First, some reminders from class.

• For the function \( A \cos(\omega t - \varphi) \), \( A \) is the amplitude, \( \frac{2\pi}{\omega} \) is the quasi-period, \( \frac{\omega}{2\pi} \) is the quasi-frequency, and \( \varphi \) is the phase lag.

• The sinusoidal response formula: let \( p(x) \) be a polynomial, and suppose that \( a \pm bi \) is not a root. Then the differential equation

\[
p(D)[y] = Ae^{at} \cos(bt - \varphi)
\]

has a particular solution given by

\[
y_p = \frac{A}{C} e^{at} \cos(bt - \varphi - \theta).
\]

To compute \( C \) and \( \theta \), use that

\[
p(a + bi) = Ce^{i\theta}.
\]

The real number \( \frac{1}{p} \) is called the gain, the angle \( \theta \) is called the (additional) phase lag, and the complex number \( \frac{1}{p(a+bi)} \) is called the complex gain. See the online link.

• Consider a differential equation of the form

\[
ay'' + by' + cy = g(t)
\]

where \( a, c > 0 \) and \( b \geq 0 \). When \( g(t) = 0 \) it is called unforced, otherwise it is forced. This is the (damped) spring equation. When \( b = 0 \) this ODE is called undamped, otherwise it is damped. When it is damped, the general homogeneous solution is called the transient solution, and a particular inhomogeneous solution is called a steady state solution. Consider the two roots of the polynomial \( ax^2 + bx + c \). If they are complex, the ODE is called underdamped. If it is a double root, it is called critically damped. If there are two distinct (negative) roots, it is called overdamped. See the book, sections 3.7 and 3.8.
1. Consider the differential equation

\[ y''' - 2y'' + 3y' - 4y = 3e^{2t} \cos(3t) - 4e^{2t} \sin(3t). \]

(a) Rewrite the forcing function in the form \( Ae^{at} \cos(bt - \varphi) \).

(b) Find a particular solution to the inhomogeneous equation. What is the amplitude? What is the gain? What is the additional phase lag?

2. Consider the differential equation \( y'' + 2y' + 3y = 10 \cos(wt) \) for some positive numbers \( A \) and \( w \).

(a) Is this overdamped, underdamped, or critically damped? Is it forced or unforced?

(b) Find the general solution.

(c) Find a formula for the amplitude of the steady state solution.

(d) Find \( w > 0 \) which maximizes the amplitude of the steady state solution.

3. (a) Find a particular solution to \( y'' + 3y' + 2y = 5e^{-t} \cos(t) \).

(b) Find a particular solution to \( y'' + 2y' + 2y = 5e^{-t} \cos(t) \).

4. Consider the differential equation \( y'' + by' + 4y = 0 \).

(a) For which values of \( b \geq 0 \) is it overdamped? Underdamped? Critically damped? Undamped?

(b) Suppose that it is underdamped. Write down a formula for the quasi-period of a solution, in terms of \( b \). As \( b \) gets smaller, what happens to the quasi-period?

(c) Suppose that it is critically damped. Find the general solution. How many times will a solution satisfy \( y(t) = 0 \)?

(d) Continue to assume that it is critically damped. Suppose that \( y(0) = 5 \) and \( y'(0) = 3 \). At what time(s) will the solution satisfy \( y(t) = 0 \)?

(e) Suppose that \( b = 5 \). Suppose that \( y(0) = 5 \). For while values of \( y'(0) \) will the solution NEVER satisfy \( y(t) = 0 \)?