Absolute Value Test The AST finally let us look at some series w/ negative values, but not all.

Ex: $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\dots$

Ex: $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ ← sometimes pos sometimes neg.

Def: We say a series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

Thm: If $\sum a_n$ is absolutely convergent, it is also convergent.

Ex: $\sum \frac{(-1)^n}{n} \xrightarrow{\text{abs value}} \sum \frac{1}{n}$ divergent

convergent by AST, NOT abs convergent

$\sum \frac{(-1)^n}{n^2} \longrightarrow \sum \frac{1}{n^2}$ convergent

convergent ←

Qn: $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\dots$? Abs convergent?

Qn: $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\dots$? Not abs convergent, don't know if convergent!! (Is it? Ask.)

Yes. In MAT 316 I'd make you prove it.

Ex: $\sum \frac{\sin(n)}{n^2} \longrightarrow 0 \le \sum \frac{|\sin n|}{n^2} \le \sum \frac{1}{n^2}$ abs value converges by comp. test so abs. convergent.

Also can use AVT to bound error: error in $\sum a_n$ is $\le$ than in $\sum |a_n|$

so error after N terms in $\sum \frac{\sin n}{n^2} \le$ error in $\sum \frac{|\sin n|}{n^2} \le$ error in $\sum \frac{1}{n^2}$

$\le \int_N^{\infty} \frac{1}{t^2}\,dt = \frac{-1}{t}\Big]_N^{\infty} = \frac{1}{N}.$

Rmks: 1) Why AV Thm is true? Convergent $\Leftrightarrow$ error goes to zero $\lim_{N\to\infty} E_N = 0$

If abs conv then

$$\underbrace{\sum_{n=N+1}^{\infty} -|a_n|}_{\downarrow 0} \le \underbrace{\sum_{n=N+1}^{\infty} a_n}_{= E_N} \le \sum_{n=N+1}^{\infty} |a_n| \downarrow 0$$

since convergent. So by squeeze, error $\to 0$.

2) Series that are NOT abs convergent are WEIRD

Ex:
$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

$++-++-$

$$1 + \frac{1}{9} - \frac{1}{4} + \frac{1}{25} + \frac{1}{49} - \frac{1}{16} \cdots = \frac{\pi^2}{12}$$

All I did was rearrange. Seems legit

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \approx .69$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots = \frac{\ln(8)}{2} \approx 1.04 \quad \text{WTF!}$$

Rearrangement does NOT work — but it does work if abs convergent.

(Similarly, reparenthesization only works if convergent)

3) Ratio test actually gives absolute convergence, if you're keeping track at home

4) Abs. Convergence is more powerful than convergence + lets you prove more things, but that's not important until 316. For now your only other tool aside from AST for partially negative sequences.

LECTURE 11 |

POWER SERIES

Ex: $1+x+x^2+x^4$ $a_2=?$ $a_3=?$ $n=?$

Recall: A polynomial is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \quad (a_n \neq 0)$$

n is the degree

Def: A power series (centered at 0) is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n.$$

Running Ex: $1+x^2+x^4+x^6+\dots$ so $a_n = 0$ for n odd

1) Think of it as a function whose values are the limit of a series when x is a specific number

$$p(10) = \sum_{n=0}^{\infty} a_n 10^n$$

in example $p(10) = 1+100+10000+\dots$ (diverges!)

$$p(\tfrac{1}{2}) = 1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \dots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

- Polynomials are defined everywhere. Power series may diverge — not defined everywhere.

Like $\frac{1}{x}$ not defined at 0

Qn 1: Where is the power series defined?

I.e. for which x is $\sum_{n=0}^{\infty} a_n x^n$ convergent? Interval of convergence

Ex: $p(0) = a_0$ always converges.

2) Think of x as a symbol that you manipulate algebraically — can add, multiply, etc power series. Like $(x+1)\cdot(x+2) = x^2+3x+2$.

(Later)

3) Think of the partial sums $p_N(x) = a_0 + a_1 x + \dots + a_N x^N$ as polynomial approximations of the function $p(x)$.

Step 1: Do ratio test first.

Ex: $1+x^2+x^4+x^6+\dots$ geometric series ratio is x^2, converges iff $|x^2|<1 \iff |x|<1$

So interval of convergence is $(-1,1)$.

Ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} \to 0$ for any x.

abs value important when $x<0$!!

