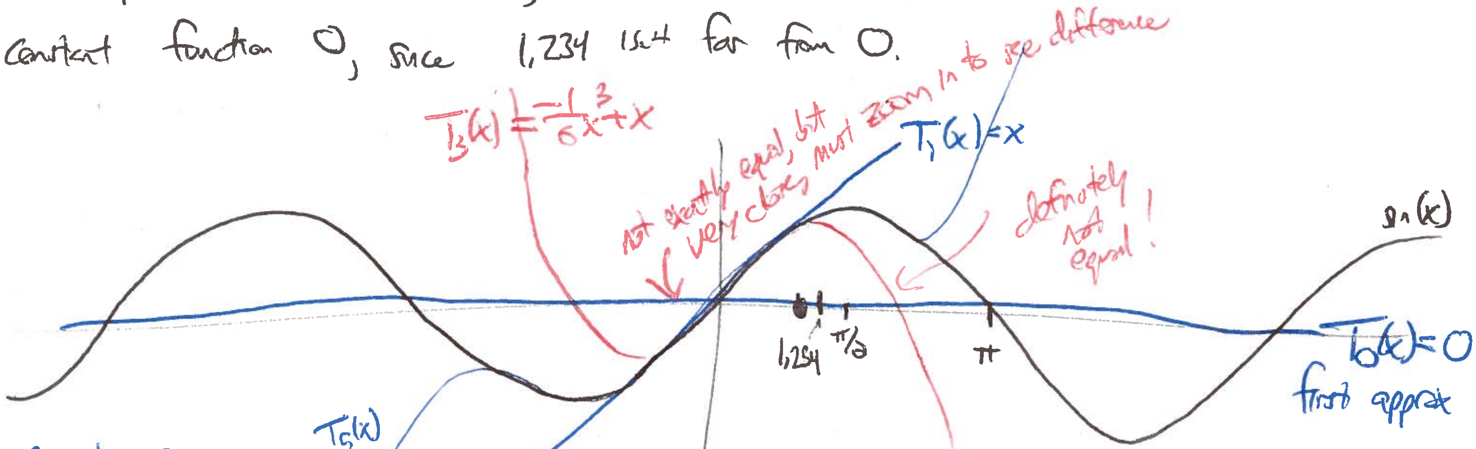
MAT 253 LECTURE NOTES by BEN ELIAS 2019 LECTURE 1

Boring stuff – day 2. Website: on board. Tea! Today – overview.

How does your computer know what $\sin(1.234)$ is? Does it have "sin" the function built in? (Yes/no) No, it approximates it, <u>NEVER</u> knows the exact answer.

Is there an exact answer (Yes/No) – Yes, but it's an irrational number w/ ∞ digits so no human will ever know it beyond X digits or in any form that's not $\sin(1.234)$... not a problem.

How to approximate it? Well, we know $\sin(0) = 0$. So let's approximate w/ constant function 0, since 1.234 isn't far from 0.



$T_0(1.234) = 0$.

Not a very good approx, but a constant function won't be great. What's the <u>best</u> constant function approx? $f(x) = \sin(1.234)$ of course, but can't find that!

Calc I: The best linear approx near 0 is the tangent line.

$\sin'(0) = \cos(0) = 1 \leftarrow$ slope $\qquad \sin(0) = 0 \leftarrow$ intercept

so $T_1(x) = 1 \cdot x + 0 = x \qquad T_1(1.234) = 1.234$

Why is this sure to be wrong? It's > 1.

(Better idea: approximate linearly near $\frac{\pi}{2}$, where we also know $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin'\left(\frac{\pi}{2}\right) = 0$
~~Better idea~~ let's postpone this idea)

T_1 is better than T_0, but still not great. What is the best quadratic polynomial approx of $\sin x$ near 0?? $T_2(x) = ax^2 + bx + c$, which is best?

What made $T_1(x)$ the best linear approx? $T_2(0) = T_1(0) = \sin(0) \leftarrow$ same intercept
$T_2'(0) = T_1'(0) = \sin'(0) \leftarrow$ same slope.

Let's choose $T_2(x)$ so that $T_2''(0) = \sin''(0)$ Note: $T_1''(0) = 0$, because linear.

Well, $\sin''(0) = -\sin(0) = 0$ so $T_2 = T_1$.

What is the best cubic poly approx of sin x near 0? $\sin'''(0) = -\cos(0) = -1$

$$T_3(x) = \frac{-1}{6}x^3 + x. \quad \text{Why } -\frac{1}{6}?$$

$$T_3'(x) = \frac{-3}{6}x^2 + 1 \qquad T_3''(x) = \frac{-6}{6}x + 0 \qquad T_3''' = \frac{-6}{6} = -1. \checkmark$$

$T_4(x) = T_3(x)$ $\sin''''(0) = 0$.

$$T_5(x) = \frac{1}{120}x^5 - \frac{1}{6}x^3 + x$$

$T_3(1.234) \approx .9208\ldots$ $(\sin(1.234) \approx .9438182093)$

$T_5(1.234) \approx .9447\ldots$

$T_7(1.234) \approx .9438001\ldots$

$T_9(1.234) \approx .943185\ldots$

$T_{11}(1.234) \approx .9438182\ldots$

Then approximations $T_n(x)$ are called the <u>n^{th} degree Taylor polynomial of sin x centered at 0</u>. They are functions, and when you plug in 1.234 you get successive approximations of $\sin(1.234)$. Computers can evaluate polynomials easily: add + multiply.

<u>Questions/Ideas</u>: 1) You can compute $T_n(x)$ so long as you can compute derivatives! It's actually pretty easy.

2) Are the numbers $T_3(1.234)$, $T_4(1.234)$, $T_5(1.234)$, $T_6(1.234)$... getting closer to some number? Sure looks like it. <u>Convergent</u>

2') Is that number $\sin(1.234)$? My calculator thinks so. But then what the calculator is doing - it's not giving me $\sin(1.234)$, it is doing $T_n(1.234)$ for some large n. Computers can add, multiply, but not do crazy things.

3) <u>How long do I need to go?</u> (Ask how many steps) It depends how accurate you want to be. Calc. has 8 digits, google has 11... how far do I need to go? (You need to buy the right amount of concrete...)

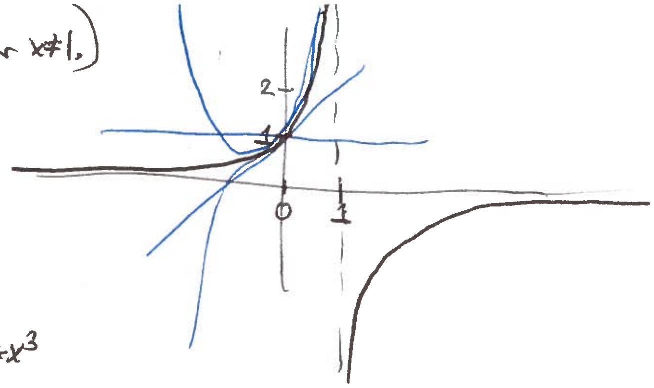
(Ask. Until stabilize? Will it stabilize?)

3') Said another way - can I know the error $|T_n(1.234) - \sin(1.234)|$?

can I <u>bound</u> the error?

This result, the Taylor Inequality estimate, is the most important thing in this class!!!

Here's another function: $f(x) = \dfrac{1}{1-x}$ (for $x \neq 1$.)



$f(0) = 1$ $\quad T_0(x) = 1$

$f'(0) = 1$ $\quad T_1(x) = 1 + x$

$f''(0) = 2$ $\quad T_2(x) = 1 + x + x^2$

$f'''(0) = 6$ $\quad T_3(x) = 1 + x + x^2 + x^3$

$\vdots \qquad\qquad \vdots$

$f(-2) = \dfrac{1}{1-(-2)} = \dfrac{1}{3}$

$$T_n(-2) = \begin{array}{rl} 1 & = 1 \\ 1-2 & = -1 \\ 1-2+4 & = 3 \\ 1-2+4-8 & = -5 \\ 1-2+4-8+16 & = 11 \\ & = -21 \end{array}$$

(are the numbers getting closer to $\frac{1}{3}$? getting closer to anything??)

divergent

When does ~~Taylor~~ taylor approximation work? For this function, $T_n(x)$ will give a better + better approx if $|x| < 1$.

~~More fundamental question: What does it even mean to get closer + closer to something? How can you tell if an approximation will have a limit? called a limit~~

What about the infinite sum? $1 + x + x^2 + x^3 + x^4 + \ldots = T(x)$ the Taylor series for $\dfrac{1}{1-x}$ centered at 0.

forever

Does this even make sense? Is it a function? Does it agree with $\dfrac{1}{1-x}$?

Certainly it encodes all the Taylor polynomials at least.

More fundamental question: What does it mean for a sequence of numbers (like $T_1(1.234), T_2(1.234), \ldots$) to have a limit (i.e. to get closer + closer to some number)?

Do infinite sums make sense? How can you tell whether they will converge to a limit?

series

This is where we begin.

This class:

sequences, series + limits ⇝ power series $1 + 2x + 3x^2 + 4x^3 + \ldots$ ⇝ Taylor series of functions → applications of Taylor series ← all of calculus as far as physics, comp. sci, etc are concerned

Second half: really ~~uses~~ just MAT 251, derivatives, similar to stuff you've done before, fairly mechanistic. BUT THE WHOLE POINT OF THE CLASS!!

