1. Consider the series
$$\frac{1}{4} - \frac{1}{8} + \frac{1}{12} - \frac{1}{16} + \dots$$

   (a) Is the series convergent? WHY?
   Alternating, decreasing with limit zero. AST (alternating series test) says it converges.

   (b) How many terms of the sum must one take in order to be within .1 of the limit?
   Error after N terms is bounded by $|a_{N+1}|$. Need $|a_{N+1}| < \frac{1}{10}$. But $\frac{1}{12} < \frac{1}{10}$ so $N = 2$ works.

   (c) Is the series absolutely convergent? WHY?
   First observe our series is
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n}.$$
   $\sum \frac{1}{4n}$ *diverges by p-test, $p = 1$. No, not absolutely convergent.*

2. What does the **ratio test** say about the following series?

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$
   $|\frac{a_{n+1}}{a_n}| = \frac{2}{n+1}$ *and the limit as $n \to \infty$ is 0. The series absolutely converges.*

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{n^3+2}$
   $|\frac{a_{n+1}}{a_n}| = \frac{((n+1)^2+3)(n^3+2)}{((n+1)^3+2)(n^2+3)} = \frac{n^5+\dots}{n^5+\dots}$ *and the limit as $n \to \infty$ is 1. The ratio test is inconclusive. (Aside: this is the classic example where you are wasting your time with the ratio test.)*

3. Does the series converge or diverge? WHY?

   (a)
$$\sum_{n=1}^{\infty} \frac{n7^n}{3^{2n+5}}$$

   Note that $3^{2n+5} = 3^5 9^n$. *The ratio test looks at*
$$|\frac{a_{n+1}}{a_n}| = \frac{n+1}{n}\frac{7}{9}$$
   which limits to $\frac{7}{9} < 1$. *Thus the series converges.*

   (b)
$$\sum_{n=0}^{\infty} \frac{n^2+2}{n^3+3}$$

   (We expect this to diverge because as n *grows this behaves like* $\sum \frac{1}{n}$*.) In fact,* $\frac{n^2+2}{n^3+3} > \frac{1}{2n}$ *for all* $n > 0$*, since (multiplying both sides by the denominator) this is equivalent to*
$$(2n)(n^2+2) > (n^3+3) \iff 2n^3+4n > n^3+3 \iff n^3+4n > 3.$$
   Since $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ *diverges by* p*-test with* $p = 1$*, and this series is bigger, it also diverges by comparison test.*

   (c)
$$-1 - \frac{1}{4} + \frac{1}{9} + \frac{1}{16} - \frac{1}{25} - \frac{1}{36} + \frac{1}{49} + \ldots$$

   This is $\sum \pm\frac{1}{n^2}$*. Its absolute value sequence* $\sum \frac{1}{n^2}$ *converges by* p*-test,* $p = 2$*. Absolute convergence implies convergence.*

   (d)
$$\sum_{n=6}^{\infty} (-1)^n \frac{2n^2+2}{n^3+3}$$

   This is an alternating series, and $\lim \frac{2n^3+2}{n^3+3} = 0$ *by examining the leading terms. Thus it converges by the Alternating Series Test (AST).*

(e)

$$\sum_{n=2}^{\infty}(-1)^n\frac{n^5-3}{2n^5+3n^3+n-1}$$

This series alternates signs, but $\lim \frac{n^5-3}{2n^5+3n^3+n-1} = \frac{1}{2} \neq 0$*, so it diverges by the Divergence Test.*

4. Find the interval of convergence of the following power series.

   (a) $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n3^n}$

   Ratio test: $|\dfrac{a_{n+1}}{a_n}| = \dfrac{|x-2|}{3}\dfrac{n}{n+1}$ *and the limit is* $\frac{|x-2|}{3}$*. This converges when* $|x-2| < 3$*, so the radius is* 3*, centered at* 2*.*
   Checking the boundary: When $x = 5$ *we have* $\sum \frac{1}{n}$ *which diverges by* p*-test,* $p = 1$*. When* $x = -1$ *we have* $\sum \frac{(-1)^n}{n}$ *which converges by AST. So the interval of convergence is* $[-1, 5)$*.*