So $IoC = (-\infty, \infty)$

(*) Remember, x is fixed number, this is limit as $n \to \infty$, not as x changes!!

Ex: $\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$ $\left|\frac{a_{n+1}}{a_n}\right| = |2x| \cdot \frac{n}{n+1} \longrightarrow 2|x|$

so $a_0 = 0$ no x^0 term.

So convergent by ratio test if $2|x|<1 \iff |x|<\frac{1}{2}$

divergent ——//—— $2|x|>1 \iff |x|>\frac{1}{2}$

ratio test inconclusive if $2|x|=1 \iff x=\pm\frac{1}{2}$.

Let's check $\pm\frac{1}{2}$. When $x=\frac{1}{2}$ get $\sum_{n=1}^{\infty}\frac{1}{n}$ divergent

When $x=\frac{-1}{2}$ get $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ convergent

So $IOC = [-\frac{1}{2}, \frac{1}{2})$.

Ex: $\sum_{n=0}^{\infty} x^n \cdot n!$ $\left|\frac{a_{n+1}}{a_n}\right| = |x| \cdot (n+1) \longrightarrow \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$

So $IOC = \{0\}$.

Thm: There is a radius of convergence $R \geq 0$ or $R=\infty$ such that

- converges when $|x|<R$
- diverges when $|x|>R$
- need to figure out $x=\pm R$ using other means

and you can always find R using the ratio test!!!

Step 1: Find R (the ratio test) Step 2: analyze $|x|=R$ ← some silly edge case... (all the other tests)!!

Ex: $\sum_{n=0}^{\infty} x^n$. $\left|\frac{a_{n+1}}{a_n}\right| = |x|$ conv for $|x|<1$, div for $|x|>1$ $\boxed{R=1}$

div for $x=\pm 1$ $I=(-1,1)$

Ex: $\sum_{n=0}^{\infty} \frac{n^2 x^n}{15^n}$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{15}\right| \cdot \left(\frac{(n+1)^2}{n^2}\right) \to \left|\frac{x}{15}\right|$ $R=15$

At $x=15$ get $\sum n^2$ div

$x=-15$ get $\sum (-1)^n n^2$ div so $I=(-15,15)$

Ex: $\sum_{n=0}^{\infty} \frac{x^n}{n^2 \cdot 15^n}$ Same story but $I=[-15,15]$ because edge cases are $\sum \frac{1}{n^2}$ and $\sum \frac{(-1)^n}{n^2}$.

There are also

Def: A <u>power series centered at c</u> is a function of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Ex: $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ centered at 5.

Same story, $\exists$ radius of conv R where

conv if $|x-c|<R$

div if $|x-c|>R$

i.e. interval is centered on c

In example $\left|\frac{a_{n+1}}{a_n}\right| = |x-5| \cdot \left(\frac{n^2}{(n+1)^2}\right) \to |x-5|$ $R=1$

so $I=[4,6]$ or $(4,6]$ or $[4,6)$ or $(4,6)$

$\xrightarrow[\text{work}]{}$ $I=[4,6]$

Weirdness: A polynomial centered at c $a_0+a_1(x-c)+a_2(x-c)^2$

But $1+2(x-5)+3(x-5)^2 = 3x^2-28x+66$

a poly centered at c is also a poly centered at 0. (ordinary poly)

BUT a power series at C may not be equal to any power series centered at 0

after all, if $I=[4,6]$ it doesn't converge at 0, but any power series at 0 does.

Also: not everything is a power series!

$$\sum_{n=0}^{\infty} (x+n)^n$$

is NOT a power series. w/ any center.

It may be a function, but doesn't have all the nice properties that power series do (e.g. radius of convergence)

(This one converges nowhere!) by div. test

Power Series and the functions they represent

Think of power series as functions. What do you do with functions?

- Add ✓
- Multiply ✓
- Differentiate ✓
- Integrate ✓
- Plug in other functions like $\sin(x^2)$ ✓?
- Evaluate ✗
  Well, what is $\sin(1)$ anyway?
- Approximate ✓✓✓
  the partial sum of a power series is a poly you can easily compute.

Soon we'll do Taylor series: the best power series "approx" to a function.

For now let's do one we know and branch out.

We know that $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \begin{cases} \frac{1}{1-x} & |x|<1 \\ \text{undefined} & |x| \geq 1 \end{cases}$

Let $g(x) = \frac{1}{1-x}$ ← defined for all x except $x=1$!!

