Exercises

Exercises will be assigned each lecture, due the next lecture. They are annotated with the week and day they are assigned, so 1F is assigned on Friday of week 1 and is due on Monday of week 2. In addition, several more exercises will be assigned weekly, due the following Wednesday; the week 1 exercises would be listed as 1.1, 1.2, etcetera, and are due on Wednesday of week 2.

Week 6  Reading: Axler Chapter 4.

6.1 Axler 3rd ed 3.16, 4.11.

6.2 Let \( L \in \text{End}(U) \) and \( M \in \text{End}(W) \) be linear operators on their respective domains. Define a linear operator \( L \oplus M \) on \( U \times W \) by the formula

\[
(L \oplus M)(u, w) = (Lu, Mw).
\]

(0.1)

What is the relationship between the matrices of \( L \) and \( M \) (with respect to some chosen bases on \( U \) and \( W \)), and the matrix of \( (L \oplus M) \) on the corresponding basis in \( U \times W \)?

6.3 We continue the glorious exercise 5W. once more. Let \( V \) be 6-dimensional, and \( L \in \text{End}(V) \) be a linear operator on \( V \). We’ve seen that when \( L \) preserves a subspace \( U \) of dimension 4, it has a particular form - in particular, a matrix with a lot of zeroes (this is as many guaranteed zeroes as possible for a general such linear operator). Let’s do some variations on this theme, no proofs required. Each problem is independent of the others.

1. Suppose that \( L \) preserves a subspace \( U \) of dimension 3. Find a matrix for \( L \) with a lot of zeroes.

2. Suppose that \( L \) preserves a subspace \( U_1 \) of dimension 2, and another subspace \( U_2 \) of dimension 2, and that \( U_1 \cap U_2 = 0 \). Find a matrix for \( L \) with a lot of zeroes.

3. Suppose that \( L \) preserves a subspace \( U_1 \) of dimension 2, and another subspace \( U_2 \) of dimension 4, and that \( U_1 \subset U_2 \). Find a matrix for \( L \) with a lot of zeroes.

6.4 The moral of this exercise is that “the notions of subspace and quotient space are dual.” Let \( V \) be a vector space, and let \( V^* = \text{Hom}(V, \mathbb{F}) \) be its dual space. Let \( U \) be a subspace of \( V \), and let \( W \subset V^* \) be defined by

\[
W = \{ f \in V^* \mid f(u) = 0 \text{ for all } u \in U \}.
\]

(0.2)

1. Show that \( W \) is a subspace of \( V^* \).

2. Construct inverse isomorphisms between \( W \) and \( (V/U)^* \). Thus “the dual of a quotient space is a subspace of the dual.”

3. Now consider the map \( V^* \to U^* \) which sends a function \( f : V \to \mathbb{F} \) to its restriction to \( U \). Is this map linear? Surjective? What is its kernel?
4. Construct an isomorphism between $U^*$ and the quotient space $V^*/W$. Thus “the dual of a subspace is a quotient of the dual space.”

6W/F. (Because of the midterm, this problem is due on Monday of week 7. Nothing is due Friday of week 6.) Let $L: V \to V$ be a linear operator, and let $U \subset V$ be a subspace, and $V/U$ be the quotient space. Suppose that one tries to define a linear operator $\bar{L}: V/U \to V/U$ by the formula

$$\bar{L}(v + U) = L(v) + U.$$  \hfill (0.3)

1. Show that this formula makes sense and defines a linear operator if and only if $L(U) \subset U$.

2. Return to the setting of 5W, where $U$ is four-dimensional and preserved by $L$. There is a basis $S = \{v_1, \ldots, v_6\}$ where $L$ has the matrix $A$ as in (0.4). Then $T = \{L(v_5), L(v_6)\}$ is a basis for $V/U$. What is the relationship between the matrix of $L$ with respect to $S$ and the matrix of $\bar{L}$ with respect to $T$?