First half: new, hard, conceptual. Good to take a break from calc + do something different!

In MAT 251-2: here are some tools, deriv + integral. hammer + saw, Now practice.

Now: here is a blueprint. Figure out what tools you need. MUCH HARDER.

More like a puzzle. GOOD LUCK.

MAT 316-7: WHY the tools work! What's inside the box? What is a number?

Before: Give me the right answer

Now: Give me an approximate answer. "Def:" An approximation is an intelligent wrong answer!

Takes smarts to be "correctly" wrong, gotta know what you're doing.

How tall are you?

Boring stuff: Office hours, library, exams W4,8 (email me if conflict in that week), quizzes weekly on HW day (wednesday)

HW, reading, syllabus etc online, CHEATING. Expect HW harder that 251-2 because takes more time to think, you're learning to think. Solution to one problem won't help you do the next.

Answer HW problems in class on Monday

Def: A sequence is an infinite list of numbers.

$$\bar{a} = (a_1, a_2, a_3, a_4, \ldots) \quad \text{or} \quad (a_n)_{n\geq 1}.$$

Example 1: $\bar{b} = (1,2,3,4,5,\ldots)$ means $b_1 = 1,\ b_2 = 2,\ \ldots,\ \boxed{b_n = n.}$

Example 2: $\bar{c} = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots)$ guess $c_n = ?$ $\quad c_n = \frac{n}{n+1}$.

Example 3: $\bar{d} = (1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \ldots)$ $\quad d_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{2^k} & \text{if } n \text{ is odd, } n = 2k-1 \end{cases}$

Example 4: $\bar{p} = (3,1,4,1,5,9,2,6,5,3,5,\ldots)$

$p_n = ?$ $\quad p_n =$ the n^{th} digit of π ← no formula.

Example 5: $\bar{q} = (5, 17, -3.7, 0, 2000, 6, \ldots)$ need not be a pattern!!

any infinite list ~~of sequences~~ of numbers is a sequence.

Convention: When you see ... there is probably a pattern that is supposed to be obvious, like Ex 1–3. If not, it probably says so.

Example 6: $a_1 = a_2 = 1$ ⋆ $\quad a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. ← a recursive ~~definition~~ formula

So $\bar{a} = (1,1,2,3,5,8,\ldots)$ fibonacci sequence

base case of the recursion

as opposed to a closed formula which says what a_n is exactly w/o reference to other values

Sequences are typically ~~given~~ described via

- list of numbers w/ ... ← guess the pattern, or maybe it doesn't matter
- closed formula
- recursive formula

but some sequences defy description.

Example 7: $\bar{a} = (5, 8, 6, 6, 6, 6, 6, \ldots)$ Ask: $a_n = \begin{cases} 5 & n=1 \\ 8 & n=2 \\ 6 & n\geq 3 \end{cases}$

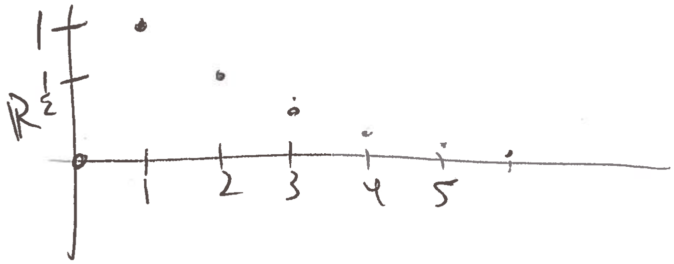
Nonex 8: $\bar{a} = (5, 8, 6)$ must be infinite to be a sequence. Why this defn?

Because we care about behavior as $n \to \infty$.

Ask: Book gives example P_n = population at year n. Hmm

Ex: $a_n = \frac{1}{2^n}$ $\bar{a} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ sequence limits to zero. $\lim_{n\to\infty} a_n = 0$

Plot sequence:



Ask: how to define limit

Keep iterating until it gets better. ✗ get closer + closer

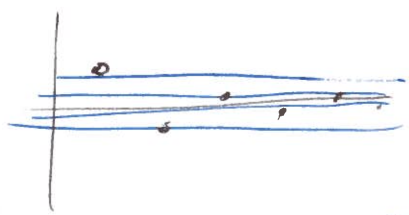
- also can get further away...

the size of the difference

Defn: A sequence $\bar{a}$ has a limit L if $|a_n - L|$ eventually gets and stays arbitrarily close to L. We say $\bar{a}$ is convergent, and converges to L. $\lim_{n\to\infty}(a_n) = L$ " $\lim \bar{a}$. $a_n \to L$.

A sequence w/o a limit is divergent, $\lim_{n\to\infty}(a_n)$ does not exist. (do not say $\lim = \infty$)

Ex: $a_n = \frac{1}{(-2)^n}$

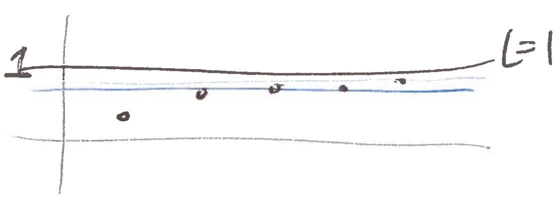


$\lim_{n\to\infty} a_n = 0$

$L = 0$ $|a_n - L| = \frac{1}{2^n}$

less than .01 when $n > 8$
.001 $n > 10$ etc
.0001 $n > 14$

Ex 2: $c_n = \frac{n}{n+1}$



$\lim_{n\to\infty} c_n = 1$

$$|a_n - L| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

less than .01 when $n > 99$
.001 $n > 999$
.0001 $n > 9999$

takes much longer to get close, but does eventually get close.

Ex: $\bar{b} = (5, 8, 6, 6, 6, 6, \ldots)$ limit = ? 6.

less than .01 $n \geq 3$
.001 $n \geq 3$
.0001 $n \geq 3$

sure!

Ex: $\bar{a} = (6,6,0,6,6,0,6,6,0,\dots)$ $\qquad a_n = \begin{cases} 0 & \text{if } n \text{ is a multiple of } 3 \\ 6 & \text{else} \end{cases}$

limit? No. Gets close to 6 but doesn't stay close to 6.

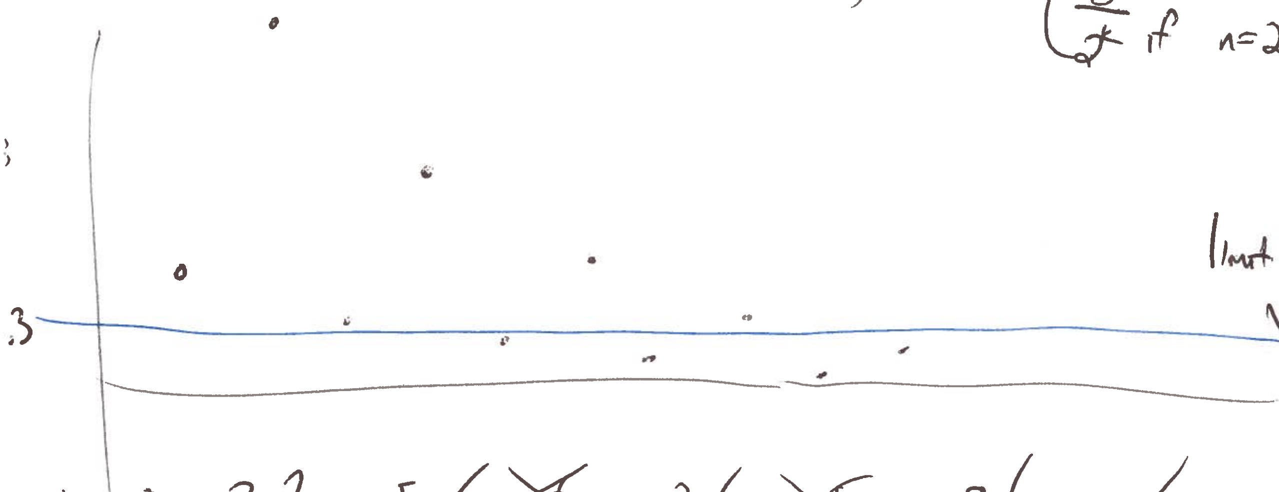
less than .01 to 6? ~~n=3~~ ~~n=4~~ ~~n=5~~ ... never stays true.

n=4, n=5 sure, but

Ex: $\bar{b} = (1,6,\frac{1}{2},6,\frac{1}{3},6,\frac{1}{4},6,\dots)$ $\qquad b_n = ?$ $\qquad$ limit?

Ex: $\bar{c} = (1,6,\frac{1}{2},\frac{6}{2},\frac{1}{4},\frac{6}{4},\frac{1}{8},\frac{6}{8},\frac{1}{16},\frac{6}{16},\dots)$ $\qquad a_n = \begin{cases} \frac{1}{2^k} & \text{if } n = 2k-1 \text{ odd} \\ \frac{6}{2^k} & \text{if } n = 2k \text{ even} \end{cases}$

Plot:



limit? Yes.

less than .3? $n=5$ ✓ ~~n=6~~ $n=7$ ✓ ~~n=8~~ $n=9$ ✓ ... ✓

$n \geq 9$.

less than .01? ...

