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I have included here some practice questions for the topics which were not included in the other midterms. The final is comprehensive, and you should use the earlier midterms and practice midterms to supplement these problems.

**Approximate List of Topics:**

- Linear systems. Augmented matrices.
- Row reduction, row echelon form, reduced row echelon form.
- Homogeneous and inhomogeneous solutions, and what they look like.
- Network flow.
- Bases.\* Linear independence and spans.\* Dimension.\*
- Parametric vector form. Finding a basis for a null space.
- Subspaces.\* The null space and column space of a matrix. The rank-nullity theorem.
- The coordinates of a vector in a given basis.\*
- Finding a basis inside a spanning set.
- Linear transformations.\* The matrix of a linear transformation.\* The meaning of the columns of a matrix.\*
- Visualizing linear transformations of the plane.
- Composing linear transformations.\* Matrix multiplication.
- Solving matrix multiplication equations.
- Word problems involving inputs and outputs, “chaining factories.”
- Difference equations.
- Injective, surjective, bijective. What it says about the row echelon form.
- Left and right inverses. Invertible matrices and 2-sided inverses.
- Finding the inverse of a matrix by row reduction. Finding the inverse of a matrix by Cramer’s rule.
- Properties of inverses (for example, Thm 8 in Ch 2.3, Thm 4 in Ch 3.2, etc).
- Computing determinants: cofactors, column and row expansion,  $2 \times 2$  and  $3 \times 3$  matrices, row reduction, triangular matrices.
- Properties of determinants. Area and volume and determinants.
- Cramer’s rule.
- Abstract vector spaces. The \*’ed items above in this context. Writing a linear transformation as a matrix, given a choice of basis for the domain and codomain.

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1. For each of the following statements, indicate whether the statement is true or false. If it is false, briefly explain why or give a counter-example. (**Note: If it is true, you need not explain why. But if you do explain why, you may get partial credit in case you were wrong and it is false.**)

(a) Every surjective linear transformation has a right inverse which is also a linear transformation.

(b) For two matrices  $A$  and  $B$ , the determinant  $\det(A+B)$  is equal to  $\det(A) + \det(B)$ .

(c) For two matrices  $A$  and  $B$ ,  $AB$  is invertible if and only if  $BA$  is invertible.

(d) The set of polynomials  $p(x) \in \mathbb{P}$  for which  $p(1) = p(4)$  is a subspace.

(e) If the determinant of  $A$  is non-zero, then the columns of  $A$  are linearly dependent.

(f) The determinants of the following two matrices are equal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+3c & b+c & c \\ d+3f & e+f & f \\ g+3i & h+i & i \end{vmatrix}$$

(g) The functions  $x^2$ ,  $\sin x$ , and  $e^{16x}$  are linearly independent.

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2. Find the determinant of the following matrix.

$$A = \begin{pmatrix} 1 & 7 & 5 & 2 & 1 \\ 0 & -10 & -3 & 3 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 3 & 9 & -1 & -1 & 1 \end{pmatrix}$$

3. Find the determinant of the following matrices.

(a)

$$\begin{pmatrix} 1 & 2 & 2 \\ 5 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

4. These questions are about the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 5 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