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{4^n(x+9)^n}{n^3+1}$

   Ratio test: $|\dfrac{a_{n+1}}{a_n}| = 4|x+9|\dfrac{(n+1)^3+1}{n^3+1}$ *and the limit is* $4|x+9|$*. This converges when* $|x+9| < \frac{1}{4}$*, so the radius is* $\frac{1}{4}$*, centered at* -9*.*
   Checking the boundary: When $x = -8.75$ *we have* $\sum \frac{1}{n^3+1}$ *which converges by comparison test to* p*-test,* $p = 3$*. When* $x = -9.25$ *we have* $\sum \frac{(-1)^n}{n^3+1}$ *which converges by AST (or because it absolutely converges). So the interval of convergence is* $[-9.25, 8.75]$*.*

5. Compute $\int_0^{1/10} \arctan(t^2)dt$ to within 10^{-9}.

   We know $\arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots$ *so* $\arctan(t^2) = t^2 - \frac{t^6}{3} + \frac{t^{10}}{5} - \ldots$*. Integrating from* $\int_0^x$*, we get*

$$\frac{x^3}{3} - \frac{x^7}{3\cdot 7} + \frac{x^{11}}{5\cdot 11} - \ldots$$

   Plugging in $x = \frac{1}{10}$ *we get an alternating series, so we are interested in finding the first term with absolute value less than* 10^{-9}*. Clearly the third term (with* x^{11}*) is small enough, so our approximation is the first two terms,* $\frac{1}{10^3\cdot 3} - \frac{1}{10^7\cdot 3\cdot 7}$*.*

6. Find a power series centered at zero for the following functions. (Note: I could also ask for the radius of convergence.)

   (a) $\dfrac{1}{4-3x}$

   $\dfrac{1}{4-3x} = \dfrac{1}{4}\dfrac{1}{(1-\frac{3}{4}x)} = \dfrac{1}{4}\sum_{n=0}^{\infty}(\dfrac{3}{4})^n x^n$. *The radius of convergence is* $\frac{4}{3}$ *(easy ratio test, or because it is a geometric series).*

   (b) $\displaystyle\int_0^x \frac{1}{1+t^6}dt$

   $\dfrac{1}{1+t^6} = \sum_{n=0}^{\infty}(-1)^n t^{6n}$. *So the integral is* $C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{6n+1}$, *and* $C = 0$ *since the integral begins at* 0. *The radius of convergence is* 1 *(easy ratio test). (Remember: radius of convergence doesn't change when you integrate or derive! However, interval of convergence may change - stuff can happen at the boundary!)*

   (c) The derivative of $\displaystyle\sum_{n=0}^{\infty}\frac{2^n(n!)x^n}{(2n)!}$.

   (This looks harder than it is - the constants don't affect the derivative at all.)

$$\sum_{n=0}^{\infty}\frac{2^n(n!)nx^{n-1}}{(2n)!}.$$

   (If you really wanted to reindex this series, unnecessary for this problem, you'd get

$$\sum_{n=0}^{\infty}\frac{2^{n+1}(n+1)!x^n}{(2(n+1))!}.$$

   If you need the radius of convergence, it's a tricky ratio test. The ratio of successive terms is

$$\frac{a_{n+1}}{a_n} = 2x\frac{n+1}{n}\frac{(n+1)!}{n!}\frac{(2n)!}{(2n+2)!} = 2x\frac{(n+1)(n+1)}{n(2n+1)(2n+2)}.$$

   This is a degree 2 polynomial over a degree 3 polynomial, so the limit is zero, no matter what x *is. Thus the radius of convergence is* ∞.

7. Find a power series centered at zero for the following functions. Write out the first three nonzero terms explicitly. (Note: I could also ask for the radius of convergence.)

   (a) e^{x^3}

   Just plug in x^3 to the formula for e^x.

   $$1 + x^3 + \frac{1}{2}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}.$$

   The radius of convergence is ∞. After all, the interval of convergence of e^x is $(-\infty, \infty)$, so e^{x^3} converges when x^3 is in $(-\infty, \infty)$, which is always. Or you could do a ratio test.

