Exercises will be assigned each lecture, due the next lecture. They are annotated with the week and day they are assigned, so **1F** is assigned on Friday of week 1 and is due on Monday of week 2. In addition, several more exercises will be assigned weekly, due the following Wednesday; the week 1 exercises would be listed as **1.1, 1.2**, etcetera, and are due on Wednesday of week 2.

**Week 5**  Reading: Axler 3rd ed 3.C-E.


4.2 This problem is a follow-up to **4W**. Assume that \( L \) preserves \( U \). Suppose that \( L \) is invertible. Prove that the restriction of \( L \) to \( U \) gives an invertible map \( U \rightarrow U \).

**4F.** Axler 3rd ed 3.E.1 and 3.E.13

**4W.** Let \( V \) be an 6-dimensional vector space, and \( L : V \rightarrow V \) a linear operator.

1. Let \( S \) be a basis of \( V \), and suppose that when \( L \) is written as a matrix \( A \) with respect to \( S \), then \( A \) has the following form

\[
\begin{pmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * & * 
\end{pmatrix}
\]

That is, the \( 2 \times 4 \) box in the lower left corner of \( A \) is zero, and the remaining entries can be arbitrary (as indicated by the \(*\)). Prove that there is a 4-dimensional subspace \( U \) of \( V \) for which \( L(U) \subset U \). (We say that \( U \) is preserved by \( L \)).

2. Conversely, suppose that \( U \) is a 4-dimensional subspace of \( V \), and \( L(U) \subset U \). Prove that there is some basis \( S \) for \( V \) such that the matrix \( A \) of \( L \) with respect to \( S \) has the above form.

4M. Let \( V \) be the vector space spanned by the functions \( S = \{e^{3t}, te^{3t}, t^2e^{3t}, t^3e^{3t}\} \). The derivative preserves \( V \); write it as a matrix with respect to the basis \( S \). Now let \( f = t^3e^{3t} \) and let \( T = \{f, f', f'', f'''\} \). This is also a basis for \( V \). Write the derivative as a matrix with respect to the basis \( T \). (You can try to do this directly or try to use a change of basis. It is worth trying to do both. You can use wolfram alpha to compute inverses of matrices if you want.)

**Week 4**  Reading: Axler Chapter 3 (2nd ed), or 3.A-D (3rd ed).


4.2 Let \( L : V \rightarrow W \) be a linear transformation, \( S = \{v_1, \ldots, v_n\} \) a subset of \( V \), and \( T = \{L(v_1), \ldots, L(v_n)\} \) the image of \( S \) in \( W \).
1. Prove that if $T$ is linearly independent then $S$ is linearly independent.

2. Give an example where $S$ is linearly independent but $T$ is not.

3. Prove that if $L$ is injective and $S$ is linearly independent then $T$ is linearly independent.

4.3

1. Let $V$ be a vector space, and $\{f_1, f_2, \ldots, f_n\} \subset V^*$ be linear functionals on $V$. Suppose we can find a vector $v_1 \in V$ such that $f_1(v) \neq 0$ but $f_2(v) = f_3(v) = \ldots = f_n(v) = 0$. Similarly, suppose that for all $1 \leq i \leq n$ we can find $v_i \in V$ such that $f_i(v_i) \neq 0$ and $f_j(v_i) = 0$ for all $j \neq i$. Prove that $\{f_1, \ldots, f_n\}$ is linearly independent in $V^*$. Prove also that the vectors $\{v_1, \ldots, v_n\}$ were linearly independent in $V$.

2. Let $V = \mathcal{P}_{\leq 2}$. Recall that $ev_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $ev_\lambda(p) = p(\lambda)$. Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in $\mathbb{R}$. Prove that $\{ev_{\lambda_1}, ev_{\lambda_2}, ev_{\lambda_3}\}$ forms a basis for $V^*$. (Hint: use the previous part of the exercise. How do you find a polynomial $p$ such that $ev_{\lambda_1}(p) \neq 0$ but $ev_{\lambda_2}(p) = ev_{\lambda_3}(p) = 0$?)