NOT THE SAME. Different domain. Same answer where both defined! on $(-1,1)$

$f(2) = 1+2+2^2+\dots \neq g(2) = \frac{1}{1-2} = -1$

$f(2)$ = DNE

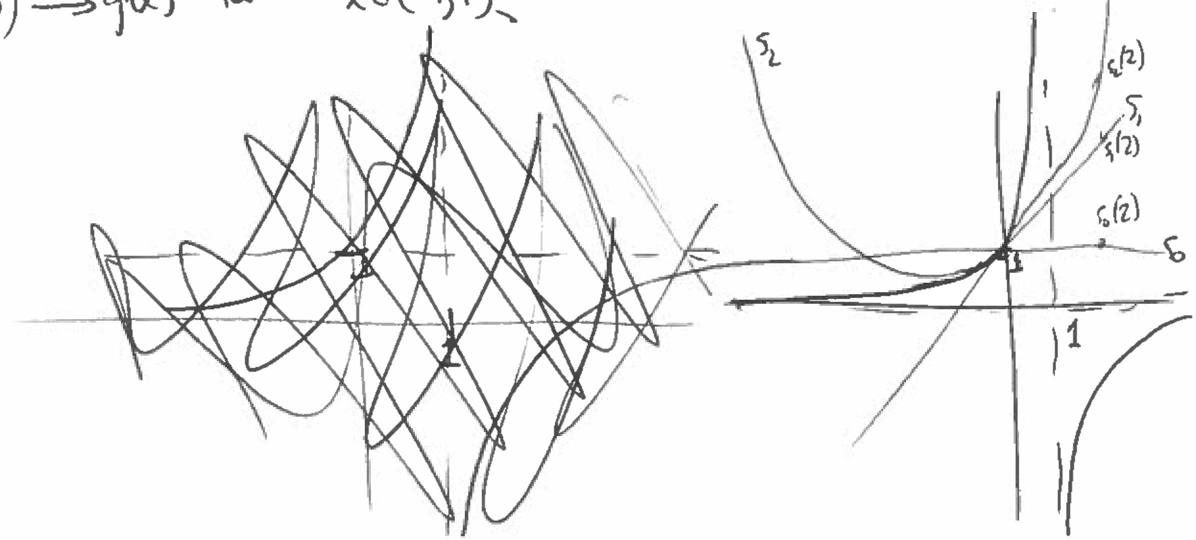
Now f is approximated by the partial sums
$f_0(x) = 1$
$f_1(x) = 1+x$
$f_2(x) = 1+x+x^2$
$\vdots$

"a function approximating a function!" Not just approximating values.

Each f_N is defined everywhere! But $(f_n(5))$ does NOT converge to $g(5)$, it diverges

$(f_n(x)) \to g(x)$ for $x \in (-1,1)$.

Picture:



Let's manipulate this power series.

Function	Power series	
$\frac{1}{1-x}$	$1+x+x^2+\ldots = \sum_{n=0}^{\infty} x^n$	
$\frac{3}{1-x}$	$3(1+x+x^2+\ldots) = \sum 3x^n$	Note: $R=\frac{1}{2}$ now because we rescaled X.
$\frac{1}{1-2x}$	$1+2x+(2x)^2+\ldots = \sum (2x)^n = \sum 2^n x^n$	
$\frac{1}{1-x^2}$	$1+x^2+x^4+\ldots = \sum x^{2n}$	(still a power series, $a_1=a_3=\ldots=0$) $R=?$
$\frac{1}{1+x} = \frac{1}{1-(-x)}$	$1-x+x^2-x^3+\ldots = \sum (-1)^n x^n$	$R=?$
$\frac{1}{1-\sin x}$	$1+\sin x+\sin^2 x+\ldots$	NOT A POWER SERIES. Can plug in power of k and still get a power series, but not a weird function!

$$\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) \qquad \frac{1}{2}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\ldots\right) = \sum \left(\frac{1}{2}\right)^{n+1} x^n \qquad R=?$$

$$\frac{4}{5+6x} = \frac{4}{5}\left(\frac{1}{1+\frac{6}{5}x}\right) \qquad \frac{4}{5}\sum \left(\frac{-6}{5}\right)^n x^n \qquad R=?$$

$$\frac{1}{1-x}+\frac{1}{1+x} = (1+x+x^2+\ldots)+(1-x+x^2-x^3+\ldots) = 2+0x+2x^2+0x^3+\ldots = 2\sum x^{2n}$$

$$= \frac{1+x+1-x}{(1-x)(1+x)} = \frac{2}{1-x^2} = 2\sum x^{2n}.$$

Hooray, we're sane.

