Math 253 (Calc III), Winter 2019
Practice Midterm 1 Solutions

Things in parentheses are either alternatives or things you don't need to say, I'm putting them in for instruction.

1. Does the sequence converge or diverge? WHY? If it converges, what is the limit.

(a)

$$a_n = \sqrt{\frac{4n^2 - 2n + 3}{2n^2 + 17}}$$

Inside converges to $\frac{4}{2} = 2$ *by leading terms (or by L'Hop).* f *of limit is limit of* f *(since* f *is continuous). Converges to* $\sqrt{2}$*.*

(b)

$$b_n = \frac{n^n}{n!}$$

$$b_n = \frac{n \cdot (n \cdot \ldots \cdot n)}{1 \cdot (2 \cdot \ldots \cdot n)} > n$$

so it diverges, goes to ∞*. (This is the inverse squeeze theorem technically.)*

(c)

$$c_n = (\frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-2}{3}, \frac{1}{4}, \frac{-3}{4}, \frac{1}{5}, \frac{-4}{5}, \ldots)$$

Half the terms go to 0, other half to -1, but doesn't stay close to one limit. Diverges.

(d)

$$d_n = 8000(\frac{99}{100})^n$$

Geometric sequence, $r = \frac{99}{100} < 1$*. Converges to zero.*
Common error: Now that you've seen geometric SERIES, you're tempted to say the limit is $\frac{a}{1-r}$... but that's a series, not a sequence.

(e)

$$e_n = \frac{\ln n}{\ln(\ln n)}$$

Use L'Hop. $\lim e_n = \lim \frac{\frac{1}{n}}{\frac{1}{n \ln n}} = \lim \ln n = \infty$*. Diverges.*

2. Consider the sequence $a_n = \frac{n^2-2}{n^2}$.

   (a) How many terms of the sequence are needed before it gets within $.01$ of the limit? Within $.001$ of the limit?

   (b) Using the mathematical definition of limit, prove that the sequence converges.

$a_n = 1 - \frac{2}{n^2}$, so $\lim a_n = 1$. (Or the leading terms rule.) $|a_n - 1| = \frac{2}{n^2}$.

For $\frac{2}{n^2} < .01 = \frac{1}{100}$, need $n^2 > 200$. First happens for $n = 15$. (Good enough to say $n > \sqrt{200}$.)

For $\frac{2}{n^2} < .001 = \frac{1}{1000}$, need $n^2 > 2000$. First happens for $n > \sqrt{2000}$.

For $\frac{2}{n^2} < \varepsilon$, need $n^2 > \frac{2}{\varepsilon}$. Happens when $n > \sqrt{\frac{2}{\varepsilon}}$. So let $T = \sqrt{\frac{2}{\varepsilon}}$ (then for $n > T$ we have $|a_n - 1| < \varepsilon$). Can find this T for any $\varepsilon > 0$. This proves that the sequence converges to 1.

3. Consider the sequence $a_1 = 10$ and $a_{n+1} = \frac{1}{4}a_n + 3$. Here are three problems to do, but you should choose what order to do them in.

   - Prove using induction that the sequence is bounded below.
   - What else should you prove to deduce that the sequence is convergent? You don't need to prove it.
   - What is the limit?

Let L be a potential limit. Then $L = \frac{1}{4}L + 3$ so $L = 4$. $a_1 = 10 > 4$ so we expect decreasing, bounded below by 4.

Suppose $a_n \geq 4$. Then $a_{n+1} = \frac{1}{4}a_n + 3 \geq \frac{1}{4}4 + 3 = 4$. Since $a_1 > 4$, induction says $a_k \geq 4$ for all k.

If the sequence is also decreasing, then it converges by the monotone convergence theorem (which says that decreasing and bounded below implies convergent).

(Extra: If you wanted to show decreasing: We have $a_{n+1} - a_n = \frac{1}{4}a_n + 3 - a_n = \frac{-3}{4}a_n + 3$. Since $a_n \geq 4$ then $a_{n+1} - a_n \leq \frac{-3}{4}4 + 3 = -3 + 3 = 0$. Since $a_{n+1} - a_n \leq 0$, the sequence is decreasing.)

4. Does the series converge or diverge? WHY?

   (a) $\sum_{n=1}^{\infty} \frac{n10^n}{13^n}$
   Converges by ratio test. $\frac{a_{n+1}}{a_n} = (\frac{n+1}{n}) \cdot \frac{10}{13}$*, so* $\lim \frac{a_{n+1}}{a_n} = \frac{10}{13}$*. Since* $\frac{10}{13} < 1$*, this converges.*

   (b) (Hint: what is the derivative of $\ln(\ln(n))$?)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

   By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \geq \int_2^{\infty} \frac{1}{t \ln t} dt = \ln(\ln(t)) \mid_2^{\infty} = \infty$*. Since the integral diverges, so does the series.*

   Said another way: The antiderivative of $\frac{1}{n \ln n}$ *is* $\ln(\ln t)$*. This goes to infinity, so* $\int_N^{\infty} \frac{1}{t \ln t} dt$ *diverges for any* N*. Integral test says our series diverges.*

   (c) $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ *Converges by ratio test.* $\frac{a_{n+1}}{a_n} = 5 \cdot \frac{1}{n+1}$*, so* $\lim \frac{a_{n+1}}{a_n} = 0$*. Since* $0 < 1$*, this converges.*

   (d) $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \dots$
   Diverges by the divergence test, since $a_n = \frac{n+2}{n+6}$ *limits to* 1*, and* $1 \neq 0$*.*

   (e) $\sum_{n=3}^{\infty} \frac{1}{n^{1.05}}$
   Converges by p*-test, with* $p = 1.05 > 1$*.*

   (f) $\sum_{n=1}^{\infty} (\frac{1}{n^{1.05}} + n^{1.05})$
   This diverges by the 2/3 rule (or algebraic limit theorem for sequences). Since $\sum \frac{1}{n^{1.05}}$ *converges by the* p*-test (*$p = 1.05$*), and* $\sum n^{1.05}$ *diverges by the* p*-test (*$p = -1.05$*) (or divergence test), the total sum can not diverge without violating the 2/3 rule.*

5. What is the sum of the following series?

$$\sum_{n=6}^{\infty} (\frac{6}{7})^n$$

Geom series. First term is $(\frac{6}{7})^6$*. Ratio* $r = \frac{6}{7}$*. Limit is*

$$\frac{(\frac{6}{7})^6}{1 - \frac{6}{7}}.$$

6. How many terms of the series $\sum_{n=1}^{\infty} \frac{4}{n^5}$ must you take to approximate the sum of the series to within 10^{-8}?

Bound error with integral test.

$$\int_N^{\infty} \frac{4}{t^5} dt = \frac{-1}{t^4}\Big|_N^{\infty} = \frac{1}{N^4}.$$

So

$$\sum_{n=N+1}^{\infty} \frac{4}{n^5} < \frac{1}{N^4}.$$

If we want $\frac{1}{N^4} \leq 10^{-8}$ *then we need* $N \geq 10^2 = 100$*. 100 terms will work.*