Math 317 (Fund. of analysis), Spring 2018
Midterm 1
Teacher: Ben Elias
Date: 5/9/2018

Name:

Q1		10
Q2		12
Q3		14
Q4		10
Q5		18
Q6		36
Total		100

General notes:

1. If you run out of space, use the other side of the paper!
2. Terminology: Supremum = least upper bound. Injective = 1-to-1 = into. Surjective = onto.
3. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
4. If I want a complete proof, I will say "Prove that ..."
5. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.
6. Make sure to specify any terms which are not in the statement of the problem. For example, when defining the term *continuous*, don't say "f is continuous if..." say "A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if..."

1. **(10 pts)** (a) Define the term *compact*.

   (b) Let $A \subset \mathbb{R}$ be compact. Prove from the definition that A is closed.

2. **(12 pts)** Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be differentiable functions. Prove directly from the definition that $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ for all $c \in \mathbb{R}$. (For this problem I have hints I can give you at the cost of points.)

3. **(14 pts)** (a) Define the term *Lipschitz*. (I can give you the answer, at the cost of points.)

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that f' is bounded. Prove that f is Lipschitz.

   (c) If $g\colon A \to \mathbb{R}$ is Lipschitz, is it uniformly continuous? If $h\colon A \to \mathbb{R}$ is uniformly continuous, is it Lipschitz? I want one theorem and one counterexample (no proof or justification required).

4. **(10 pts)** Is it true or false? Justify or give a counterexample.

   (a) If f and g are continuous on $[a, c]$, and $f(a) > g(a)$, and $f(c) < g(c)$, then there exists some point $b \in (a, c)$ where $f(b) = g(b)$.

   (b) If f and g are differentiable on $[a, c]$, and $f'(a) > g'(a)$, and $f'(c) < g'(c)$, then there exists some point $b \in (a, c)$ where $f'(b) = g'(b)$.

5. **(18 pts)** Let $f_n(x) = \frac{nx}{1+nx^2}$ for each $n \in \mathbb{N}$.

   (a) Let $f(x) = \frac{1}{x}$. Prove that $f_n \to f$ uniformly on the domain $(1, \infty)$.

   (b) On the domain $(-1, 1)$, compute the pointwise limit of f_n. (Hint: to use the algebraic limit theorem, it may help to divide the numerator and denominator by n.)

   (c) Why is it impossible that f_n converges uniformly on $(-1, 1)$?

   (d) (Extra credit) Why it is impossible that f_n converges uniformly on $(0, 1)$?

6. **(36 points, 6 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) There is no function whose derivative is $|x|$.

   (b) If $f\colon [0,1) \to \mathbb{R}$ is continuous, and $\lim_{x\to 1} f(x)$ exists, then f is bounded.

   (c) The sequence of functions

   $$f_n(x) = \begin{cases} 1 & x > n \\ -1 & x \leq n \end{cases}$$

   converges pointwise on $\mathbb{R}$.

   (d) If $(f_n) \to f$ uniformly, and for some $c \in \mathbb{R}$ the sequence $(f_n'(c))$ converges to L, then $f'(c) = L$.

   (e) If (f_n) is a sequence of functions where (f_n') converges uniformly, then (f_n) converges pointwise.

   (f) If a sequence of functions (f_n) converges pointwise to f on $\mathbb{R}$, and converges uniformly to f on $[-M, M]$ for each $M > 0$, then it converges uniformly to f on $\mathbb{R}$.