   (b) $\dfrac{1}{(1+2x)^2}$

   The power series for $\frac{1}{(1-x)^2}$ is $\sum_{n=0}^{\infty}(n+1)x^n$, since this is the derivative of $\frac{1}{1-x}$. Plugging in $-2x$ we get

   $$\sum_{n=0}^{\infty}(n+1)(-2)^n x^n$$

   The radius of convergence is $\frac{1}{2}$. After all, the radius of convergence of $\frac{1}{1-x}$ is 1, as for its derivative. Plugging in $2x$ cuts the radius in half. In other words, this series converges if $2x$ is in $(-1,1)$ which means x is in $(-1/2, +1/2)$ (I'm ignoring the boundary, since I'm just computing the radius). Or you could do a ratio test.

   (c) $\ln(1 - x^3)$

   $$\sum_{n=1}^{\infty}(-1)^{n-1}(-1)^n \frac{x^{3n}}{n} = -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \ldots$$

   The radius of convergence is 1. After all, the radius of convergence of $\ln(1+x)$ is 1, and plugging in x^3 takes the cube root of that. Or you could do a ratio test.

8. Find $\cos(.5)$ to within $\frac{1}{500}$.

   Two reasonable solutions, which are about the same. Both use the usual Taylor series for $\cos(x)$ centered at 0. Which is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

   Solution 1: Plugging in $.5$ for x, we get an alternating series. Since $\frac{1}{2^6 \cdot 6!} < \frac{1}{500}$, our estimate is just the first 3 terms, namely $1 - \frac{1}{2^2 \cdot 2!} + \frac{1}{2^4 \cdot 4!}$.

   Solution 2: We use the Taylor Remainder theorem. For $f(x) = \cos(x)$ Note that $|f^{(k)}(x)|$ is bounded above by $M = 1$ for any k. Thus

   $$|R_k(.5)| \leq \frac{(.5)^{k+1}}{(k+1)!} = \frac{1}{2^{k+1}(k+1)!}$$

   When $k = 4$, we have $|R_k(.5)| \leq \frac{1}{500}$. So our estimate is $T_4(.5)$, which is $1 - \frac{(.5)^2}{2!} + \frac{(.5)^4}{4!}$.

9. Using any method, find the first few terms of the Taylor series, up to the cubic term (i.e. the x^3 term).

   (a) $e^x \cos x$ centered at 0.
   Method 1 (easier):
   $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \ldots$ *so when we multiply we get* $1 + x + x^2(\frac{1}{2} - \frac{1}{2}) + x^3(\frac{1}{6} - \frac{1}{2}) + \ldots = 1 + x - \frac{1}{3}x^3 + \ldots$
   Method 2 (harder):
   $f(x) = e^x \cos x$ so $f(0) = 1$. $f'(x) = e^x(\cos x - \sin x)$ so $f'(0) = 1$. $f''(x) = -2e^x \sin x$ so $f''(0) = 0$. $f'''(x) = -2e^x(\cos x + \sin x)$ so $f'''(0) = -2$. Thus $f(x) = 1 + 1x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \ldots = 1 + x - \frac{1}{3}x^3 \ldots$

   (b) $\ln(x)$ centered at 2.
   (Aside: there is a tricky way to deduce this from the power series for $\ln(1+x)$*, but let's not do that.)*
   Let $g(x) = \ln(x)$*. Then* $g'(x) = \frac{1}{x}$*, and* $g''(x) = \frac{-1}{x^2}$*, and* $g'''(x) = \frac{2}{x^3}$*.*
   So $g(2) = \ln(2)$*,* $g'(2) = \frac{1}{2}$*,* $g''(2) = \frac{-1}{4}$*, and* $g'''(2) = \frac{1}{4}$*.*
   Thus the Taylor series is

$$\ln(2) + \frac{1}{2}(x-2) + \frac{-1}{4 \cdot 2!}(x-2)^2 + \frac{1}{4 \cdot 3!}(x-2)^3 + \ldots$$

   or in other words

$$\ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3.$$

   (c) e^{3x} centered at -5.
   $f^{(k)}(x) = 3^k e^{3x}$*. So we have*
   $f(x) = e^{-15} + 3e^{-15}(x+5) + \frac{3^2 e^{-15}}{2}(x+5)^2 + \frac{3^3 e^{-15}}{3!}(x+5)^3 + \ldots$

10. Find the degree three Taylor polynomial $T_3(x)$ for $\frac{1}{1-x}$ centered at 5. Bound the error on the interval $[4, 6]$.