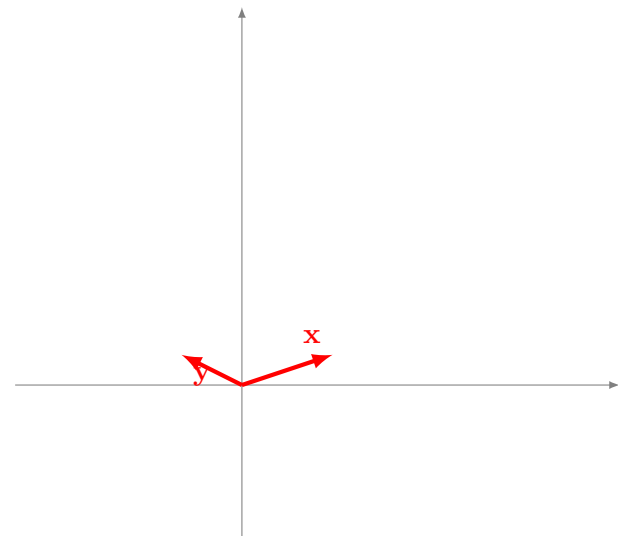
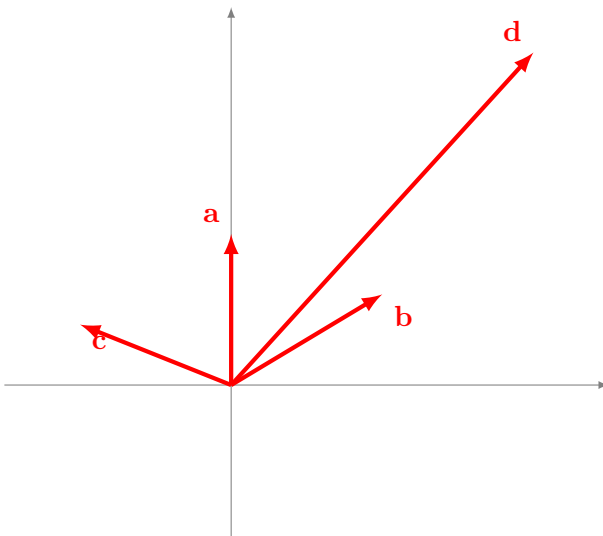
(a) Compute the cofactor matrix  $C$  attached to  $A$ . (For those who don't come to class: your book called  $C^t$  the adjugate matrix of  $A$ .)

(b) Using Cramer's rule, write down the inverse of  $A$ .

(c) Now compute the inverse of  $A$  using row reduction.

5. What is the area of the parallelogram with vertices at  $(0, 0)$ ,  $(1, 2)$ ,  $(-2, 5)$ , and  $(-1, 7)$ ?

6. The following graphs depict 6 vectors in  $\mathbb{R}^2$ . Let  $L$  be a linear transformation satisfying  $L(\mathbf{a}) = \mathbf{x}$  and  $L(\mathbf{b}) = \mathbf{y}$ . On the second graph, draw in vectors representing  $L(\mathbf{c})$  and  $L(\mathbf{d})$ .



7. What follows is a sequence of matrices in a row reduction algorithm.

$$A = \begin{pmatrix} 0 & 5 & 10 \\ 6 & 4 & 8 \\ 3 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 3 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -5 \end{pmatrix} \rightarrow$$
$$\rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & 0 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{pmatrix} = B$$

- (a) Label the arrows with the operations performed.
- (b) What is the determinant of  $B$ ?
- (c) Label each matrix with its determinant.

8. Let  $\mathcal{B} = \{x^2, \sin 2x, 1, \cos 2x, x\}$  be a set of functions on  $\mathbb{R}$ , and let  $V$  denote the span of these functions. You may assume that these functions are linearly independent (they are!).

- (a) Write the linear operator  $D: V \rightarrow V$ , which sends a function  $f$  to its derivative  $f'$ , as a matrix with respect to the basis  $\mathcal{B}$ .
  
  
  
  
  
  
  
  
  
  
- (b) Write the linear operator  $L: V \rightarrow V$ , which sends a function  $f$  to  $f' - 2f$ , as a matrix with respect to the basis  $\mathcal{B}$ .

9. Consider the vector space  $\mathbb{P}_2$  of polynomials having degree  $\leq 2$ . Let

$$\mathcal{C} = \{x^2, (x+1)^2, (x+2)^2, (x+3)^2\}$$

inside  $\mathbb{P}_2$ .

- (a) Is  $\mathcal{C}$  linearly independent? If not, find a non-trivial linear combination giving the zero polynomial.
- (b) Find a subset of  $\mathcal{C}$  which is a basis for  $\mathbb{P}_2$ .
- (c) Translation by  $+1$ , written  $T_1$ , is the linear operator on  $\mathbb{P}_2$  which sends a polynomial  $p(x)$  to the polynomial  $p(x+1)$ . For example,  $T_1(x^2+2x) = (x+1)^2+2(x+1)$ . Write down the matrix for  $T_1$  with respect to the basis you chose.