Math 316 (Fund. of analysis), Winter 2018
Midterm 1
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Date: 2/12/2018

Name:

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General notes:

1. Terminology:
   - Supremum = least upper bound.
   - Injective = 1-to-1 = into.
   - Surjective = onto.

2. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (one or two sentences).
   - To justify that “Every closed interval contains 0” is false, you should provide a counterexample, such as [1, 2].
   - To justify that “Some closed interval contains 0” is true, you should provide an example, such as [−1, 1].
   - To justify that “Every closed interval contains some nonzero number” is true, you should explain. “For any interval [a, b], either a ≠ 0 or b ≠ 0.”

3. If I want a complete proof, I will say “Prove that ...”

4. If I don’t need any justification, I will say “no justification necessary.” No justification is needed for a definition.

5. You can use any theorem from the book, from class, or from homework, unless specifically stated otherwise.
1. **(10 pts, 5 each)** Give formal definitions for the following concepts.
   
   (a) What it means for the real numbers to be *complete*.
   
   (b) What it means for a sequence \((x_n)\) of real numbers to *diverge*.

2. **(10 pts, 5 each)**

   (a) Let \(C_1\) and \(C_2\) be nonempty subsets of \(\mathbb{R}\) which are each bounded above. Does \(\sup(C_1 \cup C_2)\) exist? If so, find a formula for it (no justification required). If not, find a counterexample.

   (b) Now let \(C_1, C_2, C_3, \ldots\) be nonempty subsets of \(\mathbb{R}\) which are each bounded above. Does \(\sup(\bigcup_{n \in \mathbb{N}} C_n)\) exist? If so, find a formula for it (no justification required). If not, find a counterexample.
3. (12 pts, 3 each) Suppose that \((x_n)\) converges to \(-12\). For each of the statements below, is it true or false? (No justification needed. No partial credit.)

(a) Only finitely many of the terms \(x_n\) can be positive.
(b) At least one term \(x_n\) is less than \(-11.5\).
(c) If \(x_N = -12\) for some \(N\), then \(x_n = -12\) for all \(n \geq N\).
(d) None of the terms \(x_n\) can be less than \(-1000\).

4. (10 pts) Let \(X\) be a countably infinite subset of \(\mathbb{R}\). Fix an element \(y \in \mathbb{R}\) with \(y \notin X\). Prove that \(X \cup \{y\}\) is countably infinite, using only the definition of countability (e.g. do not use the fact that a union of countable sets is countable).
5. **(36 pts, 6 each)** For each of the following statements, is it true or false? Justification is required.

(a) Any open interval in $\mathbb{R}$ contains a rational number of the form $\frac{p}{q}$ for $q \leq 9999$.
(b) (Let $I = \mathbb{R} \setminus \mathbb{Q}$ denote the irrational numbers.) For any infinite subset $X \subset I$, $X$ is uncountable.
(c) If $\lim x_n = x$ and $\lim y_n = y \neq 0$, then $\lim \left( \frac{x_n + 3}{y_n} \right) = \frac{x + 3}{y^2}$.
(d) Every increasing sequence of negative numbers has a limit.
(e) If a sequence does not converge to 5, then infinitely many terms of the sequence must be outside the interval $(4, 6)$.
(f) Let $y_n = |x_n - x_{n+1}|$. If $y_n$ converges to zero, then $x_n$ converges.
6. **(10 pts)** Prove that

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\lim_{n \to \infty} \frac{7n + 2}{2n - 5} = \frac{7}{2}.
\]

7. **(12 pts)** If \((x_n)\) converges to 101, and \(x_n \neq 0\) for all \(n \in \mathbb{N}\), then \(\frac{1}{x_n}\) converges to \(\frac{1}{101}\). Prove this directly from the definition of convergence (that is, without using the algebraic limit theorem).