For $f(x) = \frac{1}{1-x}$ *one has* $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$*. (Or you can just compute the first 3 derivatives.) So* $f(5) = \frac{1}{-4}$*,* $f'(5) = \frac{1}{(-4)^2}$*,* $f''(5) = \frac{2!}{(-4)^3}$ *and* $f'''(5) = \frac{3!}{(-4)^4}$*. Thus*

$$T_3(x) = \frac{1}{-4} + \frac{1}{(-4)^2}(x-5) + \frac{1}{(-4)^3}(x-5)^2 + \frac{1}{(-4)^4}(x-5)^3.$$

The fourth derivative is $f''''(x) = \frac{4!}{(1-x)^5}$ *whose absolute value is a decreasing function, so the maximum is obtained at* $x = 4$*, and this maximum is* $M = \frac{4!}{3^5}$*. The radius for the interval is* $d = 1$*. Hence* $|R_3(x)| \leq \frac{4!}{4! \cdot 3^5} = \frac{1}{3^5}$*.*

11. Find the degree 5 Taylor polynomial $T_5(x)$ for $3\sin x$ centered at 0. What error bound does the Taylor Remainder Theorem give on the interval $[-.2, .2]$?

Using the standard Taylor series at 0, we have $T_5(x) = 3x - 3\frac{x^3}{3!} + 3\frac{x^5}{5!}$.

Every derivative of $3\sin x$ *is bounded in absolute value by* $M = 3$*. So on a radius of interval* d *by Taylor Remainder Theorem we have* $|R_5(x)| \leq \frac{3d^6}{6!}$*. When* $d = .2$ *we get* $\frac{3(.2)^6}{6!}$.

12. Let $f(x) = e^x \sin x$. For your convenience, we have calculated the first several derivatives of f.

$$f'(x) = e^x(\sin x + \cos x)$$
$$f''(x) = 2e^x \cos x$$
$$f'''(x) = 2e^x(\cos x - \sin x)$$
$$f''''(x) = -4e^x \sin x$$

Compute the degree 3 Taylor polynomial $T_3(x)$ for $f(x)$ centered at 0, and bound the error on the interval $[-2, 2]$.

From these formulas we compute $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, *and* $f'''(0) = 2$*. Thus*

$$T_3(x) = 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 = x + x^2 + \frac{1}{3}x^3.$$

Since $|sin(x)| \leq 1$ *for all* x*, we have* $|f''''(x)| \leq 4e^x$ *for all* x*. On the interval* $[-2, 2]$ *we get* $|f''''(x)| \leq 4e^2 = M$*. So by the Taylor Remainder theorem we get that* $|R_3(x)| \leq \frac{4e^2 \cdot 2^4}{4!}$.

13. Is this series convergent or divergent? If it is convergent, what is the sum?

   (a) $\frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} + \ldots$
   This is $\ln(1 + x)$ *with* $x = 1/2$*. The radius of convergence is* $1 > \frac{1}{2}$*, so it converges to* $\ln(3/2)$.

   (b) $3 - \frac{3^3}{3} + \frac{3^5}{5} - \frac{3^7}{7} + \ldots$
   This is $\arctan(x)$ *with* $x = 3$*. But the radius of convergence is* $1 < 3$*, so this diverges (an easy divergence test).*

   (c) $3 - \frac{3^3}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} + \ldots$
   This is $\sin(x)$ *at* $x = 3$*. The radius of convergence is* ∞*, so this converges to* $\sin(3)$.

14. Find the first few terms of a power series centered at 0 for the following function, up to the x^3 term.

$$(x^2 - 5)(\sum_{n=0}^{\infty}(n+1)x^n).$$

We have $\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \ldots$*. Multiplying by* -5*, we get*

$$-5 - 10x - 15x^2 - 20x^3 + \ldots$$

and multiplying by x^2 *we get*

$$0 + 0x + x^2 + 2x^3 + \ldots$$

so adding these we get

$$-5 - 10x - 14x^2 - 18x^3 + \ldots$$

which is the final answer.