Math 317 (Fund. of analysis), Spring 2018
Quiz 2

Teacher: Ben Elias
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Name:

Q1		16
Q2		16
Q3		18
Total		50

General notes:

1. If you run out of space, use the other side of the paper!
2. Terminology: Supremum = least upper bound. Injective = 1-to-1 = into. Surjective = onto.
3. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
4. If I want a complete proof, I will say "Prove that ..."
5. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.
6. Make sure to specify any terms which are not in the statement of the problem. For example, when defining the term *continuous*, don't say "f is continuous if..." say "A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if..."

1. **(16 pts)** (a) State the result known as the *Weierstrass M-test*.

   (b) Consider the series
   $$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2 + x + 1}.$$
   Use the Weierstrass M-test to prove that $f(x)$ is well-defined and continuous on $[-1, 1]$.

2. **(16 pts)** Let $a(x) = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots$ be a power series, of which only terms up to degree 2 are known. Assume that the radius of convergence is nonzero.

   (a) What is $a''(0)$?

   (b) How many terms of $a(x)^2$ is it possible to compute? Compute them.

   (c) Your friend claims to you that $a(x) = \sqrt{\cos(x) + x}$ inside the domain of convergence. Write down the first few terms of the power series representation of $\cos(x) + x$. Deduce that your (former) friend is incorrect.

3. **(18 points, 6 pts each)** For each of the following statements, is it true or false? Justification is required.

   (a) Suppose that $f(x)$ is a continuous function, agreeing with the power series $T(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^3}$ on the open interval $(-1, 1)$. Then $T(1) = f(1)$.

   (b) If $\sum_{n=0}^{\infty} a_n x^n$ converges at the point x_0, then so does $\sum_{n=0}^{\infty} n a_n x^{n-1}$.

   (c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and $T(x)$ be the Taylor series of $g(x)$ at zero. If R is the radius of convergence of $T(x)$, then $T(x) = g(x)$ on $(-R, R)$.

4. **(Extra Credit)** When $n = \frac{1}{3}$, what is $\binom{n}{3}$ explicitly?