Math 441 (Linear Algebra), Spring 2019
Exercises

Exercises will be assigned each lecture, due the next lecture. They are annotated with the week and day they are assigned, so **1F.** is assigned on Friday of week 1 and is due on Monday of week 2. In addition, several more exercises will be assigned weekly, due the following Wednesday; the week 1 exercises would be listed as **1.1**, **1.2**, etcetera, and are due on Wednesday of week 2.

Week 10 Reading: Axler Chapter 6.

10W.1 Two problems today. The first: now its not optional anymore. In the setup of **10M**, apply Gram-Schmidt to $\{1, x, x^2, x^3\}$ to find an orthonormal basis $\{v_0, v_1, v_2, v_3\}$ where $\deg v_i = i$.

10W.2 Find the orthogonal complement to the span of $\{v_1, v_2\}$ in $\mathbb{R}^4$, where

$$v_1 = (1, 1, 1, 1), \quad v_2 = (1, 1, 3, 5).$$

10M. This problem is about the inner product space $\mathcal{P}_{\mathbb{R}} = \mathbb{R}[x]$, with inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

1. Compute $\langle x^k, x^l \rangle$ for all $k, l \geq 0$.
2. Write down explicitly what the Cauchy-Schwartz inequality tells you about $\langle x, x \rangle$, and verify it.
3. Clearly $\langle 1, 1 \rangle = 1$. Find a vector of the form $v_1 = a + bx$ such that $\langle 1, v_1 \rangle = 0$ and $\langle v_1, v_1 \rangle = 1$. (Please doublecheck your answer!)
4. Letting $v_0 = 1$ the previous exercise shows that $\{v_0, v_1\}$ is an orthonormal basis for $\mathcal{P}_{\leq 1}$. Use inner products to efficiently compute the coordinates of x with respect to this basis (i.e. verify 6.30 from Axler 3rd ed).
5. (Optional) Find a vector of the form $v_2 = a + bx + cx^2$ which extends the above to an orthonormal basis $\{v_0, v_1, v_2\}$ for $\mathcal{P}_{\leq 2}$. Compute the coordinates of x^2 with respect to this basis.

Week 9 Reading: Axler Chapter 8.D (Jordan Normal Form), Chapter 6.A (inner products, norms)

9F. Axler 6.A.4 and 6.A.8.

9W. Axler 3rd ed 8.D.4 and 8.D.5.

9.1 Axler 6.A.1, 6.A.2, 6.A.5, 6.A.6, 6.A.7.

9.2 Let D be the derivative, acting on the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Let p be a polynomial of degree n, and let $V = \operatorname{Ker} p(D)$, the space of solutions to this "differential equation." The fundamental theorem of differential equations, stated in this special case, says that $\dim V = n$. For example, the space $\{f \mid f'' - 26f' + 3f = 0\}$ is the kernel of $p(D)$ where $p(x) = x^2 - 26x + 3$, so it has dimension 2.

Theorem 0.1. Let p be a polynomial of degree n, and D be the derivative operator on the space of infinitely differentiable functions. Then the kernel of $p(D)$ has dimension exactly n.

For this problem you are welcome to use the following fact: if an operator on a finite-dimensional real vector space has only real eigenvalues, then the Jordan normal form theorem still applies.

1. Recall from a previous exercise that D preserves V, since it commutes with $p(D)$. Prove that the minimal polynomial of D acting on V is p.

2. Prove that D acting on V can not be represented by a matrix in Jordan normal form with more than one block associated to a given eigenvalue λ.

3. Find all solutions to the differential equation $p(D)(f) = 0$, where $p(x) = (x-5)^4(x+2)(x-3)^2$.

(Hint 1: Theorem 0.1 applies to every polynomial, not just the chosen one p. So the dimension of the kernel of $(D - 5I)^3$ is three, etcetera. Hint 2: You already know how to find functions on which the derivative acts like a Jordan block.)

9.Extra Credit This exercise is a first introduction to Lagrange interpolation. Let $\lambda_1, \ldots, \lambda_d$ be distinct scalars, and let $p(x)$ be the multiplicity-free polynomial

$$q(x) = (x - \lambda_1) \cdots (x - \lambda_d).$$

Let L be an operator on an arbitrary vector space V (any dimension, any field) and suppose that $q(L) = 0$. Remember that if L preserves a subspace of V (e.g. an eigenspace) then any polynomial in L preserves that subspace.

