Math 317 (Fund. of analysis), Spring 2018
Quiz 2 Solutions
Teacher: Ben Elias
Date: 5/30/2018

1. **(16 pts)** (a) State the result known as the *Weierstrass M-test*.

   (b) Consider the series
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2 + x + 1}.$$
   Use the Weierstrass M-test to prove that $f(x)$ is well-defined and continuous on $[-1, 1]$.

a) Fix $A \subset \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$ one has a function $f_n \colon A \to \mathbb{R}$ and a number $M_n > 0$ such that $|f_n(x)| < M_n$ for all $x \in A$. If $\sum_{n \in \mathbb{N}} M_n$ converges, then $\sum_{n \in \mathbb{N}} f_n$ converges uniformly on A.

b) For any $x \in [-1, 1]$ and $n \in \mathbb{N}$ one has
$$\left| \frac{x^n}{n^2 + x + 1} \right| \le \frac{1^n}{n^2 - 1 + 1} = \frac{1}{n^2}.$$
So let $M_n = \frac{1}{n^2}$ for each $n \in \mathbb{N}$. Since $\sum M_n$ converges, we see that $\sum f_n$ converges uniformly, so that $f = \sum f_n$ is well-defined. Since each f_n is continuous on $[-1, 1]$ (the denominator never vanishes), the term-by-term continuity theorem implies that f is also continuous.

The most common error was saying that $\frac{|x^n|}{|n^2+x+1|} \le \frac{|x^n|}{n^2+1+1}$ because one plugs in $|x| \le 1$?? But of course that is not how it works. Using $x \ge -1$, you can say $\frac{1}{n^2+x+1} \le \frac{1}{n^2-1+1}$. That'll work.

Another common error was failing to use uniform continuity to prove that f was continuous. Along this road, a common error was failing to mention that each f_n is continuous (if that $+1$ was not there in the denominator, or if it was a -16, then it would not be...)

2. **(16 pts)** Let $a(x) = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots$ be a power series, of which only terms up to degree 2 are known. Assume that the radius of convergence is nonzero.

   (a) What is $a''(0)$?

   (b) How many terms of $a(x)^2$ is it possible to compute? Compute them.

   (c) Your friend claims to you that $a(x) = \sqrt{\cos(x) + x}$ inside the domain of convergence. Write down the first few terms of the power series representation of $\cos(x) + x$. Deduce that your (former) friend is incorrect.

Note: a common error was to try to guess the terms of this series, like maybe it continues $\frac{1}{8}x^3 + \ldots$ or something. Don't EVER assume that a power series is going to be one that you recognize. Maybe $a_3 = 1000000$ and $a_4 = -27$. Who knows.

a) $a''(0) = 2a_2 = -\frac{1}{2}$.

Note: For any power series $a(x) = \sum a_n x^n$, one has $a_n = \frac{1}{n!}a^{(n)}(0)$. You have this at your disposal, you don't need to recompute derivatives each time.

b) One can compute $a(x)^2$ up to degree 2 (i.e. the x^2 term). (Note: you can't compute the x^3 term of $a(x)^2$ without knowing the x^3 term of $a(x)$.) One has

$$(1 + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots)^2 = 1 + (\frac{1}{2} + \frac{1}{2})x + (\frac{-1}{4} + \frac{1}{2}\frac{1}{2} + \frac{-1}{4})x^2 + \ldots = 1 + x - \frac{1}{4}x^2.$$

c) Since $\cos(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}(-1)^{2n} = 1 - \frac{1}{2}x^2 + \ldots$, we have

$$\cos(x) + x = 1 + x - \frac{1}{2}x^2 + \ldots.$$

But this does not agree with the first terms of $a(x)^2$, so that $a(x)^2 \neq \cos(x) + x$.

Note: A weirdly common error was to write

$$\cos(x) + x = \sum \frac{1}{(2n)!}(-x)^{2n} + x = (1 + x) - (\frac{1}{2}x^2 + x) + (\frac{1}{4!}x^3 + x) - \ldots$$

or something weird like this. One should be careful when doing calculations to mentally note the "scope" of the summation sign, so you don't include stuff that isn't in there (what you get if you keep adding x like that is NOT a power series).

3. **(18 points, 6 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) Suppose that $f(x)$ is a continuous function, agreeing with the power series $T(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^3}$ on the open interval $(-1,1)$. Then $T(1) = f(1)$.
   True. The series $T(1) = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^3}$ converges by the alternating series test (or absolutely, by the p-test). Then, by Abel's theorem and continuity of f, $T(1) = \lim_{x\to 1} T(x) = \lim_{x\to 1} f(x) = f(1)$.
   Note: For a different power series, it is possible that $T(1)$ diverges. But if it converges, then it agrees with $f(1)$ by the same argument. One had better cite Abel's theorem, because this is not obvious otherwise.

   (b) If $\sum_{n=0}^{\infty} a_n x^n$ converges at the point x_0, then so does $\sum_{n=0}^{\infty} n a_n x^{n-1}$.
   False. For example, when $x_0 = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ converges at $x = 1$, but its derivative $\sum_{n=1}^{\infty}(-1)^n x^n$ diverges at $x = 1$.
   Note: Many people quoted the PSAT, which says that the RADIUS of convergence of the derivative agrees with the original power series. However, the DOMAIN of convergence can be different: a power series might converge on $(-R, R]$, while its derivative converges on $(-R, R)$, so when $x_0 = R$ you lose.

   (c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and $T(x)$ be the Taylor series of $g(x)$ at zero. If R is the radius of convergence of $T(x)$, then $T(x) = g(x)$ on $(-R, R)$.
   False. For example, let $g(x) = 0$ when $x = 0$, and $g(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$. Then every derivative of g at 0 is 0, so $T(x) = 0$ and has radius of convergence $R = \infty$. However, $T(x) \neq g(x)$ for any $x \neq 0$.
   Note: This question is NOT about the well-definedness of T (i.e. about the radius of convergence) but about the agreement or non-agreement of T with g.

4. **(Extra Credit)** When $n = \frac{1}{3}$, what is $\binom{n}{3}$ explicitly?
   Aside: For any $m \in \mathbb{R}$ we define
   $$\binom{m}{3} = \frac{m \cdot (m-1) \cdot (m-2)}{1 \cdot 2 \cdot 3},$$
   which has 3 terms on top and bottom. More generally, for $k \in \mathbb{N}$,
   $$\binom{m}{k} = \frac{m \cdot (m-1) \cdots (m-k)}{1 \cdot 2 \cdots k},$$
   with k terms on top and bottom. These binomials were important for the power series of $(1+x)^m$.
   So
   $$\binom{\frac{1}{3}}{3} = \frac{(1/3)(-2/3)(-5/3)}{1 \cdot 2 \cdot 3} = \frac{10}{27 \cdot 6}.$$