Math 253 (Calculus III), Winter 2019
Practice Final Solutions

1. **(28 pts)** Does the sequence converge or diverge? WHY? If it converges, what is the limit?

   (a) $a_n = \cos(\dfrac{3n^3 + 16n}{16n^3 + 12n^2 + 3})$

   Solution: By examination of leading terms, $\lim \frac{3n^3+16n}{16n^3+12n^2+3} = \lim \frac{3n^3}{16n^3} = \frac{3}{16}$. Since $\cos$ is a continuous function, we get $\lim a_n = \cos(\frac{3}{16})$, and the sequence converges.

   (b) $\bar{b} = (\dfrac{1}{2}, 0, \dfrac{1}{3}, 0, 0, \dfrac{1}{4}, 0, 0, 0, \dfrac{1}{5}, 0, 0, 0, 0, \ldots)$

   Solution: This sequence converges to zero. There are many reasonable explanations, and most of them use words! Something naive like "It switches (after varying amounts of time) between two sequences which both converge to zero" would be fine. I would probably accept anything reasonable and not erroneous.

   It does NOT "alternate" between the sequence $\frac{1}{n}$ and the sequence 0. The word "alternate" really implies taking turns.

   One can NOT use the following argument: we have $0 \le b_n \le \frac{1}{n}$ so the squeeze theorem implies that $\lim b_n = 0$. Why not? (Answer: because $b_6 = \frac{1}{4} > \frac{1}{6}$.) One could use $0 \le b_n \le \frac{1}{\sqrt{n}}$ but that is tricky!

   (c) $c_n = \dfrac{n^2}{e^n}$

   Solution: By extension to a function and L'Hopital's rule we have

   $$\lim \frac{n^2}{e^n} = \lim \frac{2n}{e^n} = \lim \frac{2}{e^n} = 0,$$

   since 2 is bounded and e^n increases to infinity.

   (d) $d_n = \dfrac{n - 20}{n + 3} + (-1)^n$

   Solution: This diverges. By examining leading terms, $\frac{n-20}{n+3}$ converges to 1. However, $(-1)^n$ diverges, and by the 2/3 rule, the sum of convergent and divergent is divergent.

2. **(21 pts)** Does the series converge or diverge? WHY?

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2 5^n}{3^{2n+4}}$

Solution: We do the ratio test. $\lim \left| \frac{(n+1)^2}{n^2} \frac{5^{n+1}}{5^n} \frac{3^{2n+4}}{3^{2(n+1)+4}} \right| = \lim \left| \frac{(n+1)^2}{n^2} \frac{5}{3^2} \right| = \frac{5}{9}$. Since this is less than 1, the sequence converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^5)}{n\sqrt{n}}$

Solution: We have $-1 \leq \sin(n^5) \leq 1$ so $|\sin(n^5)| \leq 1$. Now $\sum_{n=1} \frac{|\sin(n^5)|}{n\sqrt{n}}$ converges by the comparison test, since $\sum_{n=1} \frac{|\sin(n^5)|}{n\sqrt{n}} \leq \sum_{n=1} \frac{1}{n\sqrt{n}}$ which converges by the p-test, $p = 1.5$. So the original sequence converges by the absolute convergence test.

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^3+1}{n^3+2}$

Solution: Since $\lim \frac{n^3+1}{n^3+2} = 1$, the sequence $(-1)^n \frac{n^3+1}{n^3+2}$ diverges (it alternates between a sequence converging to 1 and a sequence converging to -1, so it never stays close to either one). By the divergence test, this series diverges.

3. **(21 pts)** Does the series converge or diverge? WHY?

   (a) $\displaystyle\sum_{n=15}^{\infty} \frac{n+2}{13n^2+12}$

   Solution: Looking at leading terms, we expect this to behave like $\sum \frac{1}{13n}$, which diverges. To prove it diverges, we use the comparison test:

$$\frac{n+2}{13n^2+12} \geq \frac{1}{2} \cdot \frac{1}{13n}$$

   for large enough n, and $\sum \frac{1}{2} \cdot \frac{1}{13n}$ diverges by the p-test.

