Diff Eq

Phys/Chem/Econ/etc give you a model.

Ex 1 Growth of rabbits $P(t)$ = pop. at time t.

k = "percentage of rabbits giving birth", growth rate (usually determined experimentally) (depends on units)

$$\boxed{P'(t) = k \cdot P(t)}$$

The model approximates a more chaotic phenomenon

Spose $k = \frac{1}{2}$. Anyone find P? $e^{\frac{1}{2}t}$ good. Easier? 0. (Realistic, eventually) $2e^{\frac{1}{2}t}, 26e^{\frac{1}{2}t}, -3e^{\frac{1}{2}t}$

$\boxed{Ce^{\frac{1}{2}t}}$ general solution

Right now, $P(0) = 5$ bunnies. (initial condition)   Want • $P(15)$ • qualitative behavior $5e^{\frac{1}{2}t}$ specific solution

$P(15) = 5e^{7.5}$ (a lot of bunnies) goes to infinity.

Ex 2 Growth of a business $Q(t)$ = % of city pop. which is customer.

$$Q' = kQ(1-Q)$$

some constant.   more pop = more hype   too crowded. Hipsters stop coming.

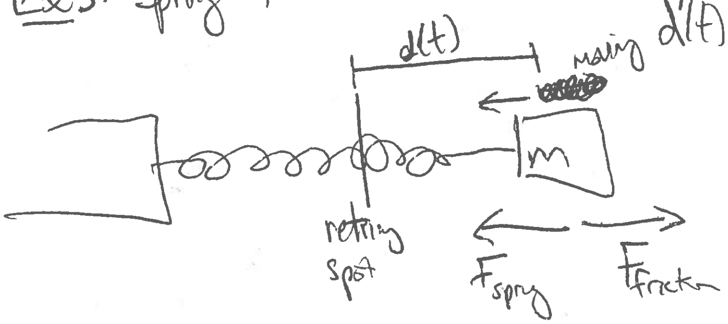
Initial: $Q(0) = .1$

General solution? Anyone?

logistic growth $\frac{1}{1+Ce^{kt}} = Q(t)$

specific solution $Q(0) = \frac{1}{1+C}$ $C = 9$

Ex 3: Spring w/ mass attached



$$d''(t) = -\frac{k}{m} d - \gamma d'$$

spring constant   mass   circled: a constant   friction coeff

General solution:

$$d(t) = e^{-at}(A\cos bt + B\sin bt)$$

a, b defined by k, m, γ, fixed   A, B parameters.

Initial condition: (at time 6) $d(6) = 5$ is that enough?   need velocity too $d'(6) = -1$

$d(0) = A$
$d'(0) = -aA + bB$
together can solve for A, B.

Def: An ~~Ordinary Differential Equation (ODE)~~ is an equation involving a function of one variable $y(t)$ and its derivatives $y^{(n)}(t)$. It's order is the highest derivative that appears

Ex 1+2: first order   Ex 3: 2^{nd} order

Ex: $y^{(5)}(t)\cdot y(t) - y''(t)\cdot \sin(t) = \dfrac{1}{y(t)^2}$ $\boxed{\checkmark}$ 5^{th} order;

↑ Don't expect to solve something like this! But

An ~~initial condition~~ initial value for an n^{th} order ODE at $t=t_0$ is the value of $y(t_0)=$ ___ , $y'(t_0)=$ ___ , $\dots$ $y^{(n-1)}(t_0)=$ ___

Ex: Initial value at $t=5$: $y(5)=1$ $y'(5)=2$ $y''(5)=\pi$ $y'''(5)=17$ $y''''(5)=-26$

(enough to determine $y^{(5)}(t)$ by equation and $y^{(6)}(t)$ and ...)

$\boxed{\text{Nonex:}}$ $y(5)=1$ $y'(3)=2$ $y''(-1)=12$ ... need same point to

Thm: Unless something goes wrong (not this class) an ODE w IV has a unique solution!

Initial value problem IVP → find the solution.

Ex: $f''=f$, $f(0)=2$, $f'(0)=4$

MAT 256 says: $f(t)=C_1e^t+C_2e^{-t}$ is the general solution

$f(0)=C_1+C_2=2$
$f'(0)=C_1-C_2=4$ $\rightsquigarrow$ $C_1=3$, $C_2=-1$ $f(t)=3e^t-e^{-t}$.

MAT 256: How to solve the easiest ODEs
MAT 253: How to approximately solve the more general ODE. "Solve" IVPs with polynomials/power series to arbitrary degree.

Ex: $f''=f$, $f(0)=2$, $f'(0)=4$

IV was at $t_0=0$ so let that be our center.

