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Math 432/532 (Diff. Top.), Winter 2016  
HW 1

Starred problems are for 532 students, and are extra credit for 432 students (not worth as much as a usual problem). 532 students must LaTeX their solutions.

You can assume the following theorem without proof in this course.

**Theorem 0.1.** If a non-empty open subset of  $\mathbb{R}^n$  is homeomorphic to a non-empty open subset of  $\mathbb{R}^m$ , then  $n = m$ .

1. G+P Chapter 1.1: 7, 8, 9, 11.
2. Prove that the following topological spaces are topological manifolds:
  - (a) An open subset of a topological manifold.
  - (b) A product of two topological manifolds.
  - (c) A coproduct of two topological manifolds with the same dimension.
3. Prove that a topological manifold is the disjoint union (with the coproduct topology) of its connected components. (Equivalently, prove that the connected components are open.) (This proves that any connected component of a topological  $n$ -manifold is a topological  $n$ -manifold.)
4. Suppose that  $X$  is Hausdorff and second countable, and has the following property: each point  $x \in X$  has a neighborhood  $U$  which is homeomorphic to an open subset of  $\mathbb{R}^k$  for some  $k$  (this  $k$  can depend on the point). Prove that  $X$  is a disjoint union of topological manifolds of various dimensions.
5. Let  $S^1 \subset \mathbb{R}^2$  be the circle of radius 1 centered at the origin. Let  $N = (0, 1)$  and  $S = (0, -1)$ . Let  $U = S^1 \setminus \{S\}$  and  $V = S^1 \setminus \{N\}$ . Let  $f_S: U \rightarrow \mathbb{R}$  and  $f_N: V \rightarrow \mathbb{R}$  be the stereographic projections away from the points  $S$  and  $N$  respectively.
  - (a) Write down the functions  $f_N$  and  $f_S$  explicitly in rectangular coordinates (i.e., where does it send the point  $(x, y) \in S^1$ , as a function of  $x$  and  $y$ ).
  - (b) What is the set  $f_S(U \cap V)$ , as a subset of  $\mathbb{R}$ ?
  - (c) Write down the transition function from  $f_S(U \cap V)$  to  $f_N(U \cap V)$  explicitly in rectangular coordinates. (Hint: your answer should have the number 4 in it.)
  - (d) (\*) Repeat this problem for two antipodal stereographic projections for  $S^2 \subset \mathbb{R}^3$ . Give the transition function in rectangular coordinates.
6. (\*) The space  $\mathbb{R}P^n$  is the set of all lines through the origin in  $\mathbb{R}^{n+1}$ . Topologically, this is the quotient of  $\mathbb{R}^{n+1} \setminus \{0\}$  by the equivalence relation that two vectors are equivalent if they are equal up to scalar. We write  $[x_0 : x_1 : \dots : x_n]$  for the equivalence class of the vector  $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ .
  - (a) Let  $U_i = \mathbb{R}P^n \setminus H_i$ , where  $H_i$  is the set of lines  $[x_0 : \dots : x_n]$  for which  $x_i = 0$ . On  $U_0$ , because  $x_i \neq 0$ , there is always a unique representative of any equivalence class where  $x_i = 1$ . Use this observation to cover  $\mathbb{R}P^n$  by  $n + 1$  charts, and prove that  $\mathbb{R}P^n$  is an  $n$ -manifold.

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- (b) Draw a picture in 3D to illustrate the similarities and differences between this chart and the stereographic projection chart of the sphere.
- (c) Compute the transition functions between these charts.
7. (\*) Suppose that  $X$  is a topological space with a countable basis. Prove that, for an arbitrary collection of open sets, there is a countable subcollection with the same union.
8. (EXTRA CREDIT) I wanted to write a problem exploring the long line (technically, according to Wikipedia, the “open long ray.”) But it ended up being too difficult, so here is some extra credit for those interested in weird topological counterexamples. The first three parts are not hard, and worth thinking about.
- (a) Let  $\mathbb{N}$  be the natural numbers. When  $\mathbb{N}$  is equipped with its usual (well-)ordering, it is often denoted  $\omega$ . Consider the set  $\omega \times [0, 1)$ , equipped with the lexicographic order, so that  $(n, x) \leq (m, y)$  if either  $n < m$  or  $n = m$  and  $x \leq y$ . Equip this set with the order topology. Prove that  $\omega \times [0, 1) \cong [0, \infty) \subset \mathbb{R}$  as topological spaces.
- (b) Now let  $(\omega + 1)$  denote the set  $\mathbb{N} \cup \{\omega\}$ , ordered so that  $\omega$  is larger than every natural number. Again, equip  $(\omega + 1) \times [0, 1)$  with the order topology, and prove that it is still homeomorphic to  $[0, \infty)$  as topological spaces. (Think about an increasing convergent sequence.)
- (c) Let  $S$  be an uncountable ordered set. Prove that  $S \times [0, 1)$  with the lexicographic order topology can not have a countable basis.
- (d) Look up the meaning of a well-ordered set and the idea of trans-finite induction. Prove that, for any countable well-ordered set  $S$ , that  $S \times [0, 1)$  with the lexicographic order topology is locally homeomorphic to  $\mathbb{R}$ .
- (e) Now let  $S$  be the first uncountable well-ordered set, often denoted  $\omega_1$  in the literature. Prove that  $S \times [0, 1)$  with the order topology is locally homeomorphic to  $\mathbb{R}$ , but does not have a countable basis. (Note: one must be careful!  $(\omega_1 + 1) \times [0, 1)$  is NOT locally homeomorphic to  $\mathbb{R}$ .)