Doesn't have to stay low the first time it gets low, but eventually has to stay low!

Ex: $a_n = \begin{cases} \frac{1}{2^n} & \text{unless } n = 10^k \text{ for some } k \\ 5 & n = 10^k \text{ for some } k \end{cases}$

looks like it goes to 0 except at $n = 1, 10, 100, 1000, 10000, \dots$

NO LIMIT.

Given a sequence
- does it converge or diverge?
- if it converges, can you find the limit?

We'll give you a toolbox, but figuring out what tool to use can be tricky. Not the only methods, but a reasonable toolkit

Tool 1: Algebra with limits. $\lim_{n\to\infty} 5+\frac{1}{n} = 5$

- If $c_n = a_n + b_n$ then $\lim c_n = \lim a_n + \lim b_n$ if those limits exist

$$\lim\left(5+\frac{1}{n}\right) = \lim 5 + \lim \frac{1}{n} = 5+0=5$$

the sequence $(5,5,5,5,5,\ldots)$ the sequence $(1,\frac{1}{2},\frac{1}{3},\ldots)$

- $c_n = \lambda a_n$ then $\lim c_n = \lambda \lim a_n$. $\lim\left(15\left(2+\frac{1}{n}\right)\right) = 15\lim\left(2+\frac{1}{n}\right) = 15\cdot 2 = 30.$
- $\lim (a_n b_n) = \lim a_n \cdot \lim b_n$

etcetera, see p557 of book.

Note: this only works if the limits exist!!

$$\lim \frac{n}{n} \neq \underbrace{\lim n}_{\text{DNE}} \cdot \underbrace{\lim \frac{1}{n}}_{=0} \overset{?}{=} 0$$

$$\lim \frac{5t+2}{12t+3} \neq \frac{\lim 5t+2}{\lim 12t+3} \quad \text{DNE, DNE}$$

Ex: $\lim_{n\to\infty}\left(n+\frac{1}{n}\right) = ?$ Ask. DNE $= \underbrace{\lim n}_{\text{DNE}} + \underbrace{\lim \frac{1}{n}}_{\text{exists}}$

To argue: If existed, then $\lim n = \lim\left(n+\frac{1}{n}\right) - \lim\left(\frac{1}{n}\right)$ would also exist

If a_n converges and b_n diverges then a_n+b_n diverges.

Tool 2: Extension to a function.

We learned in Calc I how to take $\lim_{t\to\infty} f(t)$ for a function f.

If $\bar{a}$ <u>extends</u> to f, i.e. $\exists$ function f s.t. $f(n) = a_n$ for all n, and if $\lim_{t\to\infty} f(t) = L$, then $\lim_{n\to\infty} a_n = L$.

Ex: $a_n = \frac{n}{n+1}$, $f(t) = \frac{t}{t+1}$, $\lim\limits_{t\to\infty} f(t) = \lim \frac{t}{t+1} \overset{\text{L'Hôp}}{=} \lim \frac{1}{1} = 1 = \lim\limits_{n\to\infty} a_n$.

This is the main use of Extension to Fns — it allows one to use calculus + L'Hôpital's rule !!

Ex: $\lim\limits_{n\to\infty} \frac{5n^2+3n}{12n^2-5} = \frac{5}{12}$.

Reminders: • For ratios of polynomials, $\lim \frac{ct^m + \dots}{dt^{m'} + \dots} = \begin{cases} \frac{c}{d} & \text{if } m=m' \\ 0 & \text{if } m<m' \\ \text{DNE} & \text{if } m>m' \end{cases}$

$\Rightarrow \lim\limits_{n\to\infty} \frac{5n^3+3n}{12n^2+5} = \text{DNE}$, $\lim\limits_{n\to\infty} \frac{5n+3}{12n^2-5} = 0$.

• For exponential functions, $\lim\limits_{t\to\infty} \lambda r^t = \begin{cases} 0 & \text{if } 0 \le r < 1 \\ \lambda & \text{if } r=1 \\ \text{DNE} & \text{if } r>1 \end{cases}$

$r>0$ only

For sequences this will become $\lim\limits_{n\to\infty} \lambda r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \lambda & \text{if } r=1 \\ \text{DNE} & \text{if } |r|>1 \text{ or } r=-1. \end{cases}$

geometric series

$r<0$ is now allowed

From Abs value rule, soon.

Ex: $\lim\limits_{n\to\infty} n^2 e^{-n} \overset{?}{=} \lim\limits_{t\to\infty} \frac{t^2}{e^t} = \lim\limits_{t\to\infty} \frac{2t}{e^t} = \lim\limits_{t\to\infty} \frac{2}{e^t} = 0$

Ex: $\lim\limits_{n\to\infty} \frac{n^{100000}}{(1.1)^n} = 0$ "exponentials grow faster than polynomials" eventually.

Big limitation: Factorials. $(1, 2, 6, 24, 120, 720, \dots)$

What is $n!$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $n! = n(n-1)(n-2)\cdots(2)(1)$. makes sense for $n \ge 0$

$0! = 1$ by convention

But is there a continuous function $f(t)$ with $f(n) = n!$ for each whole number n? Yes/No? Yes! In fact there's even a smooth one, gamma function.

What is $\frac{1}{2}!$? $\frac{1}{2}! = \sqrt{\frac{\pi}{4}}$.

Is it easy? NO. L'Hôpital's? Probably not.

Now $\lim_{n\to\infty} \frac{1}{n!} = 0$, even $\lim_{n\to\infty} \frac{(105)^n}{n!} = 0$, "factorials grow faster than exponentials, eventually"

but must use different tool to see it.

Why factorials? What is $f'(0)$ for $f(x)=x$ $f'=1$

$f(x)=x^2$ $f'=2x$ $f''=2$

$f(x)=x^3$ $f'=3x^2$ $f''=6x$ $f'''=6$

$f(x)=x^4$ $f'=4x^3$ $f''=4\cdot3x^2$ $f'''=4\cdot3\cdot2x$ $f''''=\underbrace{4\cdot3\cdot2\cdot1}_{=4!}$

they show up naturally when taking derivatives of polynomials. Super important for Taylor series, so they'll be everywhere in this class.

Tool 3: Evaluation under a continuous function.

If f is continuous and $b_n \to L$ then $f(b_n) \to f(L)$

i.e. $\lim f(b_n) = f(\lim b_n)$ when $\lim b_n$ exists.

(Aside: this is actually a definition of continuity!)

Ex: $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) \overset{?✓}{=} \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$

Ex: $\lim_{n\to\infty} \ln\left(\frac{2n+1}{3n+2}\right) \overset{?✓}{=} \ln\left(\lim_{n\to\infty} \frac{2n+1}{3n+2}\right) = \ln\left(\frac{2}{3}\right)$

Ex: $\lim_{n\to\infty} \frac{e^{\left(\frac{n+1}{n^2}\right)}}{\sin\left(\frac{3n}{n+1}\right)} \overset{?✓}{=} \frac{\lim e^{\left(\frac{n+1}{n^2}\right)}}{\lim \sin\left(\frac{3n}{n+1}\right)} \overset{?✓}{=} \frac{e^{\lim\left(\frac{n+1}{n^2}\right)}}{\sin\left(\lim \frac{3n}{n+1}\right)} = \frac{e^0}{\sin(3)} = \frac{1}{\sin(3)}$ ✓

need lim to exist and be nonzero

Note: If b_n diverges, it is still possible for $f(b_n)$ to converge.

Ex: $b_n = 2\pi n$. diverges

$\sin(b_n) = 0$ for all n, converges.

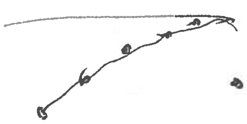
Ex: $\lim\left(\sqrt{\frac{9n^2+3n}{4n^2-n+2}} + \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)\right) = ?$

Tool 4: Monotonicity

Def: A sequence $\bar{a}$ is <u>increasing</u> if $a_{n+1} \geq a_n$ for all n
〃 <u>decreasing</u> if $a_{n+1} \leq a_n$ 〃 } <u>monotonic</u>

A sequence $\bar{a}$ is <u>bounded above</u> by M if $a_n \leq M$ for all n.
(M ← number)
<u>bounded below</u> $a_n \geq M$

Ex: $\bar{a} = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$ $a_n = \frac{n}{n+1}$. Is it monotonic? bounded above? bounded below?

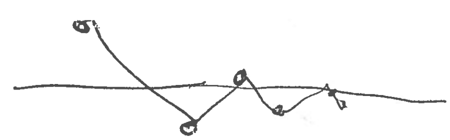


- bounded above by 1 since $\frac{n}{n+1} \leq 1 \iff n \leq n+1$
  also bounded above by 26
- increasing since $a_{n+1} \geq a_n \iff a_{n+1} - a_n \geq 0$

now $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} \geq 0$.

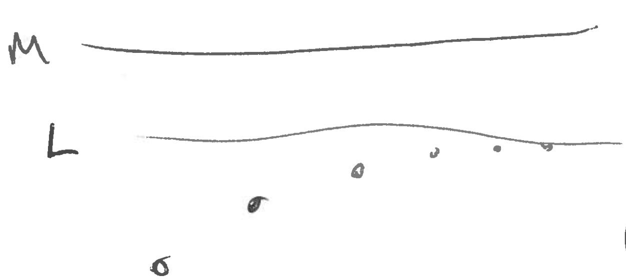
Ex: $\bar{a} = \left(1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \ldots\right)$ $a_n = \left(\frac{-1}{2}\right)^{n-1}$ ← indexing!!

