Math 317 (Fund. of analysis), Spring 2018
Final

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Name:

Q1		15
Q2		15
Q3		20
Q4		15
Q5		20
Q6		15
Q7		20
Q8		56
Xtra		24
Total		176

General notes:

1. **Please don't try extra credit until you've done all the regular problems to your satisfaction.**
2. If you run out of space, use the other side of the paper!
3. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
4. If I want a complete proof, I will say "Prove that ..."
5. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.
6. Make sure to specify any terms which are not in the statement of the problem. For example, when defining the term *continuous*, don't say "f is continuous if..." say "A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if..."
7. All Taylor series are assumed to be centered at zero.

1. **(15 pts)** (a) Define the term *compact*.

   (b) Let $A \subset \mathbb{R}$ be compact. Prove from the definition that A is bounded.

2. **(15 pts)** Let $f(x) = xe^{|x|}$. Prove that f is differentiable at 0. You may use standard facts about the function e^x.

3. **(20 pts)** Consider the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n \cos(nx)}{n}$.

   (a) Prove that $f(x)$ is continuous on $(-1, 1)$.

   (b) Prove that $f(x)$ is differentiable on $(-1, 1)$.

   For a lot of partial credit, you can do the above on the domain $(-\frac{1}{2}, \frac{1}{2})$ instead of $(-1, 1)$.

4. **(15 pts)** Prove the following result.

   Proposition. Suppose that f and g are differentiable on an interval $[a, b]$, and that $f'(x) = g'(x)$ for all $x \in [a, b]$. Then $f = g + C$ for some constant $C \in \mathbb{R}$.

   (Warning: I wouldn't use the fundamental theorem of calculus - f' need not be integrable! You can do this with just the techniques from the chapter on differentiation.)

5. **(20 pts)** Let $g \colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Assume that, for all $k \geq 1$, the k-th derivative is bounded by k, that is $|g^{(k)}(x)| \leq k$ for all $x \in \mathbb{R}$. Let $T(x) = \sum_{k=0}^{\infty} a_k x^k$ be the Taylor series of g.

   (a) Find a bound on the absolute value of a_k, for each k.

   (b) Justify that $T(x)$ converges on all of $\mathbb{R}$.

   (c) Justify that $T(x) = g(x)$ on all of $\mathbb{R}$.

6. **(15 pts)** Let $T(x)$ be the Taylor series for $f(x) = \sqrt{1+x}$, and $U(x)$ be the Taylor series for $g(x) = \frac{1}{\sqrt{1+x}}$.

   (a) Write down the terms up to degree 2 (i.e. the coefficient of x^2) of $T(x)$. (If you don't remember the formula, you can always compute derivatives.)

   (b) Write down the terms up to degree 2 of $U(x)$.

   (c) One expects that $T(x)U(x) = 1$. Verify that this is true up to degree 2.

7. **(20 pts)** (a) Define what it means for f to be integrable on $[a, b]$. (You do not need to tell me what a partition is, or to define the quantities $L(f, P)$ and $U(f, P)$ for a particular partition P.)

   (b) Let $h(x)\colon [0, 3] \to \mathbb{R}$ be the function defined as follows.

$$h(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1 \\ 2 & \text{if} \quad 1 < x < 2 \quad \text{or} \quad 2 < x \le 3, \\ 15 & \text{if} \quad x = 2. \end{cases}$$

   Let P be the partition $\{0, 1, 3\}$. Compute $U(P)$ and $L(P)$.

   (c) Let $\epsilon > 0$ be arbitrary. Find a partition P such that $U(P) - L(P) < \epsilon$. (You need not rigorously justify your answer.)

8. **(56 points, 7 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) There is no function $\mathbb{R} \to \mathbb{R}$ whose derivative is $+1$ for $x \geq 0$, and -1 for $x < 0$.

   (b) If $(f_n) \to f$ uniformly, and each f_n is discontinuous, then f is discontinuous.

   (c) If $(f_n) \to f$ pointwise, and each f_n is bounded, then f is bounded.

   (d) If (f_n) is a sequence of functions $\mathbb{R} \to \mathbb{R}$ such that (f'_n) converges uniformly to 0, and the sequence of numbers $(f_n(17))$ converges to -3, then (f_n) converges uniformly to to the constant function -3.

(e) If $P(x)$ is a power series which converges on $[-5, 5]$, then it attains a maximum value.

(f) If $Q(x)$ is a power series which converges at $x = 4$, then it converges absolutely at $x = -3$.

(g) If g is continuous, then there exists a function G such that $G' = g$.

(h) If (h_n) is a sequence of integrable functions which converges pointwise to h on $[a, b]$, and $\lim_{n\to\infty} \int_a^b h_n = L$, then $\int_a^b h = L$.

You're done! Here are some extra credit problems. Don't work on them problem until you've finished the rest of the test! The points-to-time ratio is smaller.

Xtra1. Let $f(x) = xe^{|x|}$ be the function from Problem 2. Is f twice-differentiable at 0? Justify your answer.

Xtra2. Let $T(x)$ be the Taylor series for $\sqrt{1+x}$ and $U(x)$ be the Taylor series for $\frac{1}{\sqrt{1+x}}$, as in Problem 6. One expects that $2T'(x) = U(x)$. Comparing the terms of degree n, what equality between binomial numbers do you get?

Xtra3. More true/false.

(a) If $h\colon [a,b] \to \mathbb{R}$ is integrable and $H(x) = \int_a^x h(t)dt$ is differentiable on $[a,b]$, then $H'(x) = h(x)$ for all $x \in [a,b]$.

(b) There is a sequence of functions $g_n\colon \mathbb{R} \to \mathbb{R}$ whose domain of convergence is the Cantor set.