In class, we reduced the power series formula $E(x)E(x) = E(2x)$ to a formula involving factorials or binomial numbers, namely

$$\sum_{m=0}^{k} \binom{k}{m} = 2^k. \tag{0.1}$$

One of the useful tricks when doing this was to multiply and divide by $k!$ (Hint hint). Then (0.1) was proven using the binomial formula. This first problem below asks you to reduce a power series formula to a formula involving factorials or binomial numbers, but not to prove that this formula is correct.

1. Recall that

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Take the formula $S(x)^2 + C(x)^2 = 1$, and reduce it to a formula involving factorials and binomial numbers.

2. The second problem explores the power series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

in more detail. Don't forget the tools you have at your disposal: IVT, MVT, etcetera.

(a) Show that $E(x)$ is the unique power series for which $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$. (Actually, you already did this in a previous homework, 6.5.8b, so you don't need to do it again! I'm just writing this down for the flow.)

(b) Prove that R, the radius of convergence of $E(x)$, satisfies $R = \infty$. (Usually exercise 6.5.7 is the easiest way to do this.)

(c) Prove that $E(x) > x$ for all $x \geq 0$, and hence $E(x)$ is unbounded.

(d) Prove that $E(x)E(-x) = 1$. First reduce this to a formula involving factorials or binomial numbers, and then use the binomial formula.

(e) Prove that $1 \leq E(x)$ for all $x \geq 0$, and $0 \leq E(x) \leq 1$ for all $x \leq 0$. (Proving this directly from the power series can be difficult.)

(f) Prove that $E(x)$ is a bijective map from $\mathbb{R}$ to $(0, \infty)$.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Let $s_n(x)$ and $t_n(x)$ denote their respective partial sums. In class, we talked about how the sequence $(s_n(x)t_n(x))$ limits to the product $f(x)g(x)$, but is not the sequence of partial sums of a power series. Instead, there is a unique power series

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m} \right) x^n$$

which describes the product $f(x)g(x)$ within the radius of convergence. One finds this power series by multiplying $s_i(x)$ and $t_j(x)$ and noticing that the coefficient of each x^n will "stabilize" (i.e. stop changing as i and j vary) once $i, j \geq n$. The final problem below asks the same question about the composition $f \circ g$, and then explores the inverse of $E(x)$.

3. Let f, g, s_n, and t_n be as above. Let $h_{ij}(x) = s_i(t_j(x))$, which is just a polynomial.

(a) What is the coefficient of x^0 in $h_{ij}(x)$? As i and j get large, will the coefficient of each x^n eventually stabilize?

(b) Suppose that $b_0 = 0$. Argue that the coefficient of x^n stabilizes once $i, j \geq n$. Let $h(x)$ be the power series obtained by taking the stabilized coefficients.

(c) Compute the x^0, x^1, x^2, and x^3 coefficients of $h(x)$.

(d) We know that

$$L(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

is the power series for $\log(x+1)$, and that

$$F(x) = x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

is the power series for $e^x - 1$. Thus we expect $L(x)$ and $F(x)$ to be inverse functions. Verify that $L(F(x)) = x$ and $F(L(x)) = x$ at least up to degree 3, i.e. verify that the power series for each begins with $0 + 1x + 0x^2 + 0x^3$.