Math 317 (Fund. of analysis), Spring 2018
Quiz 1 Solutions

Teacher: Ben Elias
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1. **(9 pts)** For the function $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, prove that $f'(5) = 75$. (You may use the formula $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.)

   (First the fast way)

   The difference quotient function for f at 5 is

   $$\Delta_{f,5}(x) = \frac{x^3 - 5^3}{x-5} = x^2 + 5x + 5^2.$$

   Since $\lim_{x\to 5} x = 5$, the algebraic limit theorem implies that $\lim_{x \to 5} \Delta_{f,5}(x) = 5^2 + 5^2 + 5^2 = 75$. By definition, this limit is $f'(5)$.

   (Let me be a little silly about this, for fun. Start with the same first sentence. Then)

   To prove that $f'(5) = \lim_{x\to 5} \Delta_{f,5}(x) = 75$, pick some $\epsilon > 0$. Note that, for $x \in \mathbb{R}$, we have

   $$|\Delta_{f,5}(x) - 75| = |x^2 + 5x - 50| = |(x-5)(x+10)| = |x-5||x+10|.$$

   So let $\delta = \min\{1, \frac{\epsilon}{16}\}$. If $|x-5| < 1$ then $|x+10| < 16$. Thus if $|x-5| < \delta$ then

   $$|\Delta_{f,5}(x) - 75| < 16\delta \leq \epsilon,$$

   as desired.

   (The biggest error on this problem was failing to justify why the limit of $x^2+5x+25$ as $x \to 5$ was 75. This was something which used to take real work, and still does when the function isn't super easy! But the algebraic limit theorem saves the day, in this case, if you use it! I didn't take off points for not saying "Since $\lim_{x\to 5} x = 5$," though this observation is really needed to use the algebraic limit theorem.)

2. **(11 pts)** (a) Define the term *compact*.

   (b) Let $A \subset \mathbb{R}$ be closed and bounded. Prove that A is compact.

   A subset A of $\mathbb{R}$ is compact if every sequence in A has a convergent subsequence, whose limit is also in A.

   (Common minor mistake: failing to say that this is a definition for subsets of $\mathbb{R}$, rather than random sets or something unstated.)

   Suppose that $A \subset \mathbb{R}$ is closed and bounded. Pick an arbitrary sequence in A. By the Bolzano-Weierstrass theorem, it has a convergent subsequence $(y_k)_{k \in \mathbb{N}}$. Let L denote its limit. Because A is closed, for any convergent sequence in A, its limit is also in A. Since $y_k \in A$ for all $k \in \mathbb{N}$, we see that $L \in A$, as desired.

   (A frequent bad mistake: saying something like "a bounded sequence converges." Failing to use the B-W theorem was common.)

3. **(15 pts)** (a) Let $A \subset \mathbb{R}$, and $f\colon A \to \mathbb{R}$. Define what it means for f to be *uniformly continuous*.

   (b) Briefly justify the fact that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$.

   (c) Briefly justify the fact that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1]$.

   a) f is uniformly continuous on A if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in A$, the condition $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

   b) Solution 1: The function $f(x)$ is continuous on $[1, 2]$, and $[1, 2]$ is compact. Any continuous function on a compact set is uniformly continuous, by a theorem.

   Solution 2: Note that, for $x, y \in [1, 2]$ one has

   $$|\frac{1}{x} - \frac{1}{y}| = |\frac{y - x}{xy}| \leq |y - x|,$$

   since $|xy| \geq 1$. So, for any $\epsilon > 0$, we can let $\delta = \epsilon$, and

   $$|x - y| < \delta \implies |\frac{1}{x} - \frac{1}{y}| < \delta = \epsilon.$$

   c) To show something is not uniformly continuous it is easiest to use the sequential definition. Letting $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$, we see that $(x_n - y_n) \to 0$. However, $f(x_n) - f(y_n) = -n$ does not converge to zero. Hence f is not uniformly continuous on $(0, 1]$.

   (One can also use the epsilon delta definition. Choosing some epsilon (e.g. $\epsilon = 1$) one much show that, for each $\delta > 0$, there exist some $x, y \in (0, 1]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. For example, one could choose some $n > 1$ such that $\frac{1}{2n} < \delta$, and set $x = \frac{1}{n}$ and $y = \frac{1}{2n}$.)

4. **(15 pts)** For each of the following statements, is it true or false? Justification is required.

(a) If f is continuous on $[a, c]$ then there is some $b \in (a, c)$ for which $f'(b) = \frac{f(c)-f(a)}{c-a}$.
This is false. (As many noted, it would be true if f were differentiable, by the Mean Value Theorem, but one still needs a counterexample to show that it is false without that assumption.) For example, one could choose $f(x) = |x|$, with $a > 0$ and $c = -a$. Or, one could choose $f(x)$ to be the Weierstrass monster, which is not differentiable anywhere.

(b) If f is differentiable on (a, b), and $f'(x) \neq 0$ for any $x \in (a, b)$, then either f is increasing or f is decreasing.

This is true. If $f'(z_1) > 0$ and $f'(z_2) < 0$ for $z_1, z_2 \in (a, b)$, then by Darboux's theorem, there must be some $z_3 \in (a, b)$ with $f'(z_3) = 0$. Hence either $f'(x) > 0$ for all $x \in (a, b)$, or $f'(x) < 0$ for all $x \in (a, b)$. (Common problem: not using Darboux's theorem, just asserting something like this!! Or trying some argument involving minima and maxima which wasn't quite coherent.)

(Many people just assumed at this point that if $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing. That's reasonable... I gave extra credit for a finer argument (below) that explained why this is true. Note however that you can't be "increasing at a point," that is, if $f'(c) > 0$ then you are not "increasing at c," this doesn't even make sense. In theory you can be increasing in a neighborhood of c, that at least makes sense... but an exercise on the homework showed how a function could have $f'(c) > 0$ but not be increasing in any neighborhood of c.)

If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing. This is because, if f was not increasing, there is some $x < y$ with $f(x) > f(y)$. Then, by the Mean Value Theorem, there is some z with $f'(z) < 0$, a contradiction.

(c) If C is the Cantor set and $f\colon C \to \mathbb{R}$ is continuous, then f attains a maximum value.

This is true. The Cantor set is closed (infinite intersection of closed sets) and bounded (inside $[0, 1]$), so it is compact. A continuous function on a compact set attains a maximum value by the Extremal Value Theorem. (I originally intended to take off if you didn't justify why C is compact, but decided that was silly.)