Math 253 (Calculus III), Winter 2019
Practice Final

General notes:

1. Pace yourself. Make sure you try all the problems.
2. Check your answers! Find your terrible algebra mistakes!
3. If you need more space, use the back of the previous page. Any scratch paper will need to be collected.
4. Demonstrating knowledge of what to do (and what the subtleties are) is as important as doing it without errors.
5. When you use a method (like the Alternating Series Test or the Taylor Remainder Theorem), you should name it, and make sure you've done what you need to do to use it.
6. If you don't know one of the memorizable power series and can't re-derive it, make a note and replace it with one that you do know.

1. **(28 pts)** Does the sequence converge or diverge? WHY? If it converges, what is the limit?

   (a) $a_n = \cos(\frac{3n^3 + 16n}{16n^3 + 12n^2 + 3})$

   (b) $\bar{b} = (\frac{1}{2}, 0, \frac{1}{3}, 0, 0, \frac{1}{4}, 0, 0, 0, \frac{1}{5}, 0, 0, 0, 0, \ldots)$

   (c) $c_n = \frac{n^2}{e^n}$

   (d) $d_n = \frac{n - 20}{n + 3} + (-1)^n$

2. **(21 pts)** Does the series converge or diverge? WHY?

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2 5^n}{3^{2n+4}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^5)}{n\sqrt{n}}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^3+1}{n^3+2}$

3. **(21 pts)** Does the series converge or diverge? WHY?

(a) $\displaystyle\sum_{n=15}^{\infty} \frac{n+2}{13n^2+12}$

(b) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \frac{1}{128} - \dots$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (Hint: what is the derivative of $\ln(\ln(t))$?)

4. **(14 pts)** If the series converges, find the sum, and justify your answer. If it diverges, explain why.

   (a) $\displaystyle\sum_{n=5}^{\infty} 6(\frac{11}{10})^n$

   (b) $3 + 2 + \dfrac{4}{3} + \dfrac{8}{9} + \dfrac{16}{27} + \dots$

5. **(16 pts)** Consider the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   (a) Find an upper bound for the difference between $1 + \dfrac{1}{2^4} + \dfrac{1}{3^4}$ and the sum of the series.

   (b) How many terms of the series must one add to approximate the sum to within $\frac{1}{300}$?

6. **(20 pts)** Consider the series $-\frac{3}{11}+\frac{3}{14}-\frac{3}{17}+\frac{3}{20}-\ldots$.

(a) Find an explicit formula for this series.

(b) Is the series convergent? WHY?

(c) Is the series absolutely convergent? WHY?

(d) How many terms should one add to approximate the sum to within $.01$?

7. **(27 pts)** Find the interval of convergence for the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}(6.3)^n x^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-6)^n}{2^n \cdot n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{n+2}$

8. **(13 pts)** Compute derivatives to find the Taylor series for $g(t) = 2t^3 - 5t^2 + 2$ centered at $t = 2$.

9. **(20 pts)** Find the second degree Taylor approximation of $\ln(x)$ centered at $x = 10$. Bound the error on the interval $(8, 12)$.

10. **(18 pts)** Find a power series centered at zero for the following functions. Write out the first three nonzero terms explicitly.

    (a) $x^2 \arctan(x^5)$

    (b) $\displaystyle\int_0^x \frac{1}{5+2t}\,dt$

11. **(10 pts)** Find the terms up to degree 5 in the power series centered at zero for the following function.

$$(4 - x^2)\sin x$$

12. **(30 pts)** Let $y = \sum_{n=0}^{\infty} a_n (t-3)^n$ be a power series centered at 3.

(a) Write down power series centered at 3 for y', for y'', and for $y'' - y' + y$.

(b) Suppose that y solves the differential equation $y'' - y' + y = 0$. Write down a recursive formula for the coefficients a_n.

(c) Suppose that $y(3) = 2$ and $y'(3) = -4$. Find the coefficients a_n for $n \leq 3$.

13. **(14 pts)** Let y be a solution to the differential equation $y'' = y \cdot y'$, satisfying the initial conditions $y(1) = 2$ and $y'(1) = 3$. Compute the Taylor polynomial $T_3(x)$ for y of degree 3 centered at 1. (Hint: Do NOT attempt to find the general power series solution, this is too hard.)