Moral: Adding / Rescaling / plugging in kx^l for x works fine for power series

Now derivatives!

$g(x) = \frac{1}{1-x}$ $\quad g'(x) = \frac{(-1)(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} = \frac{1}{1-2x+x^2}$ (plugging in $(2x-x^2)$ is trickier)

Well, $f(x) = 1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n$

$f'(x) = 0+1+2x+3x^2+\dots = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$ $\quad R = ?$ it's 1, by ratio test

the $n=0$ term vanishes

Does $f'(x) = g'(x)$ on $(-1,1)$? Yes.

Thm: If $f(x) = \sum a_n(x-c)^n$ converges w/ radius R around center c, then

so does $\sum n a_n (x-c)^{n-1} = \sum (n+1)a_{n+1}(x-c)^n$ SAME RADIUS!

and it agrees with $f'(x)$.

Similarly, $\sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C = C + \sum_{n=1}^{\infty} \frac{a_{n-1}(x-c)^n}{n}$, and agrees with $F(x) = \int f(t)\,dt$

the antiderivative where $F(c) = C$.

But behavior at $c \pm R$ may be wonky!!

Ex: $\ln(x+1) = g(x)$ defined on $(-1,\infty)$

Can't center $\ln(x)$ at 0.
Instead of centering $\ln(x)$ at 1, let's center $\ln(x+1)$ at 0

$g'(x) = \frac{1}{1+x} = 1-x+x^2-x^3+\dots = \sum (-1)^n x^n$. converges on $(-1,1)$

take antiderivative

$g(x) = \cancel{C} + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = \ln(x+1)}$

$C = g(0) = 0$

but $R = 1$, converges on $(-1,1]$ ← different

So finally, we see that $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ Hooray!

Ex: $h(x)=\arctan(x)$ $\quad h'(x)=\frac{1}{1+x^2}=1-x^2+x^4-x^6+\dots$

$\arctan(0)=0$

$$h(x)=\int\frac{1}{1+x^2}=0+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots \qquad I=(-1,1]$$

$\underset{\arctan(x)}{\parallel}$

So $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots=\arctan(1)=\frac{\pi}{4}$!!

Reiterate: Watch out for: • Reindexing for derivative or integral
• Adding C when antiderivative

Ex: A theorem from differential equations: $e(x)=e^x$ is the unique function such that $e(0)=1$ and $e'(x)=e(x)$.

Now consider $f(x)=\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $\qquad R=\infty$

Then $f(0)=1$ and $f'(x)=0+1+\frac{2x}{2}+\frac{3x^2}{3\cdot2\cdot1}+\frac{4x^3}{4\cdot3\cdot2\cdot1}+\frac{5x^4}{5\cdot4\cdot3\cdot2\cdot1}$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots=f(x)$$

so $f(x)=e(x)$!!!

Note: $\int e=e$ too. (up to scalar) $\quad \int f=C+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

$\boxed{C=e(0)=1.}$

Moral: Just using $(\,)'$ and $\int(\,)$ can find power series for many useful functions.

Ex: Find $\int_0^{.7}\frac{1}{1+x^4}dx$ to within .01. $\quad$ What is $\int\frac{1}{1+x^4}$? NO NICE ANSWER!!!

But we sure can get a nice power series for $\int\frac{1}{1+x^4}$. $\quad$ MOST INTEGRALS DON'T!!!

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \cdots$$

$$\int \frac{1}{1+x^4} = C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \quad \boxed{R=1}$$

Now $g(x) = \int_0^x \frac{1}{1+t^4}\,dt$ has $g(0)=0$ so $C=0$ for this antiderivative.

Now want $.7 - \frac{(.7)^5}{5} + \frac{(.7)^9}{9} - \cdots$ w/in $\frac{1}{100}$

Alternating series, so just need $\frac{(.7)^{4n+1}}{4n+1} \leq \frac{1}{100}$

So $.7 - \frac{(.7)^5}{5}$ is close enough!