- bounded above by 1, bounded below by -1000 (or $\frac{-1}{2}$)
- Neither increasing nor decreasing



Thm If $\bar{a}$ is increasing and bounded above by M, then it has a limit L and $L \leq M$.

If 〃 decreasing 〃 below 〃 $L \geq M$.



This method tells you $\bar{a}$ converges, but NOT what the limit is.

(<u>Aside</u>: the <u>smallest</u> upper bound is the true limit but hard to find.)

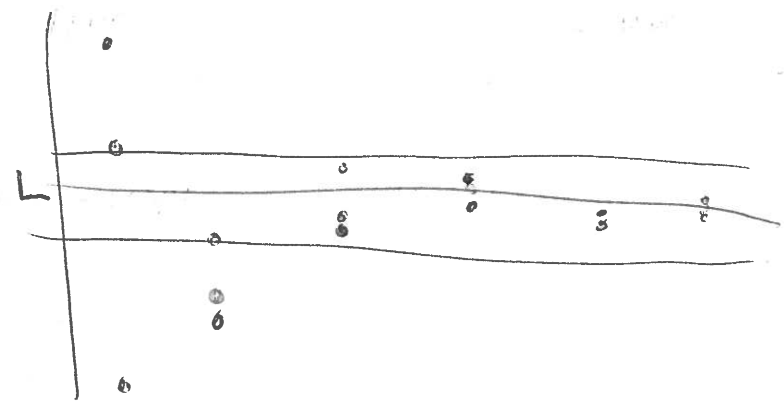
This tool is great for recursive sequences where it may not be obvious what the values are, but you can tell it is increasing + bounded.

We'll do that next week.

Tool 5: Squeezing

<u>Thm</u>: Let $\vec{a}, \vec{b}, \vec{c}$ be sequences with $a_n \le b_n \le c_n$ for all n, and suppose $\lim a_n = \lim c_n = L$. Then $\lim b_n = L$.

<u>Picture</u>



Eventually, a_n, c_n are close to stay close to L, thus so does b_n.

<u>Ex</u>: $b_n = \frac{\sin n}{n} \to 0$. Why? $-1 \le \sin n \le 1$ so $\underbrace{\frac{-1}{n}}_{\to 0} \le \frac{\sin n}{n} \le \underbrace{\frac{1}{n}}_{\to 0}$.

<u>Ex</u>: $\frac{n!}{n^n} \to 0$. (first ask) Why? $\frac{n!}{n^n} = \frac{1}{n}\left(\frac{2}{n}\frac{3}{n}\cdots\frac{n-1}{n}\frac{n}{n}\right) \le \underbrace{\frac{1}{n}}_{\to 0}$. ($\le 1$) below? $0 \le \frac{n!}{n^n}$.

<u>Ex</u>: $\frac{100^n}{n!} \to 0$ $\qquad \frac{100^n}{n!} = \frac{100}{n}\frac{100}{n-1}\cdots\frac{100}{101}\left(\frac{100}{100}\frac{100}{99}\cdots\frac{100}{2}\frac{100}{1}\right)$

($\le \frac{100}{101}$, $\le \frac{100}{101}$, …) (some number λ) (if positive, 0 is easy lower bound)

$\le \left(\frac{100}{101}\right)^{n-100} \cdot \lambda \to 0$ $\quad \frac{100}{101} = r$, $r<1$

Can use squeezing for lots of things but takes insight + creativity!! + practice

Ex: $0 \le \frac{2^n}{3^n+n^2} \le \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0$ so $\frac{2^n}{3^n+n^2} \longrightarrow 0$

b/c $3^n+n^2 > 3^n$. $r = \frac{2}{3} < 1$

Can use squeezing to isolate leading terms of numerator + denominator

Ex: $0 \le \frac{2^n+15n}{3^n}$

$2^n+15n > 2^n$, uh oh.
but $15n < 2^n$ for n big enough
so $2^n+15n < 2^n+2^n = 2\cdot 2^n$ for n big enough

$\frac{2^n+15n}{3^n} \underset{\text{for } n \text{ big enough}}{\le} 2\cdot\left(\frac{2}{3}\right)^n \longrightarrow 0$. Good enough.

(Aside: All rules apply if you can use them for n big enough, since convergence + limit only depend on the ∞ tail of the sequence!)

Ex: $0 \le \frac{2^n}{3^n-n^2} \le \frac{2^n}{\frac{1}{2}(3^n)} = 2\cdot\left(\frac{2}{3}\right)^n \longrightarrow 0$.

$3^n - n^2 > \frac{1}{2}(3^n)$ for n large enough.

PRACTICE It's hard

People need lots of help w/ algebra of inequalities!

$a \le b \Rightarrow ac \le bc$ if $c > 0$

$ac \ge bc$ if $c < 0$

$\frac{1}{a} \ge \frac{1}{b}$ if $ab > 0$ etcetera

Be careful!!

Two big consequences of squeeze theorem:

(1) Abs. Value test: If $|a_n| \to 0$ then $a_n \to 0$.

Ex! $\left(\frac{-1}{2}\right)^n \to 0$ b/c $\left(\frac{1}{2}\right)^n \to 0$.

Why? For any x, $-|x| \le x \le |x|$ so $-|a_n| \le a_n \le |a_n|$.

one of them is equal

$-|a_n| \to -0 = 0$, $|a_n| \to 0$

(2) (Not in book but awesome) Limit Ratio Test

If $\lim \frac{a_{n+1}}{a_n} = r$ exists then $\lim a_n = \begin{cases} 0 & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \\ \text{Inconclusive so use some other test!} & \text{if } r=\pm 1 \end{cases}$

Remember: $\lim r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \\ 1 & \text{if } r=1 \\ \text{DNE} & \text{if } r=-1 \end{cases}$ geometric sequence

Ex: $\lim (1.0000)^n \stackrel{?}{=}$?

Ex: $\frac{2^{2n}}{3^{2n+1}} = \frac{1}{3}\frac{(2^2)^n}{3^n} = \frac{1}{3}\left(\frac{4}{3}\right)^n$ lim DNE.

$\frac{1}{3} \ne 0$, $\frac{4}{3} = r$, $|r|>1$

Ex: $\lim \frac{2^n+5}{3^n-7} = ?$ Can use squeeze theorem or comp ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{(n+1)}+5}{3^{(n+1)}-7}}{\frac{2^n+5}{3^n-7}} = \frac{2^{n+1}+5}{2^n+5} \cdot \frac{3^n-7}{3^{n+1}-7} \longrightarrow 2 \cdot \frac{1}{3} = \frac{2}{3} = r$$

$|r|<1$

so $\lim \frac{2^n+5}{3^n-7} = 0$.

Ex: $\frac{1000^n}{n!} = a_n$ $\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}}{1000^n} \cdot \frac{n!}{(n+1)!} = \frac{1000}{n+1} \longrightarrow 0 = r$ $|r|<1$.

$\frac{n!}{(n+1)!} = \frac{\cancel{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}}{(n+1) \cdot \cancel{n \cdot (n-1) \cdots 2 \cdot 1}} = \frac{1}{n+1}$ so $\lim a_n = 0$.

Ex: $a_n = n$ $\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \longrightarrow 1 = r$ ratio test inconclusive.

Ex: $a_n = \frac{1}{n}$ $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \longrightarrow 1 = r$

One more: Memorize this $e^x = \lim \left(1 + \frac{x}{n}\right)^n = \left(\frac{n+x}{n}\right)^n$

Ex: $\lim \left(\frac{n-1}{n}\right)^n = e^{-1}$.

Ex: $\lim \left(\frac{2n+1}{n}\right)^n = \lim 2^n \cdot \left(\frac{n+\frac{1}{2}}{n}\right)^n \overset{?}{=} \lim 2^n \cdot \lim \left(\frac{n+\frac{1}{2}}{n}\right)^n = \underset{\text{DNE}}{\lim 2^n} \cdot e^{\frac{1}{2}}$

so $\lim \left(\frac{2n+1}{n}\right)^n$ DNE.

Actual defn of limit (see Appendix D)

I gave intuitive defn of limit: $(a_n) \to L$ if the sequence eventually gets + stays arbitrarily close to L. Let's make it into math.

distance to L is $|a_n - L|$. Arbitrarily close: for any degree of accuracy ε, can get closer than ε "indistinguishable to the naked eye / under microscope / etc"

eventually gets + stays: there is some time N st for all later times $n > N$ we have $|a_n - L| < \varepsilon$.

Def: $\lim a_n = L$ if for all "accuracies" $\varepsilon > 0$ there is some "moment" N such that $|a_n - L| \leq \varepsilon$ for all $n > N$.

Ex: $\lim\limits_{n\to\infty} \frac{n-1}{n} = 1$? Well, $|1 - a_n| = \left|\frac{1}{n}\right| = \frac{1}{n}$.