1. Let $1 \leq i, j \leq d$. Consider the polynomial

$$c_{ij}(x) = \frac{x - \lambda_j}{\lambda_i - \lambda_j}.$$

How does the operator $c_{ij}(L)$ act on the λ_i-eigenspace of L? How does it act on the λ_j-eigenspace of L? How does it act on the λ_k-eigenspace for some $k \neq i, j$?

2. Let $1 \leq i, j, k \leq d$. How does the operator $c_{ij}(L)c_{ik}(L)$ act on the λ_i-eigenspace of L? What about λ_j, λ_k, and λ_l for $l \neq i, j, k$?

3. Let

$$p_i(x) = \prod_{j \neq i} c_{ij}(x).$$

How does $p_i(L)$ act on the various eigenspaces of L?

4. Prove that the image of p_i is contained in the λ_i-eigenspace of L.
5. Prove that $p_i(L)^2 = p_i(L)$.
6. Prove that $p_i(L)p_j(L) = 0$ for all $i \neq j$.
7. Consider the special case where L is represented by the matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

   with $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 7$. Write down the matrices for $p_1(L)$, $p_2(L)$, and $p_3(L)$? What is $(p_1 + p_2 + p_3)(L)$? What is $(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3)(L)$?
8. Prove that $c_{ij}(x) + c_{ji}(x) = 1$.
9. Prove in general that $(p_1 + p_2 + \ldots + p_d)(L) = I$. (This is the hardest thing by far!)
10. Deduce that V is a direct sum of its λ_i-eigenspaces.

Week 8 Reading: Axler Chapter 8. You really need to read this week, several proofs and ideas will not be covered in class. We won't get to Jordan normal form.

8.1 Axler 3rd ed 8.A.4, 8.A.6, 8.A.8, 8.C.1, 8.C.4, 8.C.18 but just the case $n = 4$ for sanity, 8.C.20.

8.2 In this problem T will be a linear operator on a finite-dimensional complex vector space V of dimension n.

1. Find an example where $V \neq \operatorname{Ker} T \oplus \operatorname{Im} T$.
2. Prove that $V = \operatorname{Ker} T^n \oplus \operatorname{Im} T^n$.

8F. Axler 8.C.8 and 8.C.11. (Hint for 8.C.11: Use 8.C.8, and "solve for the constant term.")

8W. Axler 3rd ed 8.A.5.

8M. Let $S = \{v_1, v_2, \ldots, v_m\}$ be a list of vectors in V, and L be an operator on V. Suppose that v_i is a generalized eigenvector with eigenvalue λ_i, and $\lambda_i \neq \lambda_j$ if $i \neq j$. Prove that S is linearly independent. (This was the same thing I proved in class, except now they are generalized eigenvectors.) (Hint: you can use the same proof. Choose a linear combination which is zero of minimal length, and make it shorter somehow. In class I made it shorter by applying $L - \lambda_k I$ for some k. What operator should you apply this time? Addendum - this is harder than I anticipated with this proof, though it can still be done. There is a much slicker proof in 8.A.13 of Axler.)

Week 7 Reading: Axler Chapter 5.

7.1 Axler 3rd ed 5.A.6, 5.A.7, 5.A.10, 5.A.15, 5.A.19, 5.A.20, 5.A.25, 5.A.26, 5.B.9 (some of these are very short).

7F. Axler 3rd ed 5.B.4. (Hint: do it with eigenspaces.)

7W.

1. Let L be a linear operator and p a polynomial. Suppose v is an eigenvector for L with eigenvalue λ. How does the operator $p(L)$ act on v?

2. Prove that L preserves the kernel of $p(L)$. If L has an eigenvector v inside the kernel of $p(L)$, what can you say about the eigenvalue λ?

3. Let V be the space of solutions to the differential equation $f'''' - 3f''' - 3f'' + 11f' - 6f = 0$. For which λ is the function $e^{\lambda x}$ in V? (Hint: rephrase V as the kernel of a polynomial in some operator. Rephrase $e^{\lambda x}$ as an eigenvector.)

7M. Let V be a finite-dimensional complex vector space. Suppose that $S, T \colon V \to V$ are linear operators on V that commute: $ST = TS$.

1. Find an example where there is a subspace U that S preserves but T does not.

2. However, prove in general that T preserves each eigenspace of S.

3. If V is nonzero, deduce that there exists a nonzero vector v which is an eigenvector for both S and T.

Week 6 Reading: Axler Chapter 4.

