Math 317 (Fund. of analysis), Spring 2018
Midterm 1 solutions

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1. **(10 pts)** (a) Define the term *compact*.

   (b) Let $A \subset \mathbb{R}$ be compact. Prove from the definition that A is closed.

   a) A subset $A \subset \mathbb{R}$ is compact if every sequence in A has a convergent subsequence, whose limit is in A.

   b) Suppose that A is not closed. Then there is a limit point x of A, with $x \notin A$. By the definition of limit point, there is a sequence (x_n) in A with $(x_n) \to x$. But then every subsequence of this (convergent) sequence has the same limit x, so no subsequence can have a limit in A. Hence A is not compact, a contradiction. Thus A is closed.

   Common mistake: Speaking as though every sequence converged. For example: "A being compact implies that, for any sequence, there is a subsequence converging to $a \in A$. Since the subsequence has the same limit as the original sequence..." no, the original sequence need not have a limit. Of course, once you've fixed a limit point and a sequence converging to that limit point, you're good to go.

   Other common mistake: Not actually having an argument.

2. **(12 pts)** Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be differentiable functions. Prove directly from the definition that $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ for all $c \in \mathbb{R}$. (For this problem I have hints I can give you at the cost of points.)

   We compute that

$$(fg)'(c) = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} = \lim_{x \to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}$$
$$= \lim_{x \to c} \left(f(x) \frac{g(x) - g(c)}{x - c} + g(c) \frac{f(x) - f(c)}{x - c} \right).$$

   (Up to this point we haven't used the algebraic limit theorem, we've just been manipulating the object inside the limit.) Now we use the ALT to say that the above is equal to

$$\lim_{x \to c} f(x) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} + g(c) \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f(c)g'(c) + g(c)f'(c),$$

   which works because all these limits exist. Note that $\lim_{x \to c} f(x) = f(c)$ because f is continuous at c, which is true because differentiable functions are continuous.

On the whole this went very poorly! First, common mistakes:

- This is one of those proofs that has one very important trick (namely - add and subtract something) and if you missed it you wouldn't get far (which is why I offered a hint). Instead, people who missed the trick resorted to some very questionable algebra.
- Don't mess this one up again! fg denotes the PRODUCT, not the composition. The composition is $f \circ g$. (After all, this looks like the product rule, not the chain rule.)
- When asserting that $\lim_{x\to c} f(x) = f(c)$, you need to use the fact that f is continuous, and say that this is because differentiable implies continuous.
- As usual, when you use the ALT, you should say so. There is a reason for this (SEE PROBLEM 5!!!).

3. **(14 pts)** (a) Define the term *Lipschitz*. (I can give you the answer, at the cost of points.)
   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that f' is bounded. Prove that f is Lipschitz.
   (c) If $g\colon A \to \mathbb{R}$ is Lipschitz, is it uniformly continuous? If $h\colon A \to \mathbb{R}$ is uniformly continuous, is it Lipschitz? I want one theorem and one counterexample (no proof or justification required).

a) A function $A \to \mathbb{R}$ is Lipschitz if $\exists M > 0$ such that for all $x, c \in A$,

$$\left| \frac{f(x) - f(c)}{x - c} \right| \leq M.$$

(Common error: forgetting some quantifiers.)

b) Let f be differentiable, such that $|f'(x)| \leq M$ for some $M > 0$. For any $x, c \in A$, the Mean Value Theorem states that $\frac{f(x)-f(c)}{x-c} = f'(b)$ for some b between x and c. Thus

$$\left| \frac{f(x) - f(c)}{x - c} \right| = |f'(b)| \leq M.$$

(Tragically common error: not using the MVT. Every other argument I saw was mostly nonsense.)

c) If g is Lipschitz, then it is uniformly continuous. However, not every uniformly continuous function is Lipschitz. For example, $x \mapsto \sqrt{x}$ defined on $[0,1]$ (or even $x \mapsto \sqrt{|x|}$ defined on all of $\mathbb{R}$) is uniformly continuous but not Lipschitz.

