Name:

General notes:

1. If you run out of space, use the other side of the paper!


3. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (one or two sentences).

   • To justify that “Every closed interval contains 0” is false, you should provide a counterexample, such as \([1, 2]\).

   • To justify that “Some closed interval contains 0” is true, you should provide an example, such as \([-1, 1]\).

   • To justify that “Every closed interval contains some nonzero number” is true, you should explain. “For any interval \([a, b]\), either \(a \neq 0\) or \(b \neq 0\).”

4. If I want a complete proof, I will say “Prove that ...”

5. If I don’t need any justification, I will say “no justification necessary.” No justification is needed for a definition.

6. Make sure to specify any terms which are not in the statement of the problem. For example, when defining the term continuous, don’t say “\(f\) is continuous if...” say “A function \(f : \mathbb{R} \to \mathbb{R}\) is continuous if...”
1. (16 pts) (a) State the result known as the Weierstrass M-test.

(b) Consider the series

\[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2 + x + 1}. \]

Use the Weierstrass M-test to prove that \( f(x) \) is well-defined and continuous on \([-1, 1]\).

2. (16 pts) Let \( a(x) = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots \) be a power series, of which only terms up to degree 2 are known. Assume that the radius of convergence is nonzero.

(a) What is \( a''(0) \)?

(b) How many terms of \( a(x)^2 \) is it possible to compute? Compute them.

(c) Your friend claims to you that \( a(x) = \sqrt{\cos(x) + x} \) inside the domain of convergence. Write down the first few terms of the power series representation of \( \cos(x) + x \). Deduce that your (former) friend is incorrect.
3. **(18 points, 6 pts each)** For each of the following statements, is it true or false? Justification is required.

(a) Suppose that $f(x)$ is a continuous function, agreeing with the power series $T(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^3}$ on the open interval $(-1, 1)$. Then $T(1) = f(1)$.

(b) If $\sum_{n=0}^{\infty} a_n x^n$ converges at the point $x_0$, then so does $\sum_{n=0}^{\infty} na_n x^{n-1}$.

(c) Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be infinitely differentiable, and $T(x)$ be the Taylor series of $g(x)$ at zero. If $R$ is the radius of convergence of $T(x)$, then $T(x) = g(x)$ on $(-R, R)$.

4. **(Extra Credit)** When $n = \frac{1}{3}$, what is $\left( \frac{n}{3} \right)$ explicitly?