When $\varepsilon = .01 = \frac{1}{100}$ can use $N \geq 100$ for $|1 - a_n| \leq \varepsilon = \frac{1}{100}$

since $\frac{1}{n} \leq \frac{1}{100} \iff n \geq 100$. $N = 300$ also works.

When $\varepsilon = .0001 = \frac{1}{10000}$ can use $N = 10000$. or higher.

When ε is general, $\frac{1}{n} \leq \varepsilon \iff n \geq \frac{1}{\varepsilon}$ so can choose $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$.

This definition says: if you can solve for N given ε then you can confirm that $a_n \to L$.

Ex: $\lim \frac{1}{3^n} = 0$ $\left|0 - \frac{1}{3^n}\right| = \frac{1}{3^n}$. $\frac{1}{3^n} < \varepsilon \iff 3^n > \frac{1}{\varepsilon}$

$\Uparrow$

$n = \log_3(3^n) > \log_3\left(\frac{1}{\varepsilon}\right)$

So $N = \left\lceil \log_3\left(\frac{1}{\varepsilon}\right) \right\rceil$ works

$\varepsilon = .01$ $N = \lceil \log_3(100) \rceil = 5$

$\varepsilon = .001$ $N = \lceil \log_3(1000) \rceil = 7$ or 8?

This seems abstract but is really what you want in practice!!

If you're approximating something you want to know — how far do I have to go before I am within the desired accuracy bounds?

$N \longleftarrow \varepsilon$

Ex: How many terms of $a_n = \frac{2n-1}{n+3}$ must I take before I stay within .01 of the limit? Within ε of the limit?

$$\lim \frac{2n-1}{n+3} = 2. \qquad \left|\frac{2n-1}{n+3} - 2\right| = \left|\frac{2n-1-2(n+3)}{n+3}\right| = \left|\frac{-7}{n+3}\right| = \frac{7}{n+3}$$

$$\frac{7}{n+3} < \varepsilon \iff \frac{n+3}{7} > \frac{1}{\varepsilon} \iff n+3 > \frac{7}{\varepsilon} \iff n > \frac{7}{\varepsilon} - 3$$

So $N = \left\lceil \frac{7}{\varepsilon} - 3 \right\rceil$

e.g. if $\varepsilon = .01$ then $N = \lceil 700 - 3 \rceil = 697$.

a_{697} is a good enough approximation. $a_{697} = 1.99$

Again, manipulating inequalities is the hard part for many so practice. Inequalities are harder than equalities!

Rmk: Some people use the definition w/ $|a_n - L| < \varepsilon$ instead of $|a_n - L| \le \varepsilon$. Then 697 is not good enough, need $N = 698$. This is just annoying, use $N = \left\lceil \frac{7}{\varepsilon} - 3 \right\rceil + 1$ works, but this is just stupid... I will not care about this edge case, and always use $\le \varepsilon$.

What does $\lim a_n = \infty$ mean? Different + more specific than $\lim a_n$ DNE.

It means you get + stay arbitrarily larger. For any potential bound $M > 0$ there is a moment N s.t. at all later moments $n \geq N$ you surpass the bound $a_n > M$.

Ex: $a_n = n$ $\lim a_n = \infty$. Set $N = \lceil M \rceil$.

Ex: $a_n = 2^n$ $\lim a_n = \infty$ $N = \lceil \log_2 M \rceil$

Ex: $a_n = (-2)^n$ $\lim a_n$ DNE

gets negative too, don't stay $> M$.

Ex: $a_n = (-1)^n$ lim DNE

don't get large.

Similarly, $\lim a_n = -\infty$ if gets + stays arbitrarily "~~small~~". (i.e. largely negative!)

Recursive Sequences.

Here's the method which might work

Step 1: If there were a limit, compute it.
Step 2: Use the limit as a potential upper/lower bound.
Step 3: Use the limit to prove increasing / decreasing.

Ex: $a_1 = 1$
$a_{n+1} = \frac{1}{2}a_n + 3.$

replace a_n and a_{n+1} with L (i.e. take limit of both sides)

① $L = \frac{1}{2}L + 3$ $\frac{1}{2}L = 3$ $L = 6.$ If limit exists, limit is 6.

Now $a_1 < 6$, maybe 6 is an upper bound

② If $a_n \leq 6$, then $a_{n+1} = \frac{1}{2}a_n + 3 \leq \frac{1}{2}(6) + 3 = 3 + 3 = 6$ so $a_{n+1} \leq 6.$

$\underset{\text{base case}}{a_1 \leq 6} \rightsquigarrow a_2 \leq 6 \rightsquigarrow a_3 \leq 6 \rightsquigarrow \cdots \underset{\text{Induction}}{\Longrightarrow} a_n \leq 6$ for all n.

③ Maybe increasing,
$a_{n+1} - a_n \geq 0$? $a_{n+1} - a_n = \frac{1}{2}a_n + 3 - a_n = \frac{-1}{2}a_n + 3 \geq 0$?

$\frac{-1}{2}a_n + 3 \geq 0 \Leftrightarrow \frac{-1}{2}a_n \geq -3 \Leftrightarrow -a_n \geq -6 \Leftrightarrow a_n \leq 6$

but we proved that! Great. So $a_n \leq a_{n+1}$.

Increasing + Bounded $\Longrightarrow$ has a limit (And $L = 6$).

Ex: $\bar{a} = (3, 5\sqrt{3}, 5\sqrt{5\sqrt{3}}, 5\sqrt{5\sqrt{5\sqrt{3}}}, \ldots)$ $a_1 = 3$
$a_{n+1} = 5\sqrt{a_n}$

if L exists, $L = 5\sqrt{L}$ $\sqrt{L} = 5$ $L = 25.$

Now continue...

⌈BTW will limit ratio test work? No — inconclusive. Only gives limit 0.⌋

Series We make sense of an infinite sum as a special kind of sequence.

Ex: $a_n = \frac{1}{2^n}$. Then $a_1 + a_2 + a_3 + a_4 + \ldots \overset{\text{NOTATION}}{=} \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$.

Sigma notation for indexed sums ← if new to you, read appendix F and ask questions

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$\frac{1}{2}$, $3/4$, $7/8$, $15/16$

let $S_n = \sum_{i=1}^{n} a_i$ the n^{th} partial sum

this is the sequence $\overline{S} = \left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right)$

$S_n = 1 - \frac{1}{2^n}$.

$\lim S_n = 1$ so we say $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Def: Given a sequence $\overline{a} = (a_1, a_2, \ldots)$ its series is the sequence of partial sums

$\overline{S} = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$. We say the series converges/diverges if $\overline{S}$ does, and write $\sum_{n=1}^{\infty} a_n$ for the limit of $\overline{S}$.

Ex: $a_n = n$. $S_1 = 1$ $S_2 = 1+2 = 3$ $S_3 = 1+2+3 = 6$ $\lim S_n = \infty$ $\sum a_n = \infty$ (or DNE).

Big qns: Does $\sum a_n$ converge? What is the limit? Previous tools still useful, we'll adapt them to series.

Tool 1: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim a_n = 0$

i.e. if $\lim a_n \neq 0$ or DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

Because $\overline{S}$ just won't stay put!

Ex: $\sum_{n=1}^{\infty} \underbrace{\frac{n^2+1}{n^2-1}}_{\downarrow 1} = \infty$ keep adding roughly one each time

Ex: $\sum_{n=1}^{\infty} (-1)^n$ $\overline{S} = (-1, 0, -1, 0, -1, 0, \ldots)$ diverges.

WARNING: Just because $a_n \to 0$ does NOT imply $\sum_{n=1} a_n$ converges.

I CANT EMPHASIZE THIS ENOUGH. I'll highlight examples

Tool 1 says when diverges, NOT when converges.

Tool 2: Geometric series — sum of a geom. sequence

$$\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + \ldots$$

indexing!!!

a is initial term

r is successive ratio

geometric series converges when sequence converges to zero. So warning doesn't apply yet.

If $|r| \geq 1$ then $\lim ar^n \neq 0$ so $\sum_{n=0}^{\infty} ar^n$ diverges

If $|r| < 1$ it converges, and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Ex: $2 - \frac{6}{5} + \frac{18}{25} - \frac{54}{125} + \ldots$

$$= 2 - 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 - 2\left(\frac{3}{5}\right)^3 + \ldots$$

$a = 2$

$r = \frac{-3}{5}$

$$\sum = \frac{2}{1 - \frac{3}{5}} = \frac{2}{8/5} = \frac{10}{8}.$$

Why $\sum = \frac{a}{1-r}$? Recursive formula for S_n.

$$a + r(a + ar + ar^2 + \ldots + ar^n) = a + ra + \ldots + ar^{n+1}$$

$$\underset{\downarrow}{a + rS_n} \qquad\qquad \underset{\downarrow}{S_{n+1}}$$

$$a + rL = L$$

$a = L(1-r)$

$L = \frac{a}{1-r}$.

Ex: $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{n+3}} = ?$

$$\frac{3^{2n}}{2^{n+3}} = \frac{(3^2)^n}{2^3 2^n} = \frac{1}{8} \cdot \left(\frac{9}{2}\right)^n$$

$\frac{9}{2} > 1$ so diverges!

