want in practice!!
If you're approximating something you want to know how far do I have to go before I am within the desired accuracy bounds?

\[ N \leq \frac{1}{\varepsilon} \]

Ex: How many terms of \( a_n = \frac{2n-1}{n+3} \) must I take before I am within .01 of the limit? Within \( \varepsilon \) of the limit?

\[ \lim_{n \to \infty} \frac{2n-1}{n+3} = 2. \]

\[ \left| \frac{2n-1}{n+3} - 2 \right| = \left| \frac{2n-1 - 2(n+3)}{n+3} \right| = \left| \frac{-7}{n+3} \right| = \frac{7}{n+3} \]

\[ \frac{7}{n+3} < \varepsilon \iff \frac{n+3}{7} > \frac{1}{\varepsilon} \iff n+3 > \frac{7}{\varepsilon} \iff n > \frac{7-3}{\varepsilon} - 3 \]

So \( N = \left\lceil \frac{7-3}{\varepsilon} \right\rceil \)

E.g., if \( \varepsilon = .01 \) then \( N = \left\lceil \frac{700-3}{.01} \right\rceil = 697 \).

697 is a good enough approximation. \( a_{697} = 1.99 \)

Again, manipulating inequalities is the hard part for many so practice. Inequalities are harder than equalities!

Rmk: Some people use the definition \( |a_n - L| < \varepsilon \) instead of \( |a_n - L| < \varepsilon \).

Thus 697 is not a good enough, need \( N = 698 \). This is just an approximation.

\[ N = \left\lceil \frac{7}{\varepsilon} - 3 + 1 \right\rceil \]

This looks better, but this is just stupid. It will not work above this edge case, and always use \( \varepsilon \).
What does \( \lim_{n \to \infty} a_n = \infty \) mean? Differently, more specific than \( \lim a_n = \text{DNF} \).

It means you get extremely large. For any potential bound \( M > 0 \), there is a moment \( N \) such that all later moments \( n \geq N \) you surpass the bound \( a_n > M \).

**Ex:** \( a_n = n \) \( \lim a_n = \infty \).

**Ex:** \( a_n = 2^n \) \( \lim a_n = \infty \) \( N = \lceil \log_2 M \rceil \).

**Ex:** \( a_n = (2)^n \) \( \lim a_n = \text{DNF} \),

- gets negative too,
- don't stay \( > M \).

**Ex:** \( a_n = (-1)^n \) \( \lim a_n = \text{DNF} \)

- doesn't get large.

Similarly, \( \lim a_n = -\infty \) if get extremely small, "mgt" (i.e., largely negative).
Recursive Sequences

Here's the method which might work:

**Step 1:** If there were a limit, compute it.

**Step 2:** Use the limit as a potential upper/lower bound.

**Step 3:** Use the limit to prove necessity/denying.

*Example:*

\[ a_1 = 1 \]

\[ a_{n+1} = \frac{1}{2} a_n + 3 \]

**1** Replace \( a_n \) and \( a_{n+1} \) with \( L \) (i.e., take limit of both sides):

\[ L = \frac{1}{2} L + 3 \]

\[ \frac{1}{2} L = 3 \]

\[ L = 6 \]

If \( L \) exists, \( L = 6 \).

*Now \( a_n \leq 6 \), maybe \( 6 \) is an upper bound.

**2** If \( a_n \leq 6 \), then \( \frac{1}{2} a_n + 3 \leq \frac{1}{2} (6) + 3 = 3 + 3 = 6 \) so \( a_{n+1} \leq 6 \).

\[ a_1 \leq 6 \Rightarrow a_2 \leq 6 \Rightarrow a_3 \leq 6 \Rightarrow \ldots \Rightarrow a \leq 6 \text{ for all } n \]

Base case can be confirmed.

**3** Maybe it's increasing:

\[ a_{n+1} - a_n > 0? \]

\[ a_{n+1} - a_n = \frac{1}{2} a_n + 3 - a_n = -\frac{1}{2} a_n + 3 \geq 0 \]

\[ -\frac{1}{2} a_n + 3 \geq 0 \Rightarrow \frac{1}{2} a_n \geq -3 \Rightarrow -a_n \leq -6 \Rightarrow a_n \leq 6 \]

But we proved that! Great. So \( a_n \leq a_{n+1} \).

