Math 432/532 (Diff. Top.), Winter 2016
HW 9

Starred problems are for 532 students, and are extra credit for 432 students (not worth as much as a usual problem). 532 students must LaTeX their solutions.

1. Consider the subspace
$$X_{(a,b)} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a, \quad y^2 + (x - z)^2/2 = b\}.$$
This is the fiber over (a, b) for a map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$.
   (a) Describe $X_{(a,b)}$ for each $(a, b) \in \mathbb{R}^2$, whether it be a regular value or not. Which (a, b) are regular values for the map f?
   (b) If the Ehresmann theorem applied over the regular locus, which fibers would it guarantee to be diffeomorphic?
   (c) Does the Ehresmann theorem apply? Why or why not?

2. (*) Let
$$X_{(a,b,c,d)} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0, \quad ax + by + cz = d\}.$$
This is the fiber over $(a, b, c, d, 0, 0)$ for a map $f\colon \mathbb{R}^7 \to \mathbb{R}^4$. Such fibers are known as *conic sections*, because $x^2 + y^2 - z^2 = 0$ is a cone, and $X_{(a,b,c,d)}$ is the intersection of this cone with some affine plane.
   (a) Which values $(a, b, c, d, 0, 0)$ are regular?
   (b) If the Ehresmann theorem applied over the regular locus, which fibers would it guarantee to be diffeomorphic?
   (c) The Ehresmann theorem does not apply, because the fibers are not compact. Show two fibers which are not diffeomorphic, which the Ehresmann theorem would suggest are diffeomorphic.

3. Consider the equation $y^2 = x^3 + ax + b$, which cuts out a subspace $X_{(a,b)} \subset \mathbb{R}^2$. Let $Y_{(a,b)}$ denote its closure in $\mathbb{RP}^2$. These spaces $Y_{(a,b)}$ are called *real elliptic curves*.
   (a) What is the homogeneous polynomial in three variables whose zero set in $\mathbb{RP}^n$ is $Y_{(a,b)}$?
   (b) Describe the set $Y_{(a,b)} \setminus X_{(a,b)}$? For which points in $Y_{(a,b)} \setminus X_{(a,b)}$ is $Y_{(a,b)}$ locally diffeomorphic to $\mathbb{R}$?
   (c) (Extra credit) Which real elliptic curves does the Ehresmann theorem say are diffeomorphic?

4. This exercise is about an embedding of $\mathbb{RP}^n$ into $\mathbb{R}^N$ for some large N. To begin, consider the map $f\colon \mathbb{RP}^2 \to \mathbb{R}^6$ which sends $[x : y : z]$ to $\frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, xy, yz, xz)$. (Hint: some parts of this problem are simplified by viewing $\mathbb{RP}^2$ as a quotient of $S^2 \subset \mathbb{R}^3$ instead of as a quotient of $\mathbb{R}^3 \setminus 0$.)
   (a) Show that f is well-defined.

(b) Show that f is injective.

(c) Show that f is proper.

(d) Show that f is an immersion. (Hint: f is an immersion if and only if the map $S^2 \to \mathbb{R}^6$ is an immersion. Why? In turn, this is an immersion if a map $\mathbb{R}^3 \to \mathbb{R}^6$ is an immersion. Why?)

(e) (Extra credit) Generalize this to give an immersion of $\mathbb{RP}^n$ into some $\mathbb{R}^N$ for each n.

5. Let f be a homogeneous polynomial of degree k in the variables $z_0, z_1, \ldots, z_n$. Homogeneous of degree k means that each monomial $z_0^{a_0} z_1^{a_1} \cdots z_n^{a_n}$ with non-zero coefficient in f satisfies $\sum a_i = k$. For example, when $n = 2$ and $k = 3$ one might have $f = z_0^3 + z_0 z_1 z_2 + z_1^2 z_2$.

   (a) Prove the *Euler formula*:
   $$\sum_{i=0}^{n} z_i \frac{df}{dz_i} = kf.$$

   (b) Show that f is never a submersion at $0 \in \mathbb{R}^{n+1}$ so long as $k \geq 2$. Find a polynomial which fails to be a submersion at some other point in $\mathbb{R}^{n+1}$ inside $f^{-1}(0)$.