$f(t)=\sum_{n=0}^{\infty} a_nt^n$. $f(0)=a_0=2$, $f'(0)=a_1=4$ what's $a_2, a_3, \dots$?

$\cancel{f'(t)=\sum_0^\infty n a_n t^{n-1} = \sum_1^\infty n a_n t^{n-1} = \sum_0^\infty (n+1)a_{n+1}t^n}$

$\cancel{f''(t)=\sum_0^\infty n(n-1)a_nt^{n-2} = \sum_2^\infty n(n-1)a_nt^{n-2} = \sum_0^\infty (n+2)(n+1)a_{n+2}t^n}$

$f''(0)=f(0)=2=2!\,a_2$ $a_2=\frac{2}{2!}=1$. Now taking derivatives $f''=f'$!

$f'''(0)=f'(0)=4=3!\,a_3$ $a_3=\frac{4}{3!}$ $f''''(0)=f''(0)=f(0)=2$ $a_4=\frac{2}{4!}$

Approximate answer to degree 4: $2+4x+\frac{2}{2!}x^2+\frac{4}{3!}x^3+\frac{2}{4!}x^4$

(How close is this to the true answer? Can't really use TRT b/c don't know enough, but there are ways of doing this, not in this class.)

$f''(2)=7 \quad f'''(2)=3 \quad f''''(2)=7$

Ex: $f''=f \qquad f(2)=7 \quad f'(2)=3 \Rightarrow$

$f(t)=\sum a_n(t-2)^n$ then $a_n=\frac{f^{(n)}(2)}{n!}$

get $7+3(t-2)+\frac{7}{2!}(t-2)^2+\frac{3}{3!}(t-2)^3+\frac{7}{3!}(t-2)^4+\dots$

Ex: $f'=t^2f+2$ Initial at $t_0=0$: $f(0)=3 \Rightarrow f'(0)=0^2\cdot 3+2=2$

$\Rightarrow f''=?$

derivative

$f''=2tf+t^2f' \qquad f''(0)=2\cdot 0\cdot 3+0^2\cdot 2=0$

$f'''=2f+2tf'+2tf'+t^2f'' \qquad f'''(0)=2\cdot 3+0+0+0=6$ seems messy!

$f(t)\approx 3+2t+0t^2+\frac{6}{3!}t^3$ maybe we need to do this more efficiently to get degree 100.

Current skills:
- What is an ODE? An IV?
- Given an IVP, find solution to degree N as a ~~power series~~ polynomial centered at t_0 by computing derivatives.

Next: Find general solution! Sometimes easier.

Ex: $f'=t^2f+2 \qquad f(0)=a_0$ ← the general case. $f(t)=\sum a_nt^n$.

$f'=\sum_{n=0} na_nt^{n-1}=\sum_1 na_nt^{n-1}=\sum_0 (n+1)a_{n+1}t^n = a_1+2a_2t+\sum_2 (n+1)a_{n+1}t^n$

$t^2f=\sum_0 a_nt^{n+2}=\sum_2 a_{n-2}t^n \qquad 2+t^2f=2+0t+\sum_2 a_{n-2}t^n$

reindex crucial to compare

If they're equal then each coeff is equal.

$$\begin{array}{cccccccc} a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & \dots & (n+1)a_{n+1} \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & & \shortparallel \\ 2 & 0 & a_0 & a_1 & a_2 & & a_{n-2} \end{array}$$

So $\boxed{a_0+2t+0t^2+\frac{a_0}{3}t^3+\frac{2}{4}t^4+\frac{0}{5}t^5+\frac{a_0}{3\cdot 6}t^6+\frac{2}{4\cdot 7}t^7+\frac{0}{5\cdot 8}t^8+\dots}$

recursion formula!!!

The idea: The IV gives the first few coeffs (base case)
The ODE gives a recursive formula for the rest.

Ex: $f''=f$ $f(0)=a_0$ $f'(0)=a_1$ $f(t)=\sum a_n t^n$

$$f''(t)=\sum_0 a_n(n)(n-1)t^{n-2}=\sum_2 a_n n(n-1)t^{n-2}=\sum_0 a_{n+2}(n+2)(n+1)t^n$$

$$f=\sum_0 a_n t^n$$

get $a_{n+2}(n+2)(n+1)=a_n$

or $a_{n+2}=\dfrac{a_n}{(n+1)(n+2)}$

$a_0=a_0$

$a_1=a_1$

$a_2=\dfrac{a_0}{1\cdot 2}$

$a_3=\dfrac{a_1}{2\cdot 3}$ $a_5=\dfrac{a_3}{4\cdot 5}=\dfrac{a_1}{2\cdot 3\cdot 4\cdot 5}$

$a_4=\dfrac{a_2}{3\cdot 4}=\dfrac{a_0}{1\cdot 2\cdot 3\cdot 4}$

can actually solve this recursion to get

$$a_n=\begin{cases}\dfrac{a_0}{n!} & n \text{ even}\\[2mm] \dfrac{a_1}{n!} & n \text{ odd.}\end{cases}$$

Example problem: $y''+3y'-5y=0$ $y(0)=17$ $y'(0)=26$

Find a recursive formula for the Taylor coeffs of y, and give the terms up to x^4.