Increasing + Bounded \( \Rightarrow \) have a limit \( L \), (And \( L = 6 \)).

*Example:*

\[ a = (3, \sqrt{5}, \sqrt{5}, \sqrt{5}, \ldots) \]

\[ a_1 = 3 \]

\[ a_{n+1} = 5 \sqrt{a_n} \]

If \( L \) exits,

\[ L = 5 \sqrt{L} \]

\[ L^2 = 25 \]

\[ L = 5 \]

Now continue...

"If \( \sqrt{L} \) will limit ratio test work? No - inconclusive. Only gives limit 0."
Series

We make sense of an infinite sum as a special kind of sequence.

Ex: \(a_n = \frac{1}{2^n}\). Then \(a_1 + a_2 + a_3 + a_4 + \ldots = \sum_{n=1}^{\infty} a_n = 2 \cdot \frac{1}{2^n}\).

**Sigma notation for infinite sum**

\[\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\]

Let \(S_n = \sum_{k=1}^{n} a_k\) the partial sum.

This is the sequence \(S = (\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots)\)

\[S_n = 1 - \frac{1}{2^n}\]

\[\lim_{n \to \infty} S_n = 1\] so we say \(\sum_{n=1}^{\infty} \frac{1}{2^n} = 1\).

**Def:** Given a sequence \(a = (a_1, a_2, \ldots)\) its series is the sequence of partial sums \(S = (s_1, s_2, s_3, \ldots)\). We say the series converges/diverges if \(S\) does, and write \(\sum_{n=1}^{\infty} a_n\) for the limit of \(S\).

Ex: \(a_n = n\). \(S_1 = 1\) \(S_2 = 1+2=3\) \(s_3 = 1+2+3=6\) \(\lim_{n \to \infty} S_n = \infty\) \(\sum_{n=1}^{\infty} a_n = \infty\) (or DNE).

**Big question:** Does \(S\) converge? What is the limit? Previous tools still work, be ready to adapt them to find limits.

**Tool 1:** If \(\sum_{n=1}^{\infty} a_n\) converges then \(\lim_{n \to \infty} a_n = 0\).

ie. if \(\lim_{n \to \infty} a_n \neq 0\) or DNE, then \(\sum_{n=1}^{\infty} a_n\) diverges.

Because \(S\) just won't stay put!

Ex: \(\sum_{n=1}^{\infty} \frac{n^2+1}{n^2-1}\) diverges.

Ex: \(\sum_{n=1}^{\infty} (-1)^n\) \(S = (-1, 0, -1, 0, -1, 0, \ldots)\) diverges.
WARNING: Just because \( a_n \to 0 \) does NOT imply \( \sum_{n=1}^{\infty} a_n \) converges.

I CAN'T EMPHASIZE THIS ENOUGH. I'll highlight examples.

Tool 1: Say, when divergent, NOT when convergent.

Tool 2: Geometric series: sum of a geometric sequence

\[
\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + \ldots
\]

a is initial term

\( r \) is common ratio

Indexing!!!

If \( |r| \geq 1 \) the \( \lim_{n \to \infty} ar^n \neq 0 \) so \( \sum_{n=0}^{\infty} ar^n \) diverges.

If \( |r| < 1 \) it converges, and has \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \).