   (c) Show that f is a submersion at $(z_0, \ldots, z_n) \in \mathbb{R}^{n+1} \setminus 0$ if and only if it is a submersion at $(\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{R}^{\times}$.

   (d) Let $V(f) \subset \mathbb{R}^{n+1}$ denote $f^{-1}(0)$. Remove the lines on which f is not a submersion, and one obtains an n-manifold. Deduce from the Euler formula that vectors point away from the origin are in the tangent space to $V(f)$. We call these *outward tangent vectors*.

6. We continue the terminology of the previous problem, but now discuss the vanishing locus of a homogeneous polynomial in $\mathbb{RP}^n$.

   (a) Does f descend to a well-defined function on $\mathbb{RP}^n$? Why or why not?

   (b) Show that if $(x_0, x_1, \ldots, x_n) \in V(f)$ then for any real number $\lambda \in \mathbb{R}$ one also has $(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \in V(f)$. Thus it is reasonable to talk about $X(f)$, by definition the image of $V(f) \setminus 0$ in $\mathbb{RP}^n$, that is, the set of lines on which f is identically zero.

   (c) (*) For a given homogeneous polynomial f of degree k, find a function $\mathbb{RP}^n \to \mathbb{R}$ for which the preimage of 0 is $X(f)$. (Hint: Look at the embedding exercise above for inspiration.)

7. We now request two proofs that, if f is a submersion at each point in $V(f)$ except the origin, then $X(f)$ is a manifold. (Of what dimension?)

   (a) In the first proof, use charts. (Hint: for a point $[x_0 : x_1 : \cdots : x_n] \in X(f)$, some coordinate x_i is nonzero. The set of points where $x_i \neq 0$ is diffeomorphic to $\mathbb{R}^n$ by a chart discussed earlier in this class. Find a non-homogeneous polynomial on this copy of $\mathbb{R}^n$ whose vanishing set agrees with $X(f)$ on this chart. Compare derivatives.)

(b) (*) In the second proof, think of $\mathbb{RP}^n$ as a quotient of S^n. (Hint: Consider $L(f)$, which is the intersection of $V(f)$ with the unit sphere in $\mathbb{R}^{n+1}$. Is $L(f)$ a manifold? What is the relationship between $L(f)$ and $X(f)$, and how can you use it to deduce that $X(f)$ is a manifold?)

8. This problem is about $X(f) \subset \mathbb{RP}^2$ for $f = xy - z^2$. This is the projective closure of the hyperbola $xy = 1$, although it is more accurate to say it is the closure of $\frac{x}{z}\frac{y}{z} = 1$.

   (a) Show that $X(f)$ is a smooth manifold.

   (b) Let $V(f) \subset \mathbb{R}^3$ be as above. Compute its tangent space at an arbitrary point $(x, y, z) \neq 0$.

   (c) We try to justify the fact that the tangent space at $[x : y : z]$ to $X(f)$ is the same as the quotient of the tangent space at (x, y, z) to $V(f)$, modulo the outward tangent vector $[x, y, z]$. Consider a point $[x : y : z]$ for which $z \neq 0$. Compute the tangent space on the chart with coordinates $\frac{x}{z}$, $\frac{y}{z}$.

   (d) Now suppose that $y \neq 0$. Compute the tangent space on the chart with coordinates $\frac{x}{y}$ and $\frac{z}{y}$.

   (e) What is the natural way to identify these lines with the quotient of the plane $T_{(x,y,z)}V(f)$ by the line $[x, y, z]$?

9. Let X and Z be smooth manifolds in $\mathbb{RP}^n$ cut out by collections of homogeneous polynomial equations. Prove that X is transverse to Z in $\mathbb{RP}^n$ if $\tilde{X}$ is transverse to $\tilde{Z}$ in $\mathbb{R}^{n+1} \setminus 0$, where $\tilde{X}$ is the cone over X.