Math 432/532 (Diff. Top.), Winter 2016
HW 2

Starred problems are for 532 students, and are extra credit for 432 students (not worth as much as a usual problem). 532 students must LaTeX their solutions.

1. G+P Chapter 1.1, exercise 1. (Should be easy.)
2. G+P Chapter 1.1, exercise 3. Conclude that the following is actually a category: Let $\mathcal{S}$moo be the category whose objects are subspaces $X \subset \mathbb{R}^N$ for some $N$, and where $\mathrm{Mor}(X, Y)$ is the space of smooth maps from $X \subset \mathbb{R}^N$ to $Y \subset \mathbb{R}^M$.
3. G+P Chapter 1.1, exercise 17. (It may help to read and prove for yourself exercises 14-16. These are all straightforward.) (As discussed in class, the implication of this exercise is that the implicit function theorem will tell you when projection to certain coordinates is a chart.)
4. G+P Chapter 1.2, exercise 5. (Do NOT use the theorem that open subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ are not homeomorphic to $n \neq m$... that is a hard theorem. This exercise is not hard.)
5. (*) G+P Chapter 1.3, exercise 1.
6. G+P Chapter 1.3, exercise 3 and 4.
7. Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = x^3 + y^4 + z^5$.
   (a) Compute the derivative $df$.
   (b) Let $X_5 = f^{-1}(5)$ be the set of points where $x^3 + y^4 + z^5 = 5$. Describe the subset of $X$ where projection to the $(x, y)$-plane is a local diffeomorphism. Do the same for the $(x, z)$-plane and the $(y, z)$-plane.
   (c) Deduce that there is a chart for $X_5$ in a neighborhood of every point, so that $X_5$ is a smooth 2-manifold.
   (d) (*) Let $X_a = f^{-1}(a)$ be the fiber over $a$, for any $a \in \mathbb{R}$. For which $a$ can you use the same argument to deduce that $X_a$ is a smooth 2-manifold? For value(s) of $a$ where this argument will not imply that $X_a$ is a smooth 2-manifold, remove some point(s) from $X_a$ to obtain a smooth 2-manifold.
8. Let $F \colon \mathbb{R}^4 \to \mathbb{R}^3$ be defined as follows:

   $$F(x, y, z, w) = (x^2 - y^2 - z^2 + w^2, xy - zw, x^2 + y^2 + z^2 + w^2).$$

   Let $X = F^{-1}(0, 0, 1)$. That is, $X$ is the intersection of the unit sphere with the vanishing set of two homogeneous equations.

   (a) Deduce that if $(x, y, z, w) \in X$, then $x = 0$ if and only if $z = 0$. What can you say about $y$ and $w$ in this case? Prove a similar statement about the conditions $y = 0$ and $w = 0$. (This is pure algebra.)
   (b) Compute the derivative $dF$. (Your matrix has 4 columns and 3 rows.)

(c) The implicit function theorem states that projection to the $x$-coordinate is a local diffeomorphism when a certain determinant is nonzero. Compute this determinant.

(d) (*) What is the set of points in $X$ where this determinant is zero? Be very careful!

(e) (*) What is the set of points in $X$ where projection to the $w$-coordinate is a local diffeomorphism?

(f) (*) Deduce that $X$ is a smooth 1-manifold.

(g) (EXTRA CREDIT) What does $X$ look like, as a 1-manifold inside $S^3$? (Visualize this inside $\mathbb{R}^3 \cong S^3 \setminus \{p\}$ for some point $p$ not in $X$.)

9. Consider the "polar parametrization map" from $\mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}^2 \setminus \{0\}$ which sends $(t, r) \mapsto (r \cos t, r \sin t)$. Compute the derivative of this map, prove that it is a local diffeomorphism, and compute the derivative of the inverse map.