Ex: \[
2 - \frac{6}{5} + \frac{18}{25} - \frac{54}{125} + \ldots
\]

\[
= 2 - 2(\frac{3}{5}) + 2(\frac{3^2}{5^2}) - 2(\frac{3^3}{5^3}) + \ldots
\]

\[
a = 2, \quad r = \frac{-3}{5}
\]

\[
\sum = \frac{a}{1-r} = \frac{2}{1 - \frac{-3}{5}} = \frac{2}{\frac{2}{5}} = \frac{10}{8}.
\]

Why \( \sum = \frac{a}{1-r} \)? Recall formula for \( s_n \).

\[
a + r(a + ar + ar^2 + \ldots + ar^n) = a + ra + \ldots + ar^{n+1}
\]

\[
a + rs_n
\]

\[
\begin{align*}
\sum & = \frac{a}{1-r} \\
L & = a
\end{align*}
\]

Ex: \[
\sum_{n=0}^{\infty} \frac{3^n}{2^{n+3}} = ?
\]

\[
\frac{3}{2^3} = \frac{(\frac{3}{2})^n}{2^3 \cdot 2^n} = \frac{1}{8} \cdot \left(\frac{3}{2}\right)^n
\]

\[
a = \frac{3}{2}, \quad b = 2, \quad s = \frac{3}{2} > 1 \text{ so diverges!}
\]
Ex: \[ \sum_{n=0}^{\infty} 6 \left( \frac{4}{5} \right)^n = \frac{6}{1 - \frac{4}{5}} = \frac{6}{1/5} = 30 \]

Ex: \[ \sum_{n=3}^{\infty} 6 \left( \frac{4}{5} \right)^n = 6 \left( \frac{4}{5} \right)^3 + \ldots = \frac{6 \left( \frac{4}{5} \right)^3}{1 - \left( \frac{4}{5} \right)^3} = 30 \left( \frac{4}{5} \right)^3 \]

Ex: \[ \sum_{n=0}^{\infty} \left( \frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} = 2 \cdot \frac{3}{2} = \frac{3}{2} \]

not gonna

sun of limit = limit of sum
when both defined

Ex: For which \( x \) does \( \sum_{n=1}^{\infty} \left( \frac{x+2}{3} \right)^n \) converge?

\[ r = \frac{x+2}{3} \]

When is \( |r| < 1 \)?

\[ -1 < \frac{x+2}{3} < 1 \quad \Rightarrow \quad -3 < x+2 < 3 \]

|\[ -5 < x < 1 \]|

WARNING: Don't try to apply when \( |r| > 1 \).

\[ r = -1 \]

\[ \sum_{n=0}^{\infty} (-1)^n = \frac{1}{1+1} = \frac{1}{2} \quad \text{?} \quad \text{No way.} \]

Tool 3: \( p \)-test (special case of integral test, soon.)

Oh:

\[ \sum_{n=1}^{\infty} \frac{1}{n} = ? \]

Poll

It's \( \infty \)!

WARNING APPLIES.

Lots of small things add up to \( \infty \).

\[ \sum_{n=2}^{\infty} \frac{1}{n^2} = ? \]

It's \( \pi^2/6 \). Duh! (Advanced complex analysis + Fourier theory!)

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = ? \]

It's \( \infty \). \( \frac{1}{\sqrt{n}} \geq \frac{1}{n} \) so comparison test would diverge soon.

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \]

|\[ \text{converge if } p > 1 \]

|\[ \text{diverges if } 0 < p \leq 1 \]|

doesn't say what the limit is.

Giving the flavor,
Here is Oresme's proof that \( \sum_{n=1}^{\infty} \frac{1}{n} = \infty \) left the harmonic series.

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{16} + \ldots
\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots
\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \rightarrow \infty
\]

(Modern version of this argument: Cauchy Condensation test, late 19th century, MAT 316)

Euler did \( \sum \frac{1}{n^2} = \frac{\pi^2}{6} \).)

---

Having seen some of these tests, here's the big one! 

**Ratio Test:** Suppose that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \). Then if \( r > 1 \) the \( \sum_{n=1}^{\infty} a_n \) diverges, if \( r < 1 \) then \( \sum_{n=1}^{\infty} a_n \) converges, and if \( r = 1 \) the test is inconclusive— it might go either way, need to do something else.

This is your first line of defense! Only when this fails look elsewhere! (Always detester the paradox of convergence of a poor series later.)

Basically, \( \sum a_n \) is "close" to a geometric series and the behavior is similar (or \( r < 1 \) and \( \sum a_n \) is much smaller than any convergent geometric series.)

