Math 317 (Fund. of analysis), Spring 2018
Final solutions

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1. **(15 pts)** (a) Define the term *compact*.

   (b) Let $A \subset \mathbb{R}$ be compact. Prove from the definition that A is bounded.

   A subset $A \subset \mathbb{R}$ is compact if every sequence in A has a convergent subsequence, whose limit is also in A.

   Suppose that A is not bounded. Then there exists some $x_1 \in A$ such that $x_1 > 1$, some $x_2 \in A$ such that $x_2 > 2$, etcetera. The sequence $(x_n)_{n \in \mathbb{N}}$ is unbounded, and any subsequence is unbounded, so that no subsequence will be convergent. Therefore, A is not compact. Thus any compact set is bounded.

   Common error: just making the sequence increasing, or unbounded, is not enough. Both increasing and unbounded would be enough. Note: the sequence I construct above need not be increasing.

2. **(15 pts)** Let $f(x) = xe^{|x|}$. Prove that f is differentiable at 0. You may use standard facts about the function e^x.

   The difference function at 0 is $\Delta_{f,0}(x) = \frac{f(x)-f(0)}{x-0} = \frac{xe^{|x|}}{x} = e^{|x|}$. Since $|x|$ and e^x are continuous functions on all of $\mathbb{R}$, so is the composition $e^{|x|}$, meaning that $\lim_{x \to 0} e^{|x|}$ exists (and is equal to $e^{|0|} = 1$). Thus f is differentiable at 0, and $f'(0) = 1$.

   Common error: Thinking that $\lim_{x \to 0} e^{|x|}$ exists because of the algebraic limit theorem, or something like this. This really uses continuity, and it uses the continuity of composed functions, so these facts should be mentioned.

   Common error 2: Thinking you can use the product rule to compute the derivative. The function $e^{|x|}$ is not differentiable, and the product rule does not apply!

   Let's jump ahead to the extra credit: is f twice-differentiable? We have computed $f'(0) = 1$. For $x > 0$, $f(x) = xe^x$, and by the product rule $f'(x) = (x+1)e^x$. For $x < 0$, $f(x) = xe^{-x}$, and $f'(x) = (1-x)e^{-x}$. Thus, combining these computations, we see that $f'(x) = (|x|+1)e^{|x|}$, for all $x \in \mathbb{R}$.

   f is not twice-differentiable. One way is to look at the difference function of f' at 0, and check that the limit as $x \to 0$ for $x > 0$ is one thing, while the limit when $x < 0$

is another thing. A second way is to compute that f is differentiable on $\mathbb{R} \setminus 0$, with $f''(x) = (x+2)e^x$ for $x > 0$, and $(x-2)e^{-x}$ for $x < 0$. There is no value of $f''(0)$ which would allow this function to satisfy Darboux's theorem, because it would have a jump discontinuity at 0.

3. **(20 pts)** Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n \cos(nx)}{n}$.

   (a) Prove that $f(x)$ is continuous on $(-1, 1)$.

   (b) Prove that $f(x)$ is differentiable on $(-1, 1)$.

   For a lot of partial credit, you can do the above on the domain $(-\frac{1}{2}, \frac{1}{2})$ instead of $(-1, 1)$.

   Pick $c \in (0, 1)$. Note that, for any $x \in [-c, c]$ and any $n \geq 1$ we have $\left|\frac{x^n \cos(nx)}{n}\right| \leq c^n$, since $|\cos(nx)| \leq 1$ and $\frac{1}{n} \leq 1$. Note also that $\sum_{n=1}^{\infty} c^n$ is convergent. Therefore, by the Weierstrass M-test, the series converges absolutely on $[-c, c]$. Hence, since each function $\frac{x^n \cos(nx)}{n}$ is continuous, the term-by-term continuity theorem says that f is continuous on $[-c, c]$. For any $x \in (-1, 1)$ there is some c with $|x| < c < 1$, so that $x \in [-c, c]$ and f is continuous at x. Thus f is continuous on $(-1, 1)$.

   Common error: Trying to bound above with $\frac{1}{n}$ instead. Of course it is less than $\frac{1}{n}$, but it is the exponential decay which makes the sum converge, not the harmonic decay! Funny story: I was trying to make this problem *easier* by having the $\frac{1}{n}$ in the denominator, because it makes the derivative that much easier to compute. Instead, I made another hurdle for the unwary.

   To show f is differentiable, we must show that the series of derivatives converges uniformly. By the product rule, the derivative of $\frac{x^n \cos(nx)}{n}$ is $x^{n-1}(n\cos(nx) - x\sin(nx))$. For any $x \in [-c, c]$ and $n \geq 1$, this is bounded in absolute value by $c^{n-1}(n+1)$. Since $\sum_{n=1}^{\infty} c^{n-1}(n+1)$ converges for any $0 < c < 1$ (by the ratio test), the Weierstrass M-test says that the series $\sum_{n=1}^{\infty} x^{n-1}(n\cos(nx) - x\sin(nx))$ converges uniformly. Therefore, the term-by-term differentiability theorem implies that f is differentiable on $[-c, c]$. By the same argument as above, it is differentiable on $(-1, 1)$.