Ex! $\sum_{n=0}^{\infty} 6\cdot\left(\frac{4}{5}\right)^n = \frac{6}{1-\frac{4}{5}} = \frac{6}{1/5} = 30$

Ex! $\sum_{n=3}^{\infty} 6\cdot\left(\frac{4}{5}\right)^n = \underbrace{6\left(\frac{4}{5}\right)^3}_{= a} + \dots = \frac{6\left(\frac{4}{5}\right)^3}{1-\frac{4}{5}} = 30\cdot\left(\frac{4}{5}\right)^3$

Ex! $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right) \overset{?}{=} \sum_{n=0}^{\infty}\frac{1}{2^n} - \sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 2 - \frac{3}{2} = \frac{1}{2}.$

not geom

sum of limit = limit of sum
when both defined

Ex! For which x does $\sum_{n=1}^{\infty}\left(\frac{x+2}{3}\right)^n$ converge?

$r = \frac{x+2}{3}$. When is $|r|<1$? $-1 < \frac{x+2}{3} < 1 \iff -3 < x+2 < 3$
$\iff \boxed{-5 < x < 1}$

WARNING: Don't try to apply when $|r|\geq 1$. $r=-1$, $a=1$: $1-1+1-1+1-1\dots = \frac{1}{1-(-1)} = \frac{1}{2}$? No way

---

Tool 3: p-test (special case of integral test, soon.)

Qn: $\sum_{n=1}^{\infty}\frac{1}{n} = ?$ | Poll | It's ∞! WARNING APPLIES. Lots of small things add up to ∞.

$\sum_{n=1}^{\infty}\frac{1}{n^2} = ?$ | It's $\pi^2/6$. Duh! (Advanced complex analysis / number theory!!!)

Repoll

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = ?$ | It's ∞. $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ so comparison says it must diverge
soon.

p-test: $\sum_{n=1}^{\infty}\frac{1}{n^p}$ $\begin{cases}\text{converges if } p>1 \\ \text{diverges if } 0<p\leq 1\end{cases}$ doesn't say what the limit is.

Giving the flavor.

Here is Oresme's proof that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ ← the harmonic series.
(14th century)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{16} + \cdots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}} + \frac{1}{32} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \longrightarrow \infty.$$

(Modern version of this argument: Cauchy condensation test 19th century, MAT 316)
(Euler did $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.)

Having seen some of these tests, here's the big one!!! Tool 4

Ratio Test: Suppose that $\lim \left|\frac{a_{n+1}}{a_n}\right| = r$. Then if $r > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges, if $r < 1$ then $\sum_{n=1}^{\infty} a_n$ converges, and if $r = 1$ the test is inconclusive – it might go either way, need to do something else.

This is your first line of defense! Only when this fails look elsewhere! (Always determines the radius of convergence of a power series, later.)

Basically, $\sum a_n$ is "close" to a geometric series and the behavior is similar
(or $r = 0$ and $\sum a_n$ is much smaller than any convergent geometric series.)
We'll justify soon.

Ex: $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots \overset{\text{Guesses?}}{=} e \overset{??}{\approx} 2.7\ldots$

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \longrightarrow 0 = r$$ so converges.

Ex: $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ $\quad \frac{a_{n+1}}{a_n} = \frac{2}{n+1} \to 0$ Converges $(= e^2)$

Qn: For which x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent Ans: All x!!

Ex: $\displaystyle\sum_{n=0} \frac{3^n+7}{4^n+6}$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}+7}{3^n+7}\cdot\frac{4^n+6}{4^{n+1}+6} \approx \frac{3^{n+1}}{3^n}\frac{4^n}{4^{n+1}} = \frac{3}{4} = r < 1$$

Converges

$\approx$ has same limit as (can prove w/ squeeze thm)

$\displaystyle\sum_{n=0}^{\infty} \frac{4^n+6}{3^n+7}$ diverges

No idea what the limit is.

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{n+5}{n^2+3n+2}$

$$\frac{a_{n+1}}{a_n} = \frac{n+6}{n+5}\cdot\frac{n^2+3n+2}{(n+1)^2+3(n+1)+2} = \frac{n^3+\dots}{n^3+\dots} \longrightarrow 1 = r$$

ratio test inconclusive

In general $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^p}$ $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{(n)^p}{(n+1)^p} = \dfrac{n^p+\dots}{n^p+\dots} \longrightarrow 1 = r$

p-test handles one edge case where ratio test is useless!

We'll see soon that $\dfrac{n+5}{n^2+3n+2} \approx \dfrac{1}{n}$ which diverges, so former diverges too.

Ex: $\displaystyle\sum \frac{n2^n}{3^n}$

Ex: For which x is $\displaystyle\sum_{n=0}^{\infty} n!\cdot x^n$ convergent?

$$\frac{a_{n+1}}{a_n} = (n+1)x \qquad \lim_{n\to\infty}(n+1)x = \begin{cases} \infty & x>0 \leftarrow \text{div} \\ 0 & x=0 \leftarrow \text{convergent!} \\ -\infty & x<0 \leftarrow \text{div} \end{cases}$$

Only for $x=0$, $1+0+0+0\dots$

Series convergence tests — 3 main questions:

1. Can we show something converges or diverges? (Yes, or it wouldn't be a test)
2. If convergent, can we find limit exactly?
3. Can we estimate error? If we want to find N s.t. $|S_N - L| < \varepsilon$, can we do it? How many terms to sum until within .01 of limit?

Test	Find Limit	Estimate Error	Other Problem
Geom series	✓	✓	Super specific
p-test	✗	✓	—
*Ratio test	✗	✓	Great.
Integral test	✗	✓	Reasonably specific
Alternating series test	✗	✓	—
Comparison test ↳ Absolute Convergence test	✗	Depends what you're comparing to	Powerful tool, in conjunction with other tests.
Telescoping sums (a stupid trick)	✓	✓	Super specific

- You won't be finding the limit exactly very often in this class!!
- But you sure can approximate it very closely!!!!

Let's go back and look at geometric series again, focusing on error.

$$\sum_{n=0}^{\infty} ar^n = \underbrace{a + ar + ar^2 + \dots + ar^N}_{S_N} + ar^{N+1} + \dots$$

We know the exact limit, $L = \frac{a}{1-r}$ when $|r|<1$, so finding an approximation of the sum is silly. But we can still ask: how far is S_N from the true sum? $L - S_N = ar^{N+1} + \dots = \sum_{n=N+1}^{\infty} ar^n = \sum_{n=0}^{\infty} Ar^n$

for $A = ar^{N+1}$. So $L - S_N = \frac{A}{1-r} = \left(\frac{a}{1-r}\right) r^{N+1}$.

With zero terms, error is $L = \frac{a}{1-r}$. Each term multiplies error by r (e.g. $\frac{2}{3}$)

Ex: Consider the series $\sum_{n=0}^{\infty} 6\cdot\left(\frac{1}{2}\right)^n$. How many terms are needed before the partial sum is within .01 of the overall sum?

Ans: $\text{Error} = \frac{a}{1-r} r^{N+1} = \frac{6}{\frac{1}{2}}\left(\frac{1}{2}\right)^{N+1} = 6\cdot\left(\frac{1}{2}\right)^N < \frac{1}{100}$

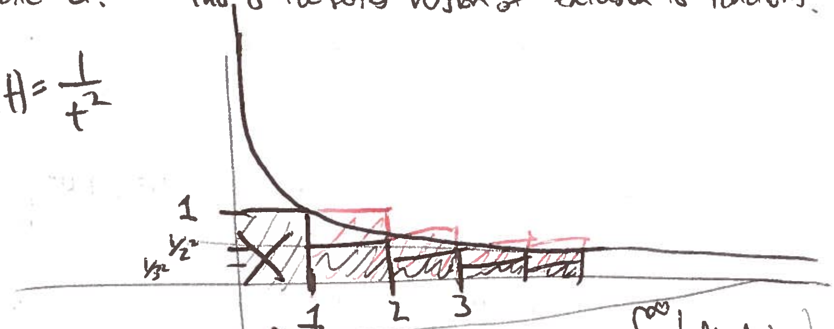
$$\iff \frac{1}{2^N} < \frac{1}{600} \iff 2^N > 600 \iff N > \log_2(600)$$ ← between 9 and 10

so $\sum_{n=0}^{10} 6\left(\frac{1}{2}\right)^n$ is close enough.

Integral test – invoke the awesome power of calc 2.

This is the series version of extension to functions.

Why does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge? Let $f(t) = \frac{1}{t^2}$



$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ = area of shaded boxes

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ = area of shaded boxes

Rmk: don't use this. Don't want $\int_0^\infty \frac{1}{t^2}\,dt$ which does not converge!!

So $$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^\infty f(t)\,dt < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

this is because f is decreasing!!

Now $$\int_1^\infty \frac{1}{t^2}\,dt = \frac{-1}{t}\Big[_1^{\infty} = 0 - \frac{-1}{1} = 1$$

Thus $$1 < \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + 1 = 2.$$

This is like the squeezing test, only it squeezes an integral between the sum or vice versa!

Does this imply that the sequence converges? Partial sums eventually between 1 and 2, but do they settle?

MONOTONE + BOUNDED = CONVERGENCE.

(Increasing) ← because $f(t) \geq 0$.