   (b) $1 - \dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{8} - \dfrac{1}{16} - \dfrac{1}{32} - \dfrac{1}{64} + \dfrac{1}{128} - \ldots$

   Solution: This converges because it absolutely converges. The sum of the absolute values is $\sum_{n=0}^{\infty} \frac{1}{2^n}$, which is a convergent geometric series with $r = \frac{1}{2}$.

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (Hint: what is the derivative of $\ln(\ln(t))$?)

   Solution: The derivative of $\ln(\ln(t))$ is $\frac{1}{t \ln t}$. Since $\lim_{t\to\infty} \ln(\ln(t)) = \infty$, we expect this series to diverge. In fact, by the integral test $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \geq \int_2^{\infty} \frac{1}{t \ln t} dt = \ln(\ln(t)) \mid_2^{\infty} = \infty$.

4. **(14 pts)** If the series converges, find the sum, and justify your answer. If it diverges, explain why.

   (a) $\displaystyle\sum_{n=5}^{\infty} 6\left(\frac{11}{10}\right)^n$

   Solution: This is a geometric series with $a = 6(\frac{11}{10})^5$ and $r = \frac{11}{10}$. Since $r > 1$, the series diverges.

   (b) $3 + 2 + \dfrac{4}{3} + \dfrac{8}{9} + \dfrac{16}{27} + \ldots$

   Solution: This is a geometric series with $a = 3$ and $r = \frac{2}{3}$. Since $r < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = 9$.

5. **(16 pts)** Consider the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   (a) Find an upper bound for the difference between $1 + \frac{1}{2^4} + \frac{1}{3^4}$ and the sum of the series.
   Solution: By the integral test,
$$\sum_{n=4}^{\infty} \frac{1}{n^4} \leq \int_3^{\infty} \frac{1}{t^4} dt = \frac{-1}{3t^3} |_3^{\infty} = \frac{1}{3 \cdot 3^3} = \frac{1}{81}.$$

   (b) How many terms of the series must one add to approximate the sum to within $\frac{1}{300}$?
   Solution: By the same argument,
$$\sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \int_N^{\infty} \frac{1}{t^4} dt = \frac{1}{3N^3}.$$
   When $N = 5$ we have $N^3 > 100$ so $3N^3 > 300$ so $\frac{1}{3N^3} < \frac{1}{300}$. Five terms is sufficient.
   It's also fine to solve for N and say that N is greater than the cube root of 100.

6. **(20 pts)** Consider the series $-\frac{3}{11} + \frac{3}{14} - \frac{3}{17} + \frac{3}{20} - \ldots$.

   (a) Find an explicit formula for this series.
   Solution: This is $\sum_{n=1}^{\infty}(-1)^n \frac{3}{8+3n}$. (One could also do $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{3}{11+3n}$.)

   (b) Is the series convergent? WHY?
   Solution: Yes, by the Alternating Series Test, because $\lim \frac{3}{8+3n} = 0$.

   (c) Is the series absolutely convergent? WHY?
   Solution: No, by the comparison test, because $\sum \frac{3}{3n+8} \geq \sum \frac{1}{2} \cdot \frac{1}{n}$ which diverges by p-test, $p = 1$.

   (d) How many terms should one add to approximate the sum to within $.01$?
   Solution: The error is bounded by the absolute value of the next term. We have $\frac{3}{11+3(N+1)} < \frac{1}{100}$ if and only if $300 < 11 + 3(N+1) = 14 + N$ if and only if $N > \frac{286}{3}$. (So $N = 96$ terms will do.)

7. **(27 pts)** Find the interval of convergence for the following power series.