   Common error: not being able to compute derivatives correctly.

4. **(15 pts)** Prove the following result.

   Proposition. Suppose that f and g are differentiable on an interval $[a, b]$, and that $f'(x) = g'(x)$ for all $x \in [a, b]$. Then $f = g + C$ for some constant $C \in \mathbb{R}$.

   (Warning: I wouldn't use the fundamental theorem of calculus - f' need not be integrable! You can do this with just the techniques from the chapter on differentiation.)

   Consider $h = f - g$. Then h is differentiable on $[a, b]$ and $h'(x) = 0$ for all $x \in [a, b]$. We claim that h is a constant function. If not, the mean value theorem would imply the existence of some point $c \in (a, b)$ with $h'(c) \neq 0$, a contradiction. So there is a constant $C \in \mathbb{R}$ such that $h(x) = C$ for all $x \in [a, b]$. Thus, $f - g = C$ or $f = g + C$.

This was the problem that made me sad. Without using the mean value theorem or some variant, I don't think you can make much progress. Without looking at $h = f - g$, it gets much harder. These were really crucial tricks in the class.

5. **(20 pts)** Let $g: \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Assume that, for all $k \geq 1$, the k-th derivative is bounded by k, that is $|g^{(k)}(x)| \leq k$ for all $x \in \mathbb{R}$. Let $T(x) = \sum_{k=0}^{\infty} a_k x^k$ be the Taylor series of g.

   (a) Find a bound on the absolute value of a_k, for each k.

   (b) Justify that $T(x)$ converges on all of $\mathbb{R}$.

   (c) Justify that $T(x) = g(x)$ on all of $\mathbb{R}$.

   Since $a_k = g^{(k)}(0)/k!$, we have $|a_k| \leq k/k! = \frac{1}{(k-1)!}$.

   We can show that $T(x)$ converges absolutely on all of $\mathbb{R}$, using the comparison test, so long as $\sum \frac{x^k}{(k-1)!}$ converges for all $x \geq 0$. This follows from the ratio test, since for any $x \in \mathbb{R}$, $\lim_{k\to\infty} \frac{x}{k} = 0$... it is also just the power series for xe^x.

   By the Taylor inequality estimate, the error $E_k(x)$ between the k-th partial sum of $T(x)$ and $g(x)$ satisfies
$$|E_k(x)| \leq \frac{|x|^{k+1}(k+1)}{(k+1)!} = \frac{|x|^{k+1}}{k!}.$$
   For any x, this goes to 0 as $k \to \infty$, because factorials grow much larger than exponentials (or again, the ratio test).

6. **(15 pts)** Let $T(x)$ be the Taylor series for $f(x) = \sqrt{1+x}$, and $U(x)$ be the Taylor series for $g(x) = \frac{1}{\sqrt{1+x}}$.

   (a) Write down the terms up to degree 2 (i.e. the coefficient of x^2) of $T(x)$. (If you don't remember the formula, you can always compute derivatives.)

   (b) Write down the terms up to degree 2 of $U(x)$.

   (c) One expects that $T(x)U(x) = 1$. Verify that this is true up to degree 2.

   $f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = \frac{-1}{4}$. So $T(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$. More generally, the coefficient in front of x^k is $\binom{\frac{1}{2}}{k}$.

   $g(0) = 1$, $g'(0) = \frac{-1}{2}$, $g''(0) = \frac{3}{4}$. So $U(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \ldots$. More generally, the coefficient in front of x^k is $\binom{\frac{-1}{2}}{k}$.

Thus the first terms of $T(x)U(x)$ are

$$1 \cdot 1 + (1 \cdot \frac{-1}{2} + \frac{1}{2} \cdot 1)x + (1 \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{-1}{2} + \frac{-1}{8} \cdot 1)x^2 + \dots$$

which is equal to $1 + 0x + 0x^2$ as desired.

Skipping straight to the extra credit, one has

$$2T'(x) = \sum_{k=0}^{\infty} 2kx^{k-1}\binom{\frac{1}{2}}{k}.$$

So if this is equal to $U(x)$, then by comparing the coefficient of x^{k-1} we have

$$2k\binom{\frac{1}{2}}{k} = \binom{\frac{-1}{2}}{(k-1)}$$

for all $k \geq 1$.

7. **(20 pts)** (a) Define what it means for f to be integrable on $[a, b]$. (You do not need to tell me what a partition is, or to define the quantities $L(f, P)$ and $U(f, P)$ for a particular partition P.)

   (b) Let $h(x)\colon [0, 3] \to \mathbb{R}$ be the function defined as follows.

$$h(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ 2 & \text{if} \quad 1 < x < 2 \quad \text{or} \quad 2 < x \leq 3, \\ 15 & \text{if} \quad x = 2. \end{cases}$$

   Let P be the partition $\{0, 1, 3\}$. Compute $U(P)$ and $L(P)$.

