1. Consider the differential equation
\[ y''' - 2y'' + 3y' - 4y = 3e^{2t} \cos(3t) - 4e^{2t} \sin(3t). \]

(a) Rewrite the forcing function in the form \( Ae^{at} \cos(bt - \varphi). \)

The complex number \( 3 - 4i \) has polar coordinates \( 5e^{i\varphi} \) where \( \varphi = \arctan\left(\frac{-4}{3}\right) \). So the forcing function equals \( 5e^{2t} \cos(3t - \varphi). \)

(b) Find a particular solution to the inhomogeneous equation. What is the amplitude? What is the gain? What is the additional phase lag?

Letting \( p(x) = x^3 - 2x^2 + 3x - 4 \), and \( r = 2 + 3i \), we have \( p(r) = 34 - 6i \) (I did this on wolfram alpha). Since \( p(r) \neq 0 \), the forcing function is not a homogeneous solution, and we can use the sinusoidal response formula. Thus our particular solution is
\[ \frac{5}{\sqrt{34^2 + 6^2}} e^{2t} \cos(3t - \varphi - \arctan\left(\frac{6}{34}\right)). \]

The amplitude is \( \frac{5}{\sqrt{34^2 + 6^2}} \), the gain is \( \frac{1}{\sqrt{34^2 + 6^2}} \), and the additional phase lag is \( \arctan\left(\frac{6}{34}\right) \).

2. Consider the differential equation \( y'' + 2y' + 3y = 10 \cos(wt) \) for some positive numbers \( A \) and \( w \).

(a) Is this overdamped, underdamped, or critically damped? Is it forced or unforced?

By the quadratic formula, the roots are \( -2 \pm \sqrt{4 - 12} = -1 \pm \sqrt{2i} \). Since these are complex with negative real part, it is underdamped. It is forced, since it is not homogeneous.

(b) Find the general solution.

Since \( \cos(wt) \) is never a homogeneous solution, the sinusoidal response formula gives a particular solution. \( p(wi) = 3 - w^2 + 2wi \), so the SRF yields
\[ \frac{10}{\sqrt{(3 - w^2)^2 + (2w)^2}} \cos(wt - \varphi) \]
where $\varphi$ is the argument of $p(wi)$. Meanwhile the general homogeneous solution is $c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t)$. Thus the general solution is

$$
\frac{10}{\sqrt{(3-w^2)^2 + (2w)^2}} \cos(wt - \varphi) + c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t).
$$

(c) Find a formula for the amplitude of the steady state solution.

The steady state solution is the (periodic) particular solution, which is given by the SRF. It has amplitude $10 \sqrt{(3-w^2)^2 + (2w)^2}$.

(d) Find $w > 0$ which maximizes the amplitude of the steady state solution.

To maximize the amplitude we could take the derivative with respect to $w$ of the above formula, and find the zeroes. However, it is easier to just minimize the part under the square root, which is $w^4 - 2w^2 + 9$. Taking the derivative of that we get $4w^3 - 4w$, which has a zero at $w = -1, 0, 1$. Taking the double derivative, it is a minimum only at $w = -1, +1$. Only $w > 0$ makes sense for frequencies. So the answer is $w = 1$.

3. (a) Find a particular solution to $y'' + 3y' + 2y = 5e^{-t} \cos(t)$.

$p(x) = x^2 + 3x + 2$ and $p(-1+i) = -1 + i$. This is nonzero, so we can use the SRF. The answer is

$$\frac{5}{\sqrt{2}} e^{-t} \cos(t - \frac{3\pi}{4}).$$

(b) Find a particular solution to $y'' + 2y' + 2y = 5e^{-t} \cos(t)$.

$p(x) = x^2 + 2x + 2$ and $p(-1 + i) = 0$, so we can not use the SRF. There is a fancier version of the SRF that would work, but we’ll just do it by hand. We guess something of the form

$$y = A e^{-t} \cos(t) + B e^{-t} \sin(t).$$

Computing derivatives, the left hand side is $2Be^{-t} \cos(t) - 2Ae^{-t} \sin(t)$. For this to equal $5e^{-t} \cos(t)$ we set $A = 0$ and $B = 2.5$.

4. Consider the differential equation $y'' + by' + 4y = 0$.

(a) For which values of $b \geq 0$ is it overdamped? Underdamped? Critically damped? Undamped?

The question is whether $b^2 - 16$ is positive, zero, or negative. For $b > 4$ it is overdamped. For $b = 4$ it is critically damped. For $0 < b < 4$ it is underdamped. For $b = 0$ it is undamped.

(b) Suppose that it is underdamped. Write down a formula for the quasi-period of a solution, in terms of $b$. As $b$ gets smaller, what happens to the quasi-period?
When $0 < b < 4$ the roots are $\frac{-b \pm \sqrt{b^2 - 16}}{2} = \frac{-b \pm \sqrt{b^2 - 16}}{2}$, and a homogeneous solution has the form $Ce^{-\frac{b}{2}t} \cos\left(\frac{\sqrt{b^2 - 16}}{2} t - \varphi\right)$. Thus the quasi-period is $\frac{4\pi}{\sqrt{b^2 - 16}}$. As $b$ approaches zero, the denominator gets larger until it approaches 4, so the quasi-period gets smaller until it approaches $\pi$.

(c) Suppose that it is critically damped. Find the general solution. How many times will a solution satisfy $y(t) = 0$?

So $b = 4$, and the repeated root is $-2$. The general solution is $(c_1 + c_2 t)e^{-2t}$. This satisfies $y(t) = 0$ exactly once at time $t_0$, when $c_1 + t_0 c_2 = 0$.

(d) Continue to assume that it is critically damped. Suppose that $y(0) = 5$ and $y'(0) = 3$. At what time(s) will the solution satisfy $y(t) = 0$?

$y(0) = c_1 = 5$ and $y'(0) = -2c_1 + c_2 = 3$ so $c_2 = 13$. Therefore, $t_0 = -\frac{c_1}{c_2} = -\frac{5}{13}$.

(e) Suppose that $b = 5$. Suppose that $y(0) = 5$. For while values of $y'(0)$ will the solution NEVER satisfy $y(t) = 0$?

The roots are $-1$ and $-4$, so the solution is $c_1 e^{-t} + c_2 e^{-4t}$. If the solution satisfies $y(t_0) = 0$ for a particular time $t_0$ then $c_1 e^{-t_0} = -c_2 e^{-4t_0}$, meaning that $\frac{c_1}{c_2} = -e^{-3t_0}$ is a negative number. Any negative number can be obtained this way for some $t_0$, so the solution will never satisfy $y(t) = 0$ if and only if $\frac{c_1}{c_2}$ is positive. Now $y'(0) = -c_1 - 4c_2$ and $y(0) = c_1 + c_2 = 5$. Thus $-3c_2 = 5 + y'(0)$ and $3c_1 = 20 + y'(0)$. In particular, $c_1$ and $c_2$ have the same sign when $20 + y'(0)$ and $5 + y'(0)$ have opposite signs. This will happen precisely for $-20 < y'(0) < -5$. 