Outline: • Use IV to give base case $a_0=17$ $a_1=26$

(If 3rd order ODE $y'''(0)=36$ then $a_3=\dfrac{36}{3!}$)

• Turn all into a power series $y(t)=\sum a_n t^n$ $3y'=?$ $-5y=?$ $y''=?$

and reindex so you can add!

$$y'=\sum_0 n a_n t^{n-1}=\sum_0 (n+1)a_{n+1}t^n$$

$$y''=\sum_0 n(n-1)a_n t^{n-2}=\sum_0 (n+2)(n+1)a_{n+2}t^n$$

$$y''+3y'-5y=\sum_0\left[(n+2)(n+1)a_{n+2}+3(n+1)a_{n+1}-5a_n\right]t^n=0$$

so $\boxed{a_{n+2}=\dfrac{5a_n-3(n+1)a_{n+1}}{(n+1)(n+2)}}$ w/ $\boxed{a_0=17 \quad a_1=26}$

see ✱ on next page

so $\boxed{n=0}$ $a_2 = \dfrac{5\cdot 17 - 3\cdot 1\cdot 26}{1\cdot 2} = \dfrac{7}{2}$

$\boxed{n=1}$ $a_3 = \dfrac{5\cdot 26 - 3\cdot 2\cdot \frac{7}{2}}{2\cdot 3} = \dfrac{109}{6}$

$\boxed{n=2}$ $a_4 = \dfrac{5\cdot\frac{7}{2} - 3\cdot 3\cdot\frac{109}{6}}{3\cdot 4} = \left(\dfrac{35-327}{2\cdot 3\cdot 4}\right) = \dfrac{-292}{24} = \dfrac{-73}{6}$

BTW, What's the general solution to $y''+3y'-5y=0$? a_0, a_1 parameters!

determined by $\boxed{a_{n+2} = \dfrac{5a_n - 3(n+1)a_{n+1}}{(n+1)(n+2)}}$ for $n \geq 0$. So $a_2 = \dfrac{5a_0 - 3a_1}{2}$

$a_3 = \dfrac{5a_1 - 3\cdot 2\cdot\left(\frac{5a_0 - 3a_1}{2}\right)}{2\cdot 3}$

$= \dfrac{-15a_0 + 14a_1}{6}$, etc.

not so easy to solve so we'll stop at recursive formula + trust computer to easily find first 100 terms.

When the center isn't zero?

Ex: $y''+3y'-5y=0$ $y(6)=17$ $y'(6)=26$ $\boxed{\text{Center at } 6.}$

$y(t) = \sum a_n(t-6)^n$ $y' = \sum a_{n+1}(n+1)(t-6)^n$ "same" formula

get same recursion $a_{n+2} = \dfrac{5a_n - 3(n+1)a_{n+1}}{(n+1)(n+2)}$ $a_0 = 17$ $a_1 = 26$

Coeffs have a different meaning now, encoding derivatives at $t=6$ not $t=0$

But shifting the center is not always so easy.

Ex: $y' - ty = 0$ centered at 0 $(y(0)=5)$ $y = \sum a_n t^n$ vs centered at 3 $(y(3)=5)$ $y = \sum a_n(t-3)^n$

$y' = \sum_0 a_{n+1}(n+1)t^n = a_1 + \sum_1 a_{n+1}(n+1)t^n$

$ty = \sum_1 a_{n-1}t^n$

$(a_0=5)$ $a_1 = 0$ $\boxed{a_{n+1} = \dfrac{a_{n-1}}{n+1}}$ so $a_2 = \dfrac{a_0}{2}$

$a_3 = 0$ $a_5 = 0$

$a_4 = \dfrac{a_2}{4} = \dfrac{a_0}{2\cdot 4}$

$y' = \sum_0 a_{n+1}(n+1)(t-3)^n$.

$\boxed{ty = ???}$ $t = (t-3)+3$ power series for t centered at 3

$y' = (t-3)y + 3y$

$\sum_0 a_{n+1}(n+1)(t-3)^n = \sum_1 a_{n-1}(t-3)^n + \sum_0 3a_n(t-3)^n$

$\boxed{a_1 = 3a_0}$ $2a_2 = a_0 + 3a_1$ $a_{n+1} = \dfrac{a_{n-1} + 3a_n}{(n+1)}$.