6.1 Axler 3rd ed 3.E.16, 4.11.

6.2 Let $L \in \mathrm{End}(U)$ and $M \in \mathrm{End}(W)$ be linear operators on their respective domains. Define a linear operator $L \oplus M$ on $U \times W$ by the formula

$$(L \oplus M)(u, w) = (Lu, Mw). \tag{0.1}$$

What is the relationship between the matrices of L and M (with respect to some chosen bases on U and W), and the matrix of $(L \oplus M)$ on the corresponding basis in $U \times W$?

6.3 We continue the glorious exercise **5W.** once more. Let V be 6-dimensional, and $L \in \mathrm{End}(V)$ be a linear operator on V. We've seen that when L preserves a subspace U of dimension 4, it has a particular form - in particular, a matrix with a lot of zeroes (this is as many guaranteed zeroes as possible for a general such linear operator). Let's do some variations on this theme, no proofs required. Each problem is independent of the others.

1. Suppose that L preserves a subspace U of dimension 3. Find a matrix for L with a lot of zeroes.

2. Suppose that L preserves a subspace U_1 of dimension 2, and another subspace U_2 of dimension 2, and that $U_1 \cap U_2 = 0$. Find a matrix for L with a lot of zeroes.

3. Suppose that L preserves a subspace U_1 of dimension 2, and another subspace U_2 of dimension 4, and that $U_1 \subset U_2$. Find a matrix for L with a lot of zeroes.

6.4 The moral of this exercise is that "the notions of subspace and quotient space are dual." Let V be a vector space, and let $V^* = \mathrm{Hom}(V, \mathbb{F})$ be its dual space. Let U be a subspace of V, and let $W \subset V^*$ be defined by

$$W = \{f \in V^* \mid f(u) = 0 \text{ for all } u \in U\}. \tag{0.2}$$

1. Show that W is a subspace of V^*.
2. Construct inverse isomorphisms between W and $(V/U)^*$. Thus "the dual of a quotient space is a subspace of the dual."
3. Now consider the map $V^* \to U^*$ which sends a function $f\colon V \to \mathbb{F}$ to its restriction to U. Is this map linear? Surjective? What is its kernel?
4. Construct an isomorphism between U^* and the quotient space V^*/W. Thus "the dual of a subspace is a quotient of the dual space."

6W/F. (Because of the midterm, this problem is due on Monday of week 7. Nothing is due friday of week 6.) Let $L\colon V \to V$ be a linear operator, and let $U \subset V$ be a subspace, and V/U be the quotient space. Suppose that one tries to define a linear operator $\bar{L}\colon V/U \to V/U$ by the formula

$$\bar{L}(v + U) = L(v) + U. \tag{0.3}$$

1. Show that this formula makes sense and defines a linear operator if and only if $L(U) \subset U$.
2. Return to the setting of **5W.**, where U is four-dimensional and preserved by L. There is a basis $S = \{v_1, \ldots, v_6\}$ where L has the matrix A as in (0.4). Then $T = \{L(v_5), L(v_6)\}$ is a basis for V/U. What is the relationship between the matrix of L with respect to S and the matrix of $\bar{L}$ with respect to T?

6M. Axler 3rd ed 4.6, which is also Axler 2nd ed 4.4. (A hint mostly to make you write it more efficiently: if λ is a root of p then write $p = qr$ as a product of two polynomials, where λ is a root of q and not a root of r.)

Week 5 Reading: Axler 3rd ed 3.C-E.

5.1 Axler 3rd ed 3.C.4, 3.C.5, 3.C.6, 3.D.3, 3.D.7, 3.E.7.

5.2 This problem is a follow-up to **5W**. Assume that L preserves U. Suppose that L is invertible. Prove that the restriction of L to U gives an invertible map $U \to U$.

5F. Axler 3rd ed 3.E.1 and 3.E.13

5W. Let V be an 6-dimensional vector space, and $L\colon V \to V$ a linear operator.

1. Let S be a basis of V, and suppose that when L is written as a matrix A with respect to S, then A has the following form

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}. \tag{0.4}$$

   That is, the 2×4 box in the lower left corner of A is zero, and the remaining entries can be arbitrary (as indicated by the $*$). Prove that there is a 4-dimensional subspace U of V for which $L(U) \subset U$. (We say that U is *preserved* by L.)

2. Conversely, suppose that U is a 4-dimensional subspace of V, and $L(U) \subset U$. Prove that there is some basis S for V such that the matrix A of L with respect to S has the above form.