**Ex:** \( \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots \) perhaps?

- \( a_{n+1} \over a_n = \frac{(n+1)!}{n!} = \frac{1}{n+1} \rightarrow 0 \) so converges.

**Ex:** \( \sum_{n=0}^{\infty} \frac{a^n}{n!} \) \( a_{n+1} \over a_n = \frac{a}{n+1} \rightarrow 0 \) so converges \( ( = e^a ) \)

For which \( x \) is \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) convergent?

An: All \( x \in \mathbb{C} \)
Ex: \[ \sum_{n=0}^{\infty} \frac{3^n+7}{4^n+6} \]

\[ \frac{a_{n+1}}{a_n} = \frac{3^{n+1}+7}{4^{n+1}+6} \cdot \frac{4^n+6}{3^n+7} \]

\[ \frac{3^{n+1}}{3^n} = \frac{9^{n+1}}{3^n} > 3 \quad \text{for } n \geq 1 \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{3}{4} < 1 \]

\[ \text{has some limit as } \text{Converge} \]

\[ \text{No idea what the limit is.} \]

Ex: \[ \sum_{n=0}^{\infty} \frac{n+5}{n^2+3n+2} \]

\[ \frac{a_{n+1}}{a_n} = \frac{n+6}{n+5} \cdot \frac{n^2+3n+2}{(n+1)^2+3(n+1)+2} \]

\[ \frac{n^2+3n+2}{n^2+3n+2} = 1 \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \]

\[ \text{Ratio Test: inconclusive} \]

In general, \[ \sum_{n=0}^{\infty} \frac{1}{n^p} \]

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p \rightarrow 1 \]

\[ \text{p-test handles one edge case where ratio test is useless!} \]

We'll see soon that \[ \frac{n+5}{n^2+3n+2} \approx \frac{1}{n} \]

which diverges, so former diverges too.

Ex: \[ \sum_{n=0}^{\infty} \frac{n+5}{3^n} \]

Ex: For which \( x \) is \[ \sum_{n=0}^{\infty} n! \cdot x^n \] convergent?

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)x}{n!} \]

\[ \lim_{n \to \infty} (n+1)x = \begin{cases} \infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases} \]

Only for \( x = 0 \), \( \frac{1}{n+0} \rightarrow 0 \).

\[ \sum_{n=0}^{\infty} \frac{1}{n+0} \rightarrow 0 \]
Series convergence tests

3 main questions:

1. Can we show something converges? (Yes, or it wouldn't be a test)
2. If convergent, can we find limit exactly?
3. Can we estimate error? If we want to find \( N \) st. \( |S_N - L| < \varepsilon \), can we do it? How many terms to sum until within .01 of limit?

<table>
<thead>
<tr>
<th>Test</th>
<th>Find Limit</th>
<th>Estimate Error</th>
<th>Other Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric sum</td>
<td>✔</td>
<td>✔</td>
<td>Super specific</td>
</tr>
<tr>
<td>p-test</td>
<td></td>
<td></td>
<td>Great</td>
</tr>
<tr>
<td>Ratio test</td>
<td></td>
<td>✔</td>
<td>Reasonably specific</td>
</tr>
<tr>
<td>Integral test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alternating sum test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Companion test</td>
<td></td>
<td></td>
<td>Powerful tool, in conjunction with other tests.</td>
</tr>
</tbody>
</table>

- You won't be finding the limit exactly very often in this class!
- But you sure can approximate it very closely! Please!

Telecopy sums
(a stupid hack)         ✔          ✔

Super specific

Let's go back and look at geometric series again, focusing on error:

\[
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + ar^{n+1} + \ldots
\]

We know the exact limit, \( L = \frac{a}{1-r} \) when \( |r| < 1 \), so finding an approximation of the sum is silly. But we can still ask: how far is \( S_N \) from the true sum? \( L - S_N = ar^{N+1} + \ldots \Rightarrow \sum_{n=N+1}^{\infty} ar^n = \sum_{n=N+1}^{\infty} ar^{n+1} \)

for \( A = ar^{N+1} \). So

\[
L - S_N = \frac{A}{1-r} = \frac{a(1-r)^{N+1}}{1-r}
\]

With 100 terms, error is \( L - S_n \approx \frac{a}{1-r} \). Each term multiples error by \( r \) (e.g. \( \frac{2}{3} \))
Ex: Consider the series \( \sum_{n=0}^{\infty} 6 \cdot (\frac{1}{2})^n \). How many terms are needed before the partial sum is within 0.1 of the overall sum?