Not only do we get convergence, we get a bound. Better still, we bound the error too!

$$L - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^\infty \frac{1}{t^2}\,dt = \underbrace{\frac{-1}{t}\Big[_N^{\infty} = \frac{1}{N}}_{\text{Draw this!}}.$$

Want within .01? Let $N = 100$.

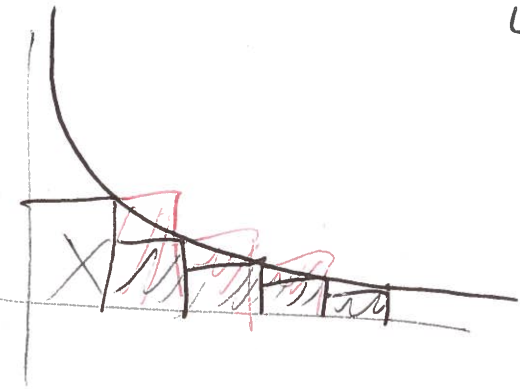
⊛ SEE INSERT

You can really see this

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.644934\ldots$$

Take 5 terms get	1.4636	error is just around $\frac{1}{N}$.
10	1.$\underline{5}$4977	
100	1.63$\underline{3}$498	
1000	1.643$\underline{3}$9...	
10000	1.644$\underline{8}$...	

Why does $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge. $f(t) = \frac{1}{t}$



$$\sum_{n=2}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{t}\,dt < \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\| $$

$$\ln t \Big|_1^{\infty} = \ln \infty - \underbrace{\ln 1}_{0} = \infty.$$

Since $\infty < \sum_{n=1}^{\infty} \frac{1}{n}$, the limit is infinity!

Think.
$\sum \frac{1}{n}$ diverges because $\ln \to \infty$
$\sum \frac{1}{n^2}$ converges because $\frac{1}{n}$ stays bounded

How many terms of $\sum \frac{1}{n}$ do I need to get over 1000?

$$\sum_{n=1}^{N} \frac{1}{n} > \int_1^{N+1} \frac{1}{t}\,dt = \ln(N+1) - \ln 1 = \ln(N+1) > 1000$$

$N+1 > e^{1000}$ ← holy cow that's a LOT.

grows to ∞ VERY slowly.

The integral test:

$f(t)$ continuous, decreasing, positive, $a_n = f(n)$.

Then

- If $\int_1^{\infty} f(t)\,dt$ converges, so does $\sum_{n=1}^{\infty} a_n$

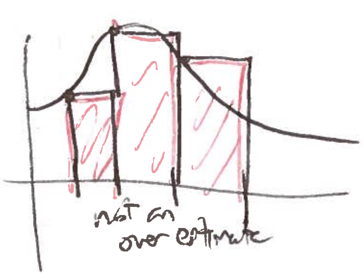
  and $\int_1^{\infty} f(t)\,dt \leq \sum_{n=1}^{\infty} a_n \leq \int_1^{\infty} f(t)\,dt + a_1$

  Also get error bound $\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(t)\,dt$.

- If $\int_1^{\infty} f(t)\,dt$ diverges, so does $\sum_{n=1}^{\infty} a_n$.

~~Application I: p-test w/ error bounds~~

Why assumptions. If f not decreasing



If f not positive, several issues

Application 1: p-test w/ error bounds

Thm: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p>1$
diverges when $0<p\leq 1$

Why? $\int_1^\infty \frac{1}{t^p}\,dt = \int_1^\infty t^{-p}\,dt \underset{\text{when } p\neq 1}{=} \frac{t^{-p+1}}{-p+1}\Big[_1^\infty = \begin{cases} \frac{1}{p-1} & \text{if } -p+1<0 \\ \infty & \text{if } -p+1>0 \end{cases}$

Conv or div:

Ex: $\sum_{n=1}^{\infty} \frac{5}{n^{.93}}$ Ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ Ex: $\sum_{n=1}^{\infty} \frac{1}{n^{100}}$

Ex: Find $\sum_{n=1}^{\infty} \frac{1}{n^4}$ correct to 3 decimal places.

$\boxed{\text{This means (error} \leq .0005)}$ error $\leq \int_N^\infty \frac{1}{t^4}\,dt = \frac{-1}{3t^3}\Big[_N^\infty = \frac{1}{3N^3} \leq .0005 = \frac{5}{10000}$

$\Leftrightarrow N^3 \geq \frac{10000}{15}$ $N > \sqrt[3]{\frac{10000}{15}} \approx 8.7$ so $N=9$ works.

$1+\frac{1}{2^4}+\frac{1}{3^4}+\dots+\frac{1}{9^4} = 1.0819$ so series is about 1.082

Is this (1.0819) an underestimate ~~or~~ overestimate?

BTW Wolfram Alpha, type "sum (n^(-4)) from n=1 to 9"

Ex: $\sum_{n=2}^{\infty} \frac{1}{n\cdot \ln n}$? $\int_2^\infty \frac{1}{t\ln t}\,dt = \ln(\ln(t))\Big[_1^\infty = \infty$! diverges

$\boxed{\text{why 2? } \ln 1 = 0,\ \frac{1}{\ln 1} \text{ not defined.}}$

Exercise: $\sum \frac{1}{n(\ln n)^p}$ $\begin{cases} \text{conv if } p>1 \\ \text{div if } p\leq 1. \end{cases}$

Comparison Test

Is it similar to something you know converges or diverges? Use the comparison test.

Ex: $\sum_{n=0}^{\infty} \frac{1}{3^n+5}$. Well, $\sum \frac{1}{3^n}$ converges, $0 < \frac{1}{3^n+5} < \frac{1}{3^n}$ so this should converge too

[What to do? Ratio test, of course.]

Thm: Spose $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.
By contrapositive, if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ $\quad \sum \frac{1}{n}$ diverges so $\frac{1}{2}\sum \frac{1}{n} \overset{ALT}{=} \sum \frac{1}{2n}$ diverges

then $\frac{1}{2n-1} > \frac{1}{2n}$ so $\sum \frac{1}{2n-1}$ diverges.

Why OK? $\quad \frac{-1}{n}$ diverges, but $\frac{-1}{n} < \frac{1}{n^2}$ ← converges.

Ex: $\sum_{n=2}^{\infty} \frac{1}{3^n-5}$. ← [Ratio test] $\quad$ Now $\frac{1}{3^n-5} > \frac{1}{3^n}$, hmmm, b/c $3^n-5 < 3^n$

all positive

But $3^n - 5 > \frac{3^n}{10}$ (for $n \geq 2$) and $\sum \frac{10}{3^n} = 10 \sum \frac{1}{3^n}$ converges

$\Longleftrightarrow \frac{1}{3^n-5} < \frac{10}{3^n}$

If you're similar to something, you're probably $a_n < \lambda b_n$ for some λ
$\quad a_n \qquad\qquad b_n \qquad\qquad a_n > \mu b_n$ for some μ

Ex: $\sum_{n=1}^{\infty} \frac{1}{2n+1} > \sum_{n=1}^{\infty} \frac{1}{3n}$ diverges

Ex: $\sum_{n=0}^{\infty} \frac{1}{3^n-5}$. Hmm, $n=0,1$, not positive
and 3^n-5 might be < than $\frac{3^n}{10}$.

Is that a problem?

No. $\sum_{n=0}^{\infty} \frac{1}{3^n-5} = \underbrace{\frac{1}{3^0-5} + \frac{1}{3^1-5}}_{\text{just two silly numbers}} + \sum_{n=2}^{\infty} \frac{1}{3^n-5}$ converges

converges

Big idea! In math, we say something is **eventually true** if, after some point in time, it is **always true**. I.e. $\lim a_n = L$ if for each $\varepsilon > 0$, $|a_n - L| < \varepsilon$ eventually.

Different from english, you'll eventually get an A in math, but just once, not forever.

If $0 \le a_n \le b_n$ eventually, then can also use comparison test.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

only need control over the tail.

some finite number

the infinite tail where things might go wrong.

Same for all tests! If eventually alternating, can use AST.

Ex! $\sum_{n=1}^{\infty} n e^{-\frac{n}{15}}$ ratio test

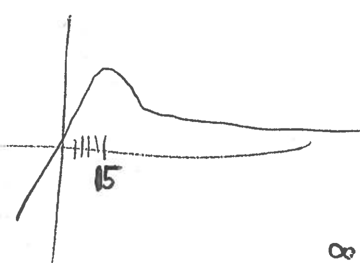
want to use integral test. Applies when $f(t) = te^{-\frac{t}{15}}$ is positive ✓

decreasing ✗

eventually decreasing ✓

$f'(t) = e^{-\frac{t}{15}}\left(1 - \frac{t}{15}\right)$

It's negative for $t > 15$



Ex! $\sum_{n=0}^{\infty} \frac{1}{3^n - 1000000}$ is eventually less than $\sum_{n=0}^{\infty} \frac{2}{3^n}$

b/c $\frac{3^n}{2}$ is eventually < than $3^n - 1000000$

takes a while though.

Also, $3^n - 1000000$ eventually ≥ 0.

The big idea makes all the tests **much** easier to apply.

Also, integral test can be used to compare error.