4.4 Recall that $\mathbb{R}^N = \{(x_1, x_2, \ldots)\}$ is the vector space of all sequences of real numbers, and $\mathbb{R}^{\oplus \infty}$ is the subspace of sequences which are eventually zero.

For each vector $v = (x_1, x_2, \ldots) \in \mathbb{R}^N$, we will define a linear transformation $f_v : \mathbb{R}^{\oplus \infty} \to \mathbb{R}$ as follows.

$$f_v(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} x_i y_i.$$ 

1. Could we use the same formula to define a linear transformation $f_v : \mathbb{R}^N \to \mathbb{R}$? Why or why not?

2. Prove that $f_{v_1 + v_2} = f_{v_1} + f_{v_2}$. (After showing a similar thing for rescaling, one could say: there is a linear transformation $\mathbb{R}^N \to (\mathbb{R}^{\oplus \infty})^*$ sending $v \mapsto f_v$.)

3. Prove that every linear transformation $\mathbb{R}^{\oplus \infty} \to \mathbb{R}$ has the form $f_v$ for some $v \in \mathbb{R}^N$. (In other words, there is an isomorphism $(\mathbb{R}^{\oplus \infty})^* \cong \mathbb{R}^N$.)

4.5 Let $U$ be a subspace of a finite-dimensional vector space $V$, with $U \neq V$. Let $W$ be another vector space.

1. Suppose one has a linear transformation $L : V \to W$. Restricting the domain of $L$ to $U$, we get a function $L|_U : U \to W$. Show that $L|_U$ is a linear transformation. Better yet, don’t write anything, but just convince yourself this is trivial!

2. Suppose one has a linear transformation $M : U \to W$. The extension by zero of $M$ is the function $M^0 : V \to W$ for which

$$M^0(v) = \begin{cases} M(v) & \text{if } v \in U, \\ 0 & \text{if } v \notin W. \end{cases}$$

Prove that $M^0$ is not a linear transformation when $M$ is nonzero.
3. Suppose one has a linear transformation $M: U \rightarrow W$. Now choose a complement $U'$ to $U$, i.e. a subspace $U' \subset V$ such that $U \oplus U' = V$. Show that there is a unique linear transformation $M': V \rightarrow W$ such that

$$
M'(v) = \begin{cases} 
M(v) & \text{if } v \in U, \\
0 & \text{if } v \in U'.
\end{cases}
$$

We call $M'$ an extension of $M$ to $V$.

4. Give an example where two different complements give rise to two different extensions of a linear transformation.

4F. Axler 3rd edition 3.D.9. Written out: If $S, T: V \rightarrow V$ are linear maps, then $S \circ T$ is invertible if and only if both $S$ and $T$ are invertible.

4W. Axler 3rd edition 3.B.4. Written out: show that

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) \mid \dim \ker(T) > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

4M. Let $V = \mathcal{P}_{\leq 2}$. Recall that $ev_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $ev_\lambda(p) = p(\lambda)$. I claim that the set $\{ev_{-1}, ev_0, ev_1, ev_2\}$ is linearly dependent in $V^*$. Find a nontrivial linear combination for 0.

**Week 3**  
Reading: Axler Chapter 2 (the rest) and start of Chapter 3. I have the idea that, even though there are more problems this week, they feel shorter to me. Let me know if I’m wrong!


3W. Two problems. I assigned this late, so feel free to hand in on Monday instead of Friday. Axler 3rd edition 2.B.7 and 2.A.17.

3M. Prove that $\mathbb{R}^N$ is infinite-dimensional.