Ans: \( \text{Error} = \frac{a}{1-r} = \frac{6}{1 - \frac{1}{2}} = 12 \cdot (\frac{1}{2})^N < \frac{1}{100} \)

\( \Rightarrow \frac{\frac{1}{10}}{\text{area of } f(t)} < \frac{1}{600} \Rightarrow 2^N > 600 \Rightarrow N > \log_2(600) \approx 9.1 \)

So \( \sum_{n=0}^{10} 6 (\frac{1}{2})^n \) is close enough.

Integral test - invoke the average power of Calc 2. This is the series version of extrema to functions.

Why does \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converge? Let \( f(t) = \frac{1}{t^2} \)

\( \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \text{area of } \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \text{area of } \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots 
\)

So \( \sum_{n=2}^{\infty} \frac{1}{n^2} < \int_{1}^{\infty} f(t) \, dt < \sum_{n=1}^{\infty} \frac{1}{n^2} \)

Now \( \int_{1}^{\infty} \frac{1}{t^2} \, dt = \left[ -\frac{1}{t} \right]_{1}^{\infty} = 0 - \frac{1}{1} = -1 \)

This \( 1 < \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + 1 = 2 \).

This is like the spreading test, only it squeezes an integral between a sum or vice versa!

Does this imply that the sequence converges? Partial sum eventually between 1 and 2 but do they settle?

**MONOTONE + BOUNDED = CONVERGENCE**

(Increasing) \( \rightarrow \) \( \text{became } f(t) \geq 0 \).

Not only do we get convergence, we get a bound. Better still, we bound the error too!

\[ L - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \sum_{n=N}^{\infty} \frac{1}{t^2} \, dt = \frac{1}{t} \left[ \frac{1}{n} \right]_{N}^{\infty} = \frac{1}{N} \]

Want within 0.01? Let \( N = 100 \). \( \Box \) S66 INSERT
You can really see this:

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.644934... \]

Take 5 terms get:

- 10: 1.4636
- 100: 1.54977
- 1000: 1.63498
- 10000: 1.6439
- 100000: 1.6448...

Error is just around \( \frac{1}{n} \).
Why does \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverge.

\[
\sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{t} \, dt < \sum_{n=1}^{\infty} \frac{1}{n}
\]

\[
\text{Int} [1, \infty] = \ln \infty - \ln 1 = \infty.
\]

Since \( \infty = \sum_{n=1}^{\infty} \frac{1}{n} \), the limit is infinity!

How many terms of \( \sum_{n=1}^{\infty} \frac{1}{n} \) do I need to get over 1000?

\[
\sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{n+1} \frac{1}{t} \, dt = \ln(n+1) - \ln 1 = \ln(n+1) \approx 1000
\]

\( n+1 > e^{1000} \) \( \Rightarrow \) helps us think a LOT.

\( n+1 \) grows too \( \infty \) \( \Rightarrow \) very slowly.

The integral test:

- \( f(t) \) continuous, decreasing, positive, \( a_n = f(n) \).

Then:
- If \( \int_{1}^{\infty} f(t) \, dt \) converges, so does \( \sum_{n=1}^{\infty} a_n \).
- Also get error bound.
- \( \int_{1}^{\infty} f(t) \, dt < \sum_{n=1}^{\infty} a_n < \int_{1}^{\infty} f(t) \, dt + q \).
- \( \sum_{n=1}^{\infty} a_n \) diverges, so does \( \sum_{n=1}^{\infty} \frac{1}{n} \).

Application: \( p \)-test with error bounds.