   (a) $\sum_{n=0}^{\infty} \sqrt{n}(6.3)^n x^n$

   Solution: Doing the ratio test, we have $r = \lim \frac{\sqrt{n+1}}{\sqrt{n}} \cdot 6.3x = 6.3x$. So the sequence converges when $|x| < \frac{1}{6.3}$, and diverges when $|x| > \frac{1}{6.3}$.
   When $x = \frac{1}{6.3}$ the series is $\sum \sqrt{n}$ which diverges by the divergence test.
   When $x = -\frac{1}{6.3}$ the series is $\sum (-1)^n \sqrt{n}$ which diverges by the divergence test.
   So the interval of convergence is $(-\frac{1}{6.3}, +\frac{1}{6.3})$.

   (b) $\sum_{n=0}^{\infty} \frac{(x-6)^n}{2^n \cdot n!}$

   Solution: Doing the ratio test, we have $r = \lim \frac{(x-6)}{2} \frac{n!}{(n+1)!} = \lim \frac{(x-6)}{2(n+1)} = 0$. Since $r < 1$ always, this sequence converges everywhere.

   (c) $\sum_{n=0}^{\infty} \frac{(x+4)^n}{n+2}$

   Solution: Doing the ratio test, we have $r = \lim (x+4)\frac{n+2}{n+3} = x+4$. So the sequence converges when $|x+4| < 1$ and diverges when $|x+4| > 1$.
   When $x = -3$ the series is $\sum \frac{1}{n+2}$ which diverges by (comparison to) p-test, $p = 1$ (or by the integral test).
   When $x = -5$ the series is $\sum \frac{(-1)^n}{n+2}$ which converges by the alternating series test.
   So the interval of convergence is $[-5, 3)$.

8. **(13 pts)** Compute derivatives to find the Taylor series for $g(t) = 2t^3 - 5t^2 + 2$ centered at $t = 2$.

Solution: We have

$$g(t) = 2t^3 - 5t^2 + 2, \qquad g(2) = 16 - 20 + 2 = -2,$$
$$g'(t) = 6t^2 - 10t, \qquad g'(2) = 24 - 20 = 4,$$
$$g''(t) = 12t - 10, \qquad g''(2) = 24 - 10 = 14,$$
$$g'''(t) = 12, \qquad g'''(2) = 12.$$

All higher derivatives are zero. So the taylor series is equal to the third taylor polynomial, which is

$$\frac{-2}{0!} + \frac{4}{1!}(t-2) + \frac{14}{2!}(t-2)^2 + \frac{12}{3!}(t-2)^3 (+\frac{0}{4!}(t-2)^4 + 0 + 0 + \ldots)$$
$$= -2 + 4(t-2) + 7(t-2)^2 + 2(t-2)^3.$$

9. **(20 pts)** Find the second degree Taylor approximation of $\ln(x)$ centered at $x = 10$. Bound the error on the interval $(8, 12)$.

Solution: Letting $f(x) = \ln(x)$ we have $f(10) = \ln 10$, and

$$f'(x) = \frac{1}{x}, \qquad f'(10) = \frac{1}{10},$$
$$f''(x) = \frac{-1}{x^2}, \qquad f''(10) = \frac{-1}{100}.$$

So the second degree Taylor polynomial is

$$\ln 10 + \frac{1}{10}(x - 10) + \frac{-1}{200}(x - 10)^2.$$

Computing one more derivative we have

$$f'''(x) = \frac{2}{x^3}$$

which is a decreasing function. On the interval $[8, 12]$ the maximal value of $f'''(x)$ is obtained at $x = 8$, and is $\frac{2}{8^3}$. We can let this be M.

The Taylor Remainder Theorem states that the error $R_N(x)$ is less than $M\frac{d^{N+1}}{(N+1)!}$ on the interval $[c - d, c + d]$. Here $N = 2$, $d = 2$, $c = 10$, and $M = \frac{2}{8^3}$, so the error is bounded by $\frac{2 \cdot 2^3}{3! \cdot 8^3} = \frac{1}{192}$.

10. **(18 pts)** Find a power series centered at zero for the following functions. Write out the first three nonzero terms explicitly.