Reminder:

$\int \frac{1}{1+x}$ is $\ln(1+x)$

$\int \frac{1}{1+x^2}$ is $\arctan(x)$

$\int \frac{1}{1+x^k}$, $k \geq 3$ is some crazy function! But can approximate it with a power series, within $(-1,1]$.

Multiplying power series

Multiplying polynomials

$$(\overset{a_0}{1}+\overset{a_1}{5}x+\overset{a_2}{23}x^2+\overset{a_3}{2}x^3)(\overset{b_0}{6}-\overset{b_1}{3}x+\overset{b_2}{8}x^2+\overset{b_3}{17}x^3)$$

$$= \underset{a_0b_0}{6} + \underset{a_0 b_1 + a_1 b_0}{(1\cdot(-3) + 5\cdot 6)}x + \underset{a_0b_2+a_1b_1+a_2b_0}{(1\cdot 8+5(-3)+23\cdot 6)}x^2 + \underset{a_0b_3+a_1b_2+a_2b_1+a_3b_0}{(1\cdot 17+5\cdot 8+23\cdot(-3)+2\cdot 6)}x^3$$

$$+ (\quad)x^4 + (\quad)x^5 + \underset{\cancel{a_0b_6}+\cancel{a_1b_5}+\cancel{a_2b_4}+a_3b_3+\cancel{a_4b_2}+\cancel{a_5b_1}+\cancel{a_6b_0}}{(2\cdot 17)}x^6$$

why only one term?

Multiplying power series:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

How to express the above idea in symbols

Ex:

$$\frac{1+3x+2x^2}{1-x} = (1+3x+2x^2)(1+x+x^2+x^3+\dots)$$

$$\begin{aligned} = \;& 1+x+x^2+x^3+\dots \\ +\;& 3x+3x^2+3x^3+\dots \\ +\;& 2x^2+2x^3+2x^4+\dots \\ \hline & 1+4x+6x^2+6x^3+6x^4+\dots \end{aligned}$$

Ex:

$$\left(\frac{1}{1-x}\right)^2 = (1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)$$

$$\begin{aligned} = \;& 1+x+x^2+x^3+x^4+\dots \\ +\;& x+x^2+x^3+x^4+\dots \\ +\;& x^2+x^3+x^4+\dots \\ +\;& x^3+x^4+\dots \\ +\;& x^4+\dots \\ \hline & 1+2x+3x^2+4x^3+5x^4+\dots \end{aligned}$$

or said another way, coeff of x^2 is $a_0b_2+a_1b_1+a_2b_0$
$= 1\cdot 1+1\cdot 1+1\cdot 1 = 3$

Rmk: We already derived this using $\left(\frac{1}{1-x}\right)^2 = \left(\frac{1}{1-x}\right)'$ derivative of

Ex:
$$(1+x)\ln(1+x) = (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$
$$+ x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots$$
$$x + \left(1-\frac{1}{2}\right)x^2 + \left(\frac{-1}{2}+\frac{1}{3}\right)x^3 + \left(\frac{1}{3}-\frac{1}{4}\right)x^4 + \left(\frac{-1}{4}+\frac{1}{5}\right)x^5 + \cdots$$
$$= x + \underset{\uparrow 1\cdot 2}{\frac{1}{2}x^2} - \underset{\uparrow 2\cdot 3}{\frac{1}{6}x^3} + \underset{\uparrow 3\cdot 4}{\frac{1}{12}x^4} - \underset{\uparrow 4\cdot 5}{\frac{1}{20}x^5} \quad \left(+\frac{1}{30}x^6 - \frac{1}{42}x^7 + \cdots\right)$$
$$= x + \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}.$$

Confirm: This is also the integral of $1+\ln(x+1)$!

Another Kind of question I might ask on a quiz/test:

Find the terms up to degree 4 in the power series for $\dfrac{\arctan x}{1-2x}$

maybe too high for test

$$\left(1+2x+4x^2+8x^3+16x^4\right)\left(x - \frac{x^3}{3} + \frac{x^5}{7} \cdots\right)$$

$16x^4$: ~~very~~ not important

$\frac{x^5}{7}$: not important

$$= x + 2x^2 + 4x^3 + 8x^4 + \cdots$$
$$- \frac{x^3}{3} - \frac{2x^4}{3} + \cdots$$
$$x + 2x^2 + \left(4-\frac{1}{3}\right)x^3 + \left(8-\frac{2}{3}\right)x^4 + \cdots$$

don't expect a super nice formula since this function sucks