4. **(10 pts)** Is it true or false? Justify or give a counterexample.

   (a) If f and g are continuous on $[a, c]$, and $f(a) > g(a)$, and $f(c) < g(c)$, then there exists some point $b \in (a, c)$ where $f(b) = g(b)$.

   (b) If f and g are differentiable on $[a, c]$, and $f'(a) > g'(a)$, and $f'(c) < g'(c)$, then there exists some point $b \in (a, c)$ where $f'(b) = g'(b)$.

a) True. Consider the function $h = f - g$. Then $h(a) > 0$ and $h(c) < 0$, and h is continuous, so by the Intermediate Value Theorem there is some $b \in (a, c)$ with $h(b) = 0$. Therefore, $f(b) = g(b)$.

b) True. Consider the function $h = f - g$. Then h is differentiable, $h' = f' - g'$. Thus $h'(a) > 0$ and $h'(c) < 0$. By Darboux's theorem, there exists some $b \in (a, c)$ with $h'(b) = 0$. Therefore, $f'(b) = g'(b)$.

This problem was here to test the IVT, Darboux's theorem, common misconceptions around these, and most importantly, the crucial trick of using the function $f - g$!! We did this many times, and will continue to do so - when comparing two functions, it is best to look at the function which measures their difference, rather than applying any theorem to the functions individually. It was tragically common not to mention $f - g$, and therefore, to make at best a hand-waving argument. (It was also common to not mention Darboux's theorem.)

5. **(18 pts)** Let $f_n(x) = \frac{nx}{1+nx^2}$ for each $n \in \mathbb{N}$.

   (a) Let $f(x) = \frac{1}{x}$. Prove that $f_n \to f$ uniformly on the domain $(1, \infty)$.

   (b) On the domain $(-1, 1)$, compute the pointwise limit of f_n. (Hint: to use the algebraic limit theorem, it may help to divide the numerator and denominator by n.)

   (c) Why is it impossible that f_n converges uniformly on $(-1, 1)$?

   (d) (Extra credit) Why it is impossible that f_n converges uniformly on $(0, 1)$?

a) For $n \in \mathbb{N}$ and $x \in (1, \infty)$ we have

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|\frac{nx^2 - (1+nx^2)}{x(1+nx^2)}\right| = \frac{1}{x(1+nx^2)} < \frac{1}{n}.$$

So let $\epsilon > 0$ be arbitrary, and choose some $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Then for any $n \geq N$ and any $x \in (1, \infty)$, we have

$$|f_n(x) - f(x)| < \frac{1}{n} < \epsilon.$$

Thus $f_n \to f$ uniformly.

(For those who knew what to do, part (a) went pretty well!)

(Meanwhile, part (b) was DISASTROUS.)

b) We have

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x^2}.$$

When $x \neq 0$, the limit of the denominator is nonzero, so we can apply the ALT to equate the above with

$$\frac{\lim_{n\to\infty} x}{\lim_{n\to\infty} \frac{1}{n} + x^2} = \frac{x}{x^2} = \frac{1}{x}.$$

Meanwhile, when $x = 0$, the numerator is zero for any n, so $\lim_{n\to\infty} 0 = 0$. Hence the pointwise limit of f_n is the function g where $g(0) = 0$ and $g(x) = \frac{1}{x}$ for $x \neq 0$.

Tragically common mistake: There is a reason I ask you to write that you're using the ALT when you're using the ALT, and it is precisely because you can NOT use the ALT when it does not apply! In particular, when trying to apply the ALT to fractions, you need the limit of the denominator to be nonzero. (A similar thing happens when looking at $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$... you'd better not replace this with $\frac{\lim_{x\to c}(f(x)-f(c))}{\lim_{x\to c} x-c}$.) (Note that L'Hopital's rule exists precisely to deal with this problem for functional limits (rather than limits as $n \to \infty$ of sequences, as we have in this problem.)) Very often you need to be careful, and check limits on a case by case basis (e.g. what happens for $x > 0$, or $x = 0$, or $x < 0$, etcetera.)