6M. Axler 3rd ed 4.6, which is also Axler 2nd ed 4.4. (A hint mostly to make you write it more efficiently: if $\lambda$ is a root of $p$ then write $p = qr$ as a product of two polynomials, where $\lambda$ is a root of $q$ and not a root of $r$.)

**Week 5**  
Reading: Axler 3rd ed 3.C-E.


5.2 This problem is a follow-up to 5W. Assume that $L$ preserves $U$. Suppose that $L$ is invertible. Prove that the restriction of $L$ to $U$ gives an invertible map $U \to U$.

5F. Axler 3rd ed 3.E.1 and 3.E.13

5W. Let $V$ be an 6-dimensional vector space, and $L: V \to V$ a linear operator.

1. Let $S$ be a basis of $V$, and suppose that when $L$ is written as a matrix $A$ with respect to $S$, then $A$ has the following form

$$
\begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
\end{pmatrix}.
$$  \hfill (0.4)

That is, the $2 \times 4$ box in the lower left corner of $A$ is zero, and the remaining entries can be arbitrary (as indicated by the $*$). Prove that there is a 4-dimensional subspace $U$ of $V$ for which $L(U) \subset U$. (We say that $U$ is preserved by $L$.)

2. Conversely, suppose that $U$ is a 4-dimensional subspace of $V$, and $L(U) \subset U$. Prove that there is some basis $S$ for $V$ such that the matrix $A$ of $L$ with respect to $S$ has the above form.
5M. Let $V$ be the vector space spanned by the functions $S = \{e^{3t}, te^{3t}, t^2 e^{3t}, t^3 e^{3t}\}$. The derivative preserves $V$; write it as a matrix with respect to the basis $S$. Now let $f = t^3 e^{3t}$ and let $T = \{f, f', f'', f'''\}$. This is also a basis for $V$. Write the derivative as a matrix with respect to the basis $T$. (You can try to do this directly or try to use a change of basis. It is worth trying to do both. You can use wolfram alpha to compute inverses of matrices if you want.)

**Week 4** Reading: Axler Chapter 3 (2nd ed), or 3.A-D (3rd ed).


4.2 Let $L: V \to W$ be a linear transformation, $S = \{v_1, \ldots, v_n\}$ a subset of $V$, and $T = \{L(v_1), \ldots, L(v_n)\}$ the image of $S$ in $W$.

1. Prove that if $T$ is linearly independent then $S$ is linearly independent.
2. Give an example where $S$ is linearly independent but $T$ is not.
3. Prove that if $L$ is injective and $S$ is linearly independent then $T$ is linearly independent.

4.3

1. Let $V$ be a vector space, and $\{f_1, f_2, \ldots, f_n\} \subset V^*$ be linear functionals on $V$. Suppose we can find a vector $v_1 \in V$ such that $f_1(v) \neq 0$ but $f_2(v) = f_3(v) = \ldots = f_n(v) = 0$. Similarly, suppose that for all $1 \leq i \leq n$ we can find $v_i \in V$ such that $f_i(v_i) \neq 0$ and $f_j(v_i) = 0$ for all $j \neq i$. Prove that $\{f_1, \ldots, f_n\}$ is linearly independent in $V^*$. Prove also that the vectors $\{v_1, \ldots, v_n\}$ were linearly independent in $V$.

2. Let $V = \mathcal{P}_{\leq 2}$. Recall that $ev_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $ev_\lambda(p) = p(\lambda)$. Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in $\mathbb{R}$. Prove that $\{ev_{\lambda_1}, ev_{\lambda_2}, ev_{\lambda_3}\}$ forms a basis for $V^*$. (Hint: use the previous part of the exercise. How do you find a polynomial $p$ such that $ev_{\lambda_1}(p) \neq 0$ but $ev_{\lambda_2}(p) = ev_{\lambda_3}(p) = 0$?)