Why assumed? If \( f \) not decreasing,

If \( f \) not positive, several issues.
Application 1: p-test w/ error bounds.

Then: \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges when \( p > 1 \)

diverges when \( 0 < p \leq 1 \)

Why? \( \int_1^{\infty} \frac{1}{t^p} \, dt = \int_1^{\infty} t^{-p} \, dt = \frac{t^{-p+1}}{-p+1} \bigg|_1^{\infty} = \frac{1}{p-1} \)

Conv or diver?:

Ex: \( \sum_{n=1}^{\infty} \frac{5}{n^{0.3}} \)

Ex: \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

Ex: \( \sum_{n=1}^{\infty} \frac{1}{n^{100}} \)

Ex: Find \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) correct to 3 decimal places.

This means \( \text{error} \leq 0.0005 \)

\[ \text{error} \leq \int_1^N \frac{1}{t^4} \, dt = \frac{1}{3t^3} \bigg|_1^N = \frac{1}{3N^3} \leq 0.0005 \]

\( \iff N^3 \geq \frac{10000}{15} \quad N > \sqrt[3]{\frac{10000}{15}} \approx 8.7 \) so \( N = 9 \) works.

Ex: \( \sum_{n=1}^{\infty} \frac{1}{n \cdot \ln n} \) ?

\[ \int_2^{\infty} \frac{1}{x \ln x} \, dx = \ln (\ln x) \bigg|_1^{\infty} = \infty \quad \text{diverged} \]

Why 2? \( \ln 1 = 0 \)

\( \frac{1}{\ln x} \) not defined.

Ex: \( \sum_{n=1}^{\infty} \frac{1}{n (\ln n)^p} \)

Conv if \( p > 1 \)

Div if \( p \leq 1 \).
Comparison Test

Is it similar to something you know converges or diverges?

**Use the Comparison Test.**

**Ex:** \[ \sum_{n=0}^{\infty} \frac{1}{3^n+5} \]

Well, \[ \sum_{n=0}^{\infty} \frac{1}{3^n} \text{ converges, } 0 < \frac{1}{3^n+5} < \frac{1}{3^n} \] so this should converge too.

**Thm:** Suppose \( 0 \leq a_n \leq b_n \) for all \( n \). If \( \sum_{n=0}^{\infty} b_n \text{ converges}, \) then \( \sum_{n=0}^{\infty} a_n \) converges.

By contrapositive, if \( \sum_{n=0}^{\infty} a_n \text{ diverges}, \) then \( \sum_{n=0}^{\infty} b_n \) diverges.

**Ex:** \[ \sum_{n=0}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \]

\( \sum_{n=0}^{\infty} \frac{1}{n} \) diverges so \( \sum_{n=0}^{\infty} \frac{1}{2n} \) diverges. But \( \frac{1}{2n} \to \frac{1}{2n} = \sum_{n=0}^{\infty} \frac{1}{2n-1} \) diverges.

Why \( 0 < \frac{1}{n} \) always, but \( \frac{1}{n^2} < \frac{1}{n} \) always converges.

**Ex:** \[ \sum_{n=2}^{\infty} \frac{1}{3^n-5} \]

Now \( \frac{1}{3^n} > \frac{1}{3^n-5} \), \( \text{ for } 3 < 3^n \)

But \( 3^n-5 > \frac{3^n}{10} \) (for \( n \geq 2 \)) and \( \sum_{n=2}^{\infty} \frac{1}{3^n} \text{ converges} \)

\( \Leftrightarrow \frac{1}{3^n-5} < \frac{10}{3^n} \)

If you're similar to something, you're probably \( a_n < \lambda b_n \) for some \( \lambda \), 

\( a_n > \mu b_n \) for some \( \mu \)

**Ex:** \[ \sum_{n=1}^{\infty} \frac{1}{a_n+1} \]

\( \sum_{n=1}^{\infty} \frac{1}{3^n} \) diverges.

\( \sum_{n=0}^{\infty} \frac{1}{3^n-5} \). Hmm, \( n=0 \), not positive and \( 3^n-5 \) might be \( > \frac{3^n}{10} \).