   (c) Let $\epsilon > 0$ be arbitrary. Find a partition P such that $U(P) - L(P) < \epsilon$. (You need not rigorously justify your answer.)

f is integrable if $\sup_P L(P) = \inf_P U(P)$, where the supremum and infimum are taken over the set of all partitions of $[a, b]$.

For $P = \{0, 1, 3\}$, we have

$$U(P) = 1 \cdot (1 - 0) + 15 \cdot (3 - 1) = 31.$$

$$L(P) = 1 \cdot (1 - 0) + 1 \cdot (3 - 1) = 3.$$

Common error: forgetting that $h(1) = 1$, so the lower sum on the interval $[1, 3]$ takes the value 1, not 2.

Now let $\epsilon > 0$ be arbitrary. Consider the partition $P = \{0, 1, 1+\delta, 2-\delta, 2+\delta, 3\}$, where δ is assumed to be less than $\frac{1}{2}$ so that this partition is listed in order. The difference $U(P) - L(P)$ is equal to

$$U(P) - L(P) = (2-1)((1+\delta) - 1) + (15-2)((2+\delta) - (2-\delta)) = \delta + 13(2\delta) = 27\delta.$$

Hence, if $\delta < \epsilon/27$ and $\delta < \frac{1}{2}$, then P will work. (This included the unnecessary justification.)

8. **(56 points, 7 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) There is no function $\mathbb{R} \to \mathbb{R}$ whose derivative is $+1$ for $x \geq 0$, and -1 for $x < 0$.
   True. This would violate Darboux's theorem.

   (b) If $(f_n) \to f$ uniformly, and each f_n is discontinuous, then f is discontinuous.
   False. For example, $f_n(x) = 0$ for $x \leq 0$, and $f_n(x) = \frac{1}{n}$ for $x > 0$. Then f_n converges uniformly to 0.

   (c) If $(f_n) \to f$ pointwise, and each f_n is bounded, then f is bounded.
   False. For example, f_n could agree with the function $\frac{1}{x}$ on the interval $(\frac{1}{n}, \infty)$, and be the zero function on the interval $(-\infty, \frac{1}{n}]$. Then f_n converges pointwise to the unbounded function f which agrees with $\frac{1}{x}$ on $(0, \infty)$ and zero on $(-\infty, 0]$.

   (d) If (f_n) is a sequence of functions $\mathbb{R} \to \mathbb{R}$ such that (f_n') converges uniformly to 0, and the sequence of numbers $(f_n(17))$ converges to -3, then (f_n) converges uniformly to to the constant function -3.
   True. By the differentiable limit theorem (upgrade), f_n converges uniformly to some function f, with $f' = 0$ and $f(17) = -3$. But then f is a constant function, so $f = -3$.

   (e) If $P(x)$ is a power series which converges on $[-5, 5]$, then it attains a maximum value.
   True. By the PSAT and Abel's theorem, $P(x)$ is continuous on $[-5, 5]$. Since $[-5, 5]$ is compact, $P(x)$ attains a maximum value.

   (f) If $Q(x)$ is a power series which converges at $x = 4$, then it converges absolutely at $x = -3$.
   True. We proved (en route to the PSAT) that a power series converging at x_0 will converge absolutely for all x with $|x| < |x_0|$.

   (g) If g is continuous, then there exists a function G such that $G' = g$.
   True, by the fundamental theorem of calculus. One can let $G(x) = \int_a^x g$ for some $a \in \mathbb{R}$.

   (h) If (h_n) is a sequence of integrable functions which converges pointwise to h on $[a, b]$, and $\lim_{n\to\infty} \int_a^b h_n = L$, then $\int_a^b h = L$.
   False. (It is true for uniform convergence.) One can let

$$h_n(x) = \begin{cases} n & \text{if} \quad x \in (0, \frac{1}{n}) \\ 0 & \text{else.} \end{cases}$$

   Then $\int_0^1 h_n = 1$ for all n, but h_n converges pointwise to 0.

Xtra3. More true/false.

(a) If $h\colon [a,b] \to \mathbb{R}$ is integrable and $H(x) = \int_a^x h(t)dt$ is differentiable on $[a,b]$, then $H'(x) = h(x)$ for all $x \in [a,b]$.

False. h could be Thomae's function, or just any continuous function with one removable discontinuity thrown in. (If it were true for some function h, it would be false if one took h and altered a single point $h(c)$ to be some other value, which doesn't change the integral H or its derivative H'.)

(b) There is a sequence of functions $g_n\colon \mathbb{R} \to \mathbb{R}$ whose domain of convergence is the Cantor set.

True. Denote by C the Cantor set. Let

$$g_n(x) = \begin{cases} n & \text{if} \quad x \notin C \\ 0 & \text{if} \quad x \in C. \end{cases}$$