4.4 Recall that $\mathbb{R}^N = \{(x_1, x_2, \ldots)\}$ is the vector space of all sequences of real numbers, and $\mathbb{R}^{\oplus \infty}$ is the subspace of sequences which are eventually zero.

For each vector $v = (x_1, x_2, \ldots) \in \mathbb{R}^N$, we will define a linear transformation $f_v: \mathbb{R}^{\oplus \infty} \to \mathbb{R}$ as follows.

$$f_v(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} x_i y_i. \quad (0.5)$$

1. Could we use the same formula to define a linear transformation $f_v: \mathbb{R}^N \to \mathbb{R}$? Why or why not?

2. Prove that $f_{v_1 + v_2} = f_{v_1} + f_{v_2}$. (After showing a similar thing for rescaling, one could say: there is a linear transformation $\mathbb{R}^N \to (\mathbb{R}^{\oplus \infty})^*$ sending $v \mapsto f_v$)

3. Prove that every linear transformation $\mathbb{R}^{\oplus \infty} \to \mathbb{R}$ has the form $f_v$ for some $v \in \mathbb{R}^N$. (In other words, there is an isomorphism $\mathbb{R}^{\oplus \infty}^* \cong \mathbb{R}^N$.)
Let $U$ be a subspace of a finite-dimensional vector space $V$, with $U \neq V$. Let $W$ be another vector space.

1. Suppose one has a linear transformation $L : V \to W$. Restricting the domain of $L$ to $U$, we get a function $L|_U : U \to W$. Show that $L|_U$ is a linear transformation. Better yet, don’t write anything, but just convince yourself this is trivial!

2. Suppose one has a linear transformation $M : U \to W$. The extension by zero of $M$ is the function $M^0 : V \to W$ for which

$$M^0(v) = \begin{cases} M(v) & \text{if } v \in U, \\ 0 & \text{if } v \notin U. \end{cases} \quad (0.6)$$

Prove that $M^0$ is not a linear transformation when $M$ is nonzero.

3. Suppose one has a linear transformation $M : U \to W$. Now choose a complement $U'$ to $U$, i.e. a subspace $U' \subset V$ such that $U \oplus U' = V$. Show that there is a unique linear transformation $M' : V \to W$ such that

$$M'(v) = \begin{cases} M(v) & \text{if } v \in U, \\ 0 & \text{if } v \in U'. \end{cases} \quad (0.7)$$

We call $M'$ an extension of $M$ to $V$.

4. Give an example where two different complements give rise to two different extensions of a linear transformation.

4F. Axler 3rd edition 3.D.9. Written out: If $S, T : V \to V$ are linear maps, then $S \circ T$ is invertible if and only if both $S$ and $T$ are invertible.

4W. Axler 3rd edition 3.B.4. Written out: show that

$$\{T \in L(\mathbb{R}^5, \mathbb{R}^4) \mid \dim \ker(T) > 2\}$$

is not a subspace of $L(\mathbb{R}^5, \mathbb{R}^4)$.

4M. Let $V = P_{\leq 2}$. Recall that $ev_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $ev_\lambda(p) = p(\lambda)$. I claim that the set $\{ev_{-1}, ev_0, ev_1, ev_2\}$ is linearly dependent in $V^*$. Find a nontrivial linear combination for 0.

**Week 3** Reading: Axler Chapter 2 (the rest) and start of Chapter 3. I have the idea that, even though there are more problems this week, they feel shorter to me. Let me know if I’m wrong!


3W. Two problems. I assigned this late, so feel free to hand in on Monday instead of Friday. Axler 3rd edition 2.B.7 and 2.A.17.
3M. Prove that $\mathbb{R}^N$ is infinite-dimensional.

**Week 2**  
Reading: Axler Chapter 1 (sums and direct sums), Chapter 2 (spans, linear independence, bases)


2.2 Let $U$ be a subspace of $V$. For any vector $v \in V$, define $U + v$ to be the subset

$$U + v = \{ z \in V \mid z = u + v \text{ for some } u \in U \}. \tag{0.8}$$

(Note: Now that we’ve discussed quotient spaces, we call this set $v + U$.)