Is that a problem?
No. \[ \sum_{n=0}^{\infty} \frac{1}{3^{n} - 5} = \frac{1}{3^0 - 5} + \frac{1}{3^1 - 5} + \sum_{n=2}^{\infty} \frac{1}{3^{n} - 5} \]

**Big Idea:** In math, we say something is **eventually true** if, after some point, it is always true. I.e., \( \lim a_n = L \) if for each \( \varepsilon > 0 \), \( |a_n - L| < \varepsilon \) eventually.

Different from English, you'll eventually get an A in math, but just once, not forever.

If \( a_n \leq b_n \) eventually, then can also use comparison test.

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n
\]

Somefoot note: the infinite tail when things might go wrong.

Same for all tests! If eventually alternating, can use AST.

**Ex:** \[ \sum_{n=1}^{\infty} \frac{n}{e^{15}} \]

**Ratio Test:**

\[
f(t) = \frac{n+1}{e^{15}}
\]

Test is positive, increasing, eventually decreasing.

\( f(t) = e^{15} \)

It's negative for \( t > 15 \)

**Ex:** \[ \sum_{n=0}^{\infty} \frac{1}{3^{n} - 1000000} \]

Is eventually less than \( \sum_{n=0}^{\infty} \frac{2}{3^n} \)

\( \frac{1}{2} \) is eventually less than \( 3^{n} - 1000000 \)

Also, \( 3^{n} - 1000000 \) eventually \( \geq 0 \).

The big idea make all the tests much easier to apply.
Also, integral test can be used to compare error.

If \( 0 < a_n \leq b_n \), then the error after \( N \) terms is

\[
\sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} b_n.
\]

Ex:
Find a reasonably efficient \( N \) such that \( \sum_{n=0}^{N} \frac{1}{3^n + n} \) is within .001.

Well \( \sum_{n=0}^{\infty} \frac{1}{3^n} \leq \sum_{n=0}^{\infty} \frac{1}{3^n} \) \(<\) geometric, \( a=1 \), \( r=\frac{1}{3} \)

\[
\text{error} = \frac{a}{1-r} \left( r^n \right) = \frac{1}{3} \left( \frac{1}{3} \right)^{N+1} = \frac{1}{3} \cdot \left( \frac{1}{3} \right)^N
\]

so \( \frac{1}{2} \cdot \left( \frac{1}{3} \right)^N < \frac{1}{1000} \) iff \( \left( \frac{1}{3} \right)^N < \frac{1}{500} \)

\( \Rightarrow 3^N > 500 \)

\( \Rightarrow N > \log_3(500) \approx 5.5 \)

so \( N=6 \) will suffice for geometric series

\( \Rightarrow \) suffice for \( \frac{1}{3^n + n} \) too.

Ex:
\( \sum \frac{n2^n}{15^n} \). Another good trick — make \( r \) slightly bigger (but still < 1).

Should be like \( \frac{2^n}{15^n} = \left( \frac{2}{15} \right)^n \).

Ratio test:

\[
\frac{\frac{n2^n}{15^n}}{\frac{3^n}{15^n}} < \left( \frac{2}{3} \right)^n \text{ when } n \left( \frac{3}{2} \right)^n
\]

Eventually \( r^n > n \) for any \( r > 1 \)

"Expected local polynomial"

For comparison test — first find something similar. Then modify slightly (rescale, change \( r \))

to make a true comparator (eventually).
A slightly looser version of this is the limit comparison test:

If \( \lim \frac{a_n}{b_n} = c \), \( c \neq 0, \infty \)

then \( \sum a_n \) converges \( \Leftrightarrow \) \( \sum b_n \) converges.

I.e., comparable (up to mutltiples) they share convergence properties.