Once you do something silly and say that the pointwise limit of f_n is the function $\frac{1}{x}$, you'd better do something not silly like say "The pointwise limit of f_n on $(-1, 1)$ does not exist!" Because the function $\frac{1}{x}$ is not defined at zero, this says that f_n does NOT converge at zero, i.e. the functions do not have a pointwise limit. Unfortunately, failing to do this point was another common error.

c) (f_n) can not converge to g uniformly on $(-1, 1)$, because f_n is continuous for each n, while g is not continuous; the uniform limit of continuous functions is continuous.

d) (f_n) can not converge to $\frac{1}{x}$ uniformly on $(0, 1)$, because f_n is bounded for each n, while $\frac{1}{x}$ is not bounded; the uniform limit of bounded functions is bounded.

Note: If you used boundedness to talk about (f_n) not converging uniformly to $\frac{1}{x}$ on $(-1, 1)$, I took off some points because that's not the problem, the problem is the lack of pointwise convergence (making the function $\frac{1}{x}$ on $(-1, 1)$ not even a valid function...)

6. **(36 points, 6 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) There is no function whose derivative is $|x|$.

   False. The function $\frac{1}{2}x|x|$ will work. You could also write this as $\frac{x^2}{2}$ for $x \geq 0$, and $\frac{-x^2}{2}$ for $x \leq 0$.

   (This problem was mostly here because someone might incorrectly say "no, the derivative is too weird", or "doesn't Darboux's theorem say that can't happen" or something like that.)

(b) If $f\colon [0,1) \to \mathbb{R}$ is continuous, and $\lim_{x\to 1} f(x)$ exists, then f is bounded.

True. If $\lim_{x\to 1} = L$ then we can extend f to a continuous function on $[0,1]$ by setting $f(1) = L$. A continuous function defined on a compact set (like $[0,1]$) is bounded.

Common error: having no clue. In particular, the idea that when a functional limit exists, one can extend the function continuously is a really important concept! This is what functional limits are really about.

(c) The sequence of functions

$$f_n(x) = \begin{cases} 1 & x > n \\ -1 & x \leq n \end{cases}$$

converges pointwise on $\mathbb{R}$.

True. For any given $x \in \mathbb{R}$, the sequence $(f_n(x))$ is eventually just the constant sequence -1. So $(f_n) \to -1$.

(d) If $(f_n) \to f$ uniformly, and for some $c \in \mathbb{R}$ the sequence $(f_n'(c))$ converges to L, then $f'(c) = L$.

False. For example, $f_n(x) = \frac{\sin(nx)}{n}$ is a sequence which converges to $f = 0$ uniformly. Then $f_n'(0) = 1$ for all n, but $f'(0) = 0$. (This is one of several things which is NOT the DLT.)

(e) If (f_n) is a sequence of functions where (f_n') converges uniformly, then (f_n) converges pointwise.

False. For example, the sequence of functions $f_n(x) = x + n$ does not converge pointwise, but $f_n' = 1$ for all n, which converges uniformly. (For more counterexamples, you could take any sequence g_n for which the DLT applies, and let $f_n(x) = g_n(x) + n$.) (This is one of several things which is NOT the DLT. To be the DLT, you also need $f_n(x_0)$ to converge at some point x_0.)

(f) If a sequence of functions (f_n) converges pointwise to f on $\mathbb{R}$, and converges uniformly to f on $[-M, M]$ for each $M > 0$, then it converges uniformly to f on $\mathbb{R}$.

False. For example, $f_n(x) = \frac{x^2}{n}$. (Common error: giving something which doesn't even converge pointwise on $\mathbb{R}$.)