1. For which $v \in V$ is $U + v$ a subspace?

2. Let $v, v' \in V$. Prove that the sets $U + v$ and $U + v'$ are either equal or disjoint. (Hint: first prove that if $U + v \cap U + v' \neq \emptyset$ then $v' \in U + v$.)

3. Think about the connection between this exercise and the notion of parallel lines, or parallel planes in 3D. Say something meaningful and brief, both about the definition of $U + v$, and about the result you just proved.


2W. Axler 3rd edition exercise 1.C.24. (Hint: ONLY READ AFTER THINKING AND BEING STUMPED. Given an arbitrary function $f(x)$, consider the function $g(x) = \frac{f(x) + f(-x)}{2}$.)

2M. Do Axler (3rd edition) exercise 1.C.20 and 1.C.21. Here they are written out in case you only have 2nd ed. Both exercises give you a subspace $U$ and ask for $W$ such that $U \oplus W$ is the whole space.

- $U = \{(x, x, y, y)\} \subset \mathbb{F}^4$,
- $U = \{(x, y, x + y, x - y, 2x)\} \subset \mathbb{F}^5$.

**Week 1**  
Reading: Axler Chapter 1

1.1 Do Axler exercises 1.2, 1.4, 1.5, 1.6, 1.9. You only need rigorous proofs for 1.4 and 1.9.

1.2 Consider $\mathbb{Q}(\sqrt{2})$ defined abstractly as $\{(a, b) \mid a, b \in \mathbb{Q}\}$ with addition and multiplication coming from the formulas in exercise 1W. For $z = (a, b)$ in $\mathbb{Q}(\sqrt{2})$ let us define $N(z) \in \mathbb{R}$ by the formula $N(z) = a^2 - 2b^2$. Confirm that $N(z \cdot w) = N(z) \cdot N(w)$ for any $z, w \in \mathbb{Q}(\sqrt{2})$.

Aside: For a complex number $z = a + bi$ we can define $N(z) = a^2 + b^2 = |z|^2$. The exercise above is essentially the same computation one would use to show that $N(z \cdot w) = N(z) \cdot N(w)$ for the complex numbers, or that $|z \cdot w| = |z| \cdot |w|$.

1F. Do Axler exercise 1.8.

1W. Consider the set $X = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$. Equip $X$ with an addition and multiplication structure just like we did for $\mathbb{Q}(\sqrt{2})$ in class, namely
• \((a, b) + (c, d) = (a + c, b + d)\), and

• \((a, b) \cdot (c, d) = (ac + 2bd, ad + bc)\).

Sadly, \(X\) is not a field, and we will NOT denote it by \(\mathbb{R}(\sqrt{2})\).

• Which properties of a field hold for \(X\)? (You need not provide the proof.)

• Which properties of a field fail? (Give an example.)

1M. Prove that the set of all twice-differential functions \(f : \mathbb{R} \to \mathbb{R}\) which satisfy

\[ f'' - 26f' + 3f = 0 \]

is closed under addition and rescaling.

Note: This is not a hard proof, but I want to see your style of proof-writing. Don’t be too verbose or too sketchy please. Feel free to use basic facts from calculus, but you should call them out.

Proof-writing hint: When you want to avoid ambiguity and convoluted sentences, name things. For instance, I didn’t name the set above, but when you write up the proof, an excellent first sentence is “Let \(Y\) denote the set of all twice-differentiable ...” It is a lot easier to refer to \(Y\) than “the set in question” or “it” or whatever else one might say. Similarly, the equation \(f'' - 26f' + 3f = 0\) might itself be named something, like \((\ast)\), which enables you to say “Suppose that \(f\) satisfies \((\ast)\). Then ...”