Thus won't compare \( \frac{n2^n}{5^n} \) w/ \( \frac{2^n}{5^n} \) b/c ratio is \( n \to \infty \) (or \( \frac{1}{n} \to 0 \))

but it will compare \( \frac{n^2+26n-12}{n^4-17n+3000} \) with \( \frac{1}{n^2} \) \( (c=1) \)

without needing to do any nasty algebra (which is bigger?)
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots
\]

**Plot**

\[
0 \quad \frac{1}{2} \quad 1
\]

- Decreasing and bounded below
- More or less banding above
- More or less banding below
- Both curves must meet in the middle

\[
1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < \text{limit} < 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}
\]

\[
\text{error term}
\]

**More over**

\[
|a_{N+1}| \text{ is a bound on the error.}
\]

\[
\text{since it goes past the limit!}
\]

**AST:** If \( a \) is a sequence alternating between positive and negative entries, and \( |a_n| \) is decreasing to zero, then \( \sum_{n=1}^{\infty} a_n \) converges.

| \[ |a_{N+1}| \leq |a_N| \] |

\[
\text{error at step } N
\]

**Note:** Pretty easy for an AS to converge if all terms decrease to zero.

\[
\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges!!}
\]

**Exercise:** Compute \( \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \) to within \( 0.1 \). Is \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \approx 0.693 \) anyone?

1 - \( \frac{1}{2} - \frac{1}{4} - \ldots \) = 0.745

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \approx \ln 2. \quad \text{we'll see!}
\]
Exercise: \( \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \)? Converges. To get within 0.1 need \( \frac{1}{\ln (N+1)} < 0.1 \implies \ln (N+1) > 10 \implies N+1 > e^{10} \approx \text{very large!} \)

Does \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converge or diverge? \( \frac{1}{n \ln n} > \frac{1}{n} \) which diverges, comparison test soon. \( \approx \) an \( \infty \times \infty \) form. 

**Example:** Believe it or not, we have 

\[
4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \pi .
\]

Compute 3 decimal places of \( \pi \).

Need to sum until \( \frac{4}{2N+1} < 0.0005 \). \( \approx \frac{40000}{5} = 8000 \approx 4000 \times \text{term} \)

Let me do it in my head... ah, \( 3.141 \).

A stupid quick trick: telecopy suns.

**Example:** Compute \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) exactly. Anyone.

Ok, \( \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \). So our sum is 

\[
\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots
\]

\( s_4 = 1 - \frac{1}{2} \)

\( s_5 = 1 - \frac{1}{3} \)

\( s_6 = 1 - \frac{1}{4} \)

\( s_N = 1 - \frac{1}{N} \to 1. \)

Cancellation makes life easy!

**Definition:** Let \( a_n \) be a sequence. Let \( b_n = a_n - a_{n+1} \).

The sum \( \sum_{n=1}^{\infty} b_n \) is called a **telescoping sum**.

If \( (a_n) \) converges then \( \sum_{n=1}^{\infty} b_n = a_1 - \lim a_n \) converges.

Else, \( \sum_{n=1}^{\infty} b_n \) diverges.

\( s_N = a_1 - a_{N+1} \to a_1 - \lim a_n \)
Ex: \( \sum_{n=1}^{\infty} \left( \frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1} \right) = \frac{\sin(1)}{1} \) since \( \frac{\sin(n)}{n} \to 0 \)

Ex: \( \sum_{n=1}^{\infty} \sin(n) - \sin(n+1) \) diverges, but does not go to \( \infty \)! \( \sin(n) \to 0 \), never settles on one thing!

Ex: \( \sum_{n=1}^{\infty} \ln \left( \frac{n^2+2}{n^2+2n+3} \right) = \sum_{n=1}^{\infty} \ln \left( \frac{n^2+2}{(n+1)^2+2} \right) = \sum \ln(n^2+2) - \ln((n+1)^2+2) \)

\( a_n = \ln(n^2+2) \to \infty \)

So diverges.

\( a_1 - a_N \) goes to \( -\infty \).

Telescoping sums are easy to recognize

- \( \sin(n) - \sin(n+1) \)
- \( a_n = \ln(n^2+2) \)

Stupidly hard to recognize

Either way they are rare and special, party tricks.

Some very useful applications

In higher math, start reading on Wikipedia maybe.

Have you read about Grandi's series yet?