(a) $x^2 \arctan(x^5)$

Solution: We have

$$\arctan(x) = \sum_0 (-1)^n \frac{x^{2n+1}}{2n+1},$$

so

$$\arctan(x^5) = \sum_0 (-1)^n \frac{(x^5)^{2n+1}}{2n+1} = \sum_0 (-1)^n \frac{x^{10n+5}}{2n+1}$$

and

$$x^2 \arctan(x^5) = \sum_0 (-1)^n \frac{x^{10n+7}}{2n+1}.$$

The first three terms are

$$\frac{x^7}{1} - \frac{x^{17}}{3} + \frac{x^{27}}{5}.$$

(b) $\int_0^x \frac{1}{5+2t} dt$

Solution: We have

$$\frac{1}{5+2t} = \frac{1}{5} \cdot \frac{1}{1-(\frac{-2t}{5})} = \sum_0 \frac{1}{5}(\frac{-2t}{5})^n = \sum_0 (-1)^n \frac{2^n t^n}{5^{n+1}}.$$

Integrating this we get

$$\sum_0 (-1)^n \frac{2^n}{5^{n+1}} \frac{t^{n+1}}{n+1}.$$

The first three terms are

$$\frac{t}{5} - \frac{2t^2}{50} + \frac{4t^3}{375}.$$

11. **(10 pts)** Find the terms up to degree 5 in the power series centered at zero for the following function.

$$(4-x^2)\sin x$$

Solution: We have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ so that

$$4\sin x = 4x - \frac{4}{3!}x^3 + \frac{4}{5!}x^5 - \ldots$$

$$-x^2 \sin x = 0x - x^3 + \frac{1}{3!}x^5 - \ldots$$

and adding we get

$$(4-x^2)\sin x = 4x - (\frac{4}{3!}+1)x^3 + (\frac{4}{5!}+\frac{1}{3!})x^5 - \ldots$$

12. **(30 pts)** Let $y = \sum_{n=0}^{\infty} a_n (t-3)^n$ be a power series centered at 3.

(a) Write down power series centered at 3 for y', for y'', and for $y'' - y' + y$.
Solution: We have

$$y' = \sum_0 (n+1)a_{n+1}(t-3)^n,$$

$$y'' = \sum_0 (n+2)(n+1)a_{n+2}(t-3)^n,$$

$$y'' - y' + y = \sum_0 ((n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n)(t-3)^n.$$

(b) Suppose that y solves the differential equation $y'' - y' + y = 0$. Write down a recursive formula for the coefficients a_n.
Solution: For all $n \geq 0$ we have $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$, or

$$a_{n+2} = \frac{(n+1)a_{n+1} - a_n}{(n+1)(n+2)},$$

which is sufficient to determine all a_n from the base cases $a_0 = y(3)$ and $a_1 = y'(3)$.

(c) Suppose that $y(3) = 2$ and $y'(3) = -4$. Find the coefficients a_n for $n \leq 3$.
Solution: We have $a_0 = 2$, $a_1 = -4$,

$$a_2 = \frac{1a_1 - a_0}{2} = -3,$$

$$a_3 = \frac{2a_2 - a_1}{6} = \frac{-8}{6}.$$

13. **(14 pts)** Let y be a solution to the differential equation $y'' = y \cdot y'$, satisfying the initial conditions $y(1) = 2$ and $y'(1) = 3$. Compute the Taylor polynomial $T_3(x)$ for y of degree 3 centered at 1. (Hint: Do NOT attempt to find the general power series solution, this is too hard.)

Solution: We have

$$y'' = y \cdot y', \qquad y''(1) = 2 \cdot 3 = 6,$$

and by the product rule we have

$$y''' = y' \cdot y' + y \cdot y'', \qquad y'''(1) = 3 \cdot 3 + 2 \cdot 6 = 21.$$

So

$$T_3(x) = \frac{2}{0!} + \frac{3}{1!}(x-1) + \frac{6}{2!}(x-1)^2 + \frac{21}{3!}(x-1)^3 = 2 + 3(x-1) + 3(x-1)^2 + \frac{7}{2}(x-1)^3.$$