**Week 2**  
Reading: Axler Chapter 1 (sums and direct sums), Chapter 2 (spans, linear independence, bases)


2.2 Let $U$ be a subspace of $V$. For any vector $v \in V$, define $U_{+v}$ to be the subset

$$U_{+v} = \{z \in V \mid z = u + v \text{ for some } u \in U\}.$$

1. For which $v \in V$ is $U_{+v}$ a subspace?
2. Let \( v, v' \in V \). Prove that the sets \( U + v \) and \( U + v' \) are either equal or disjoint. (Hint: first prove that if \( U + v \cap U + v' \neq \emptyset \) then \( v' \in U + v \).

3. Think about the connection between this exercise and the notion of parallel lines, or parallel planes in 3D. Say something meaningful and brief, both about the definition of \( U + v \), and about the result you just proved.


2W. Axler 3rd edition exercise 1.C.24. (Hint: ONLY READ AFTER THINKING AND BEING STUMPED. Given an arbitrary function \( f(x) \), consider the function \( g(x) = \frac{f(x) + f(-x)}{2} \).

2M. Do Axler (3rd edition) exercise 1.C.20 and 1.C.21. Here they are written out in case you only have 2nd ed. Both exercises give you a subspace \( U \) and ask for \( W \) such that \( U \oplus W \) is the whole space.

- \( U = \{ (x, x, y, y) \} \subset \mathbb{F}^4 \),
- \( U = \{ (x, y, x + y, x - y, 2x) \} \subset \mathbb{F}^5 \).

Week 1  
Reading: Axler Chapter 1

1.1 Do Axler exercises 1.2, 1.4, 1.5, 1.6, 1.9. You only need rigorous proofs for 1.4 and 1.9.

1.2 Consider \( \mathbb{Q}(\sqrt{2}) \) defined abstractly as \( \{(a, b) \mid a, b \in \mathbb{Q}\} \) with addition and multiplication coming from the formulas in exercise 1W. For \( z = (a, b) \) in \( \mathbb{Q}(\sqrt{2}) \) let us define \( N(z) \in \mathbb{R} \) by the formula \( N(z) = a^2 - 2b^2 \). Confirm that \( N(z \cdot w) = N(z) \cdot N(w) \) for any \( z, w \in \mathbb{Q}(\sqrt{2}) \).

Aside: For a complex number \( z = a + bi \) we can define \( N(z) = a^2 + b^2 = |z|^2 \). The exercise above is essentially the same computation one would use to show that \( N(z \cdot w) = N(z) \cdot N(w) \) for the complex numbers, or that \( |z \cdot w| = |z| \cdot |w| \).

1F. Do Axler exercise 1.8.

1W. Consider the set \( X = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\} \). Equip \( X \) with an addition and multiplication structure just like we did for \( \mathbb{Q}(\sqrt{2}) \) in class, namely

- \( (a, b) + (c, d) = (a + c, b + d) \), and
- \( (a, b) \cdot (c, d) = (ac + 2bd, ad + bc) \).

Sadly, \( X \) is not a field, and we will NOT denote it by \( \mathbb{R}(\sqrt{2}) \).

- Which properties of a field hold for \( X \)? (You need not provide the proof.)
- Which properties of a field fail? (Give an example.)

1M. Prove that the set of all twice-differential functions \( f : \mathbb{R} \to \mathbb{R} \) which satisfy

\[
 f'' - 26f' + 3f = 0
\]
is closed under addition and rescaling.

Note: This is not a hard proof, but I want to see your style of proof-writing. Don’t be too verbose or too sketchy please. Feel free to use basic facts from calculus, but you should call them out.

Proof-writing hint: When you want to avoid ambiguity and convoluted sentences, name things. For instance, I didn’t name the set above, but when you write up the proof, an excellent first sentence is “Let $Y$ denote the set of all twice-differentiable ...” It is a lot easier to refer to $Y$ than “the set in question” or “it” or whatever else one might say. Similarly, the equation $f'' - 26f' + 3f = 0$ might itself be named something, like $(\star)$, which enables you to say “Suppose that $f$ satisfies $(\star)$. Then ...”