If $0 \le a_n \le b_n$ then the error after N terms is

$$\sum_{n=N+1}^{\infty} a_n \le \sum_{n=N+1}^{\infty} b_n .$$

Ex: Find a reasonably efficient N such that $\sum_{n=0}^{N} \frac{1}{3^n+n}$ is within .001 of the series sum

Well $\sum_{n=0}^{\infty} \frac{1}{3^n+n} \le \sum_{n=0}^{\infty} \frac{1}{3^n}$ ← geometric, $a=1$, $r=\frac{1}{3}$

error is $\frac{a}{1-r}\left(r^{N+1}\right) = \frac{1}{\frac{2}{3}} \cdot \left(\frac{1}{3}\right)^{N+1} = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^{N}$

so $\frac{1}{2} \cdot \left(\frac{1}{3}\right)^{N} < \frac{1}{1000}$ iff $\left(\frac{1}{3}\right)^{N} < \frac{1}{500} \iff 3^N > 500$

$\iff N > \log_3(500)$
5.5

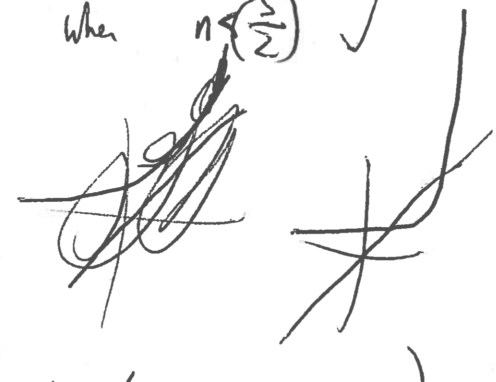
so $N=6$ will suffice for geom series

$\Rightarrow$ suffices for $\frac{1}{3^n+n}$ too.

Ex: $\sum \frac{n2^n}{15^n}$. Another good trick — make r slightly bigger (but still <1.)

Should be like $\frac{2^n}{15^n} = \left(\frac{2}{15}\right)^n$. Ratio test.

$\frac{n2^n}{15^n} < \frac{3^n}{15^n}$ when $n < \left(\frac{3}{2}\right)^n$ ✓



eventually, $r^n > n$ for any $r>1$

"exponential beats polynomial"

For comparison test — first find something similar. Then modify slightly (rescale, or change r) to make a true comparison (eventually).

A slightly boiled-down version of this is the

Limit comparison test If $\lim \frac{a_n}{b_n} = c$, $c \neq 0, \infty$

then $\sum a_n$ converges $\Leftrightarrow \sum b_n$ converges.

I.e. comparable (up to mult, almost) things share convergence properties.

This won't compare $\frac{n2^n}{5^n}$ w/ $\frac{2^n}{5^n}$ b/c ratio is $n \to \infty$ (or $\frac{1}{n} \to 0$)

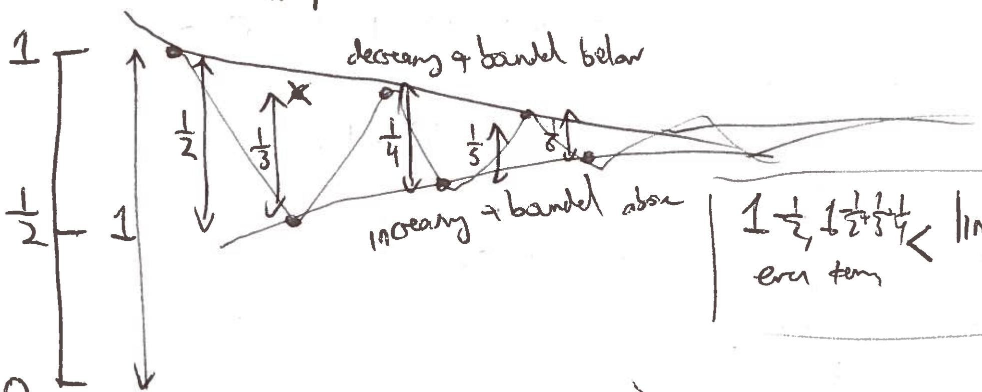
but it will compare $\frac{n^2+26n-12}{n^4-17n+3000}$ with $\frac{1}{n^2}$ $(c=1)$

without needing to do any nasty algebra (which is bigger?)

Alternating series test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Plot

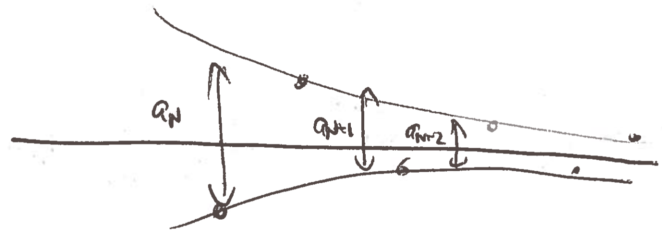


both converge, must meet in the middle!

$1-\frac{1}{2},\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} <$ limit $< 1,\ 1-\frac{1}{2}+\frac{1}{3},\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

even terms — odd terms

Moreover a_{N+1} is a bound on the error since it goes past the limit!



AST: If $\bar{a}$ is a sequence alternating between positive and negative entries, and $|a_n|$ is decreasing to zero, then $\sum_{n=1}^{\infty} a_n$ converges

and $\left|\sum_{n=N+1}^{\infty} a_n\right| \leq |a_{N+1}|$. ← pretty good estimate

error after N steps

Note: Pretty easy for an AS to converge, just need terms to decrease to zero.
MUCH harder for a positive sequence to converge. $a_n \to 0$ not enough!

$\sum \frac{1}{n}$ diverges $\qquad \sum \frac{(-1)^n}{n}$ converges!!

Exercise: Compute $\sum \frac{(-1)^{n+1}}{n}$ to within $.1 = \frac{1}{10}$

Need 9 terms, then error $\leq \frac{1}{10}$ $\qquad 1 - \frac{1}{2} + \dots + \frac{1}{9} = .745$

anyone? $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \approx .693$

$= \ln 2$. ← we'll see!

Exercise: $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$? Converges. To get within .1

need $\frac{1}{\ln(N+1)} < \frac{1}{10} \iff \ln(N+1) > 10 \iff N+1 > e^{10}$ very large!

Does $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converge or diverge? $\frac{1}{\ln n} > \frac{1}{n}$ which diverges, comparison test, soon.

goes to ∞

even larger!

Ex: Believe it or not, we have

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots = \pi.$$

Compute 3 decimal places of π,

Need to sum until $\frac{4}{2N+1} < .0005 = \frac{5}{10000} \iff 2N+1 > \frac{40000}{5} = 8000$

≈ 4000 term

Let me do it in my head... ah, 3.141.

A stupid quick trick: telescoping sums.

Ex: Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ exactly. Anyone.

Ok, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So our sum is

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$S_1 = 1 - \frac{1}{2}$

Cancellation makes life easy!

$S_2 = 1 - \frac{1}{3}$

$S_3 = 1 - \frac{1}{4}$

$\vdots$

$S_N = 1 - \frac{1}{N} \longrightarrow 1.$

Def: Let a_n be a sequence. Let $b_n = a_n - a_{n+1}$.

The series $\sum_{n=1}^{\infty} b_n$ is called a telescoping sum.

If (a_n) converges then $\sum_{n=1}^{\infty} b_n = a_1 - \lim a_n$ converges.

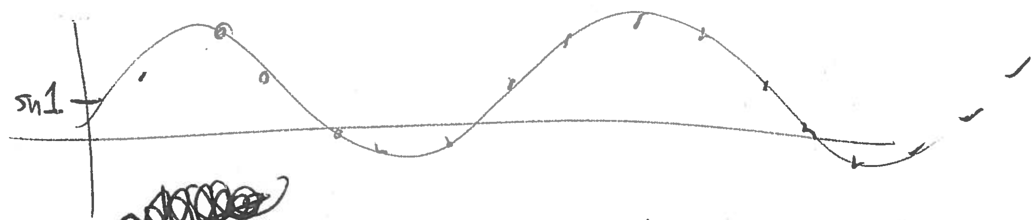
Else, $\sum_{n=1}^{\infty} b_n$ diverges.

$S_N = a_1 - a_{N+1} \longrightarrow a_1 - \lim a_n$

Ex: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin(n)}{n}-\frac{\sin(n+1)}{n+1}\right)=\frac{\sin(1)}{1}$ since $\frac{\sin(n)}{n}\to 0$

Ex: $\displaystyle\sum_{n=1}^{\infty}\sin(n)-\sin(n+1)$ diverges, but does not go to ∞!! $\overset{\sin(n)}{\overset{\shortparallel}{a_n}}\not\to 0$.

$\sin(1)-\sin(N)$



never settles on one thing!

Ex: $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n^2+2}{n^2+2n+3}\right)=\sum_{n=1}^{\infty}\ln\left(\frac{n^2+2}{(n+1)^2+2}\right)=\sum \ln(n^2+2)-\ln((n+1)^2+2)$

this goes to 1

$\ln 1=0$

so terms go to zero.

does it converge?

$a_n=\ln(n^2+2)\longrightarrow\infty$.

so diverges.

a_1-a_N goes to $-\infty$.

Telescoping sums are either
- stupidly easy to recognize $\sin(n)-\sin(n+1)$
- stupidly hard to recognize

Either way they are rare + special, party tricks. Some very useful applications in higher math, start reading on wikipedia maybe.

Have you read about Grandi's series yet?