5M. Let V be the vector space spanned by the functions $S = \{e^{3t}, te^{3t}, t^2e^{3t}, t^3e^{3t}\}$. The derivative preserves V; write it as a matrix with respect to the basis S. Now let $f = t^3e^{3t}$ and let $T = \{f, f', f'', f'''\}$. This is also a basis for V. Write the derivative as a matrix with respect to the basis T. (You can try to do this directly or try to use a change of basis. It is worth trying to do both. You can use wolfram alpha to compute inverses of matrices if you want.)

Week 4 Reading: Axler Chapter 3 (2nd ed), or 3.A-D (3rd ed).

4.1 Axler 3rd ed, 3.B.29, 3.B.22 (Hint: use preimages to understand $\mathrm{Ker}(ST)$), 3.B.20, 3.B.21.

4.2 Let $L\colon V \to W$ be a linear transformation, $S = \{v_1, \ldots, v_n\}$ a subset of V, and $T = \{L(v_1), \ldots, L(v_n)\}$ the image of S in W.

1. Prove that if T is linearly independent then S is linearly independent.

2. Give an example where S is linearly independent but T is not.

3. Prove that if L is injective and S is linearly independent then T is linearly independent.

4.3

1. Let V be a vector space, and $\{f_1, f_2, \ldots, f_n\} \subset V^*$ be linear functionals on V. Suppose we can find a vector $v_1 \in V$ such that $f_1(v) \neq 0$ but $f_2(v) = f_3(v) = \ldots = f_n(v) = 0$. Similarly, suppose that for all $1 \leq i \leq n$ we can find $v_i \in V$ such that $f_i(v_i) \neq 0$ and $f_j(v_i) = 0$ for all $j \neq i$. Prove that $\{f_1, \ldots, f_n\}$ is linearly independent in V^*. Prove also that the vectors $\{v_1, \ldots, v_n\}$ were linearly independent in V.

2. Let $V = \mathcal{P}_{\leq 2}$. Recall that $\mathrm{ev}_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $\mathrm{ev}_\lambda(p) = p(\lambda)$. Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in $\mathbb{R}$. Prove that $\{\mathrm{ev}_{\lambda_1}, \mathrm{ev}_{\lambda_2}, \mathrm{ev}_{\lambda_3}\}$ forms a basis for V^*. (Hint: use the previous part of the exercise. How do you find a polynomial p such that $\mathrm{ev}_{\lambda_1}(p) \neq 0$ but $\mathrm{ev}_{\lambda_2}(p) = \mathrm{ev}_{\lambda_3}(p) = 0$?).

4.4 Recall that $R^{\mathbb{N}} = \{(x_1, x_2, \ldots)\}$ is the vector space of all sequences of real numbers, and $\mathbb{R}^{\oplus\infty}$ is the subspace of sequences which are eventually zero.

For each vector $v = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, we will define a linear transformation $f_v \colon \mathbb{R}^{\oplus\infty} \to \mathbb{R}$ as follows.

$$f_v(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} x_i y_i. \tag{0.5}$$

1. Could we use the same formula to define a linear transformation $f_v \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$? Why or why not?

2. Prove that $f_{v_1+v_2} = f_{v_1} + f_{v_2}$. (After showing a similar thing for rescaling, one could say: there is a linear transformation $\mathbb{R}^{\mathbb{N}} \to (\mathbb{R}^{\oplus\infty})^*$ sending $v \mapsto f_v$.)

3. Prove that every linear transformation $\mathbb{R}^{\oplus\infty} \to \mathbb{R}$ has the form f_v for some $v \in \mathbb{R}^{\mathbb{N}}$. (In other words, there is an isomorphism $(\mathbb{R}^{\oplus\infty})^* \cong \mathbb{R}^{\mathbb{N}}$.)

4.5 Let U be a subspace of a finite-dimensional vector space V, with $U \neq V$. Let W be another vector space.

1. Suppose one has a linear transformation $L \colon V \to W$. Restricting the domain of L to U, we get a function $L|_U \colon U \to W$. Show that $L|_U$ is a linear transformation. Better yet, don't write anything, but just convince yourself this is trivial!

2. Suppose one has a linear transformation $M \colon U \to W$. The *extension by zero* of M is the function $M^0 \colon V \to W$ for which

$$M^0(v) = \begin{cases} M(v) & \text{if } v \in U, \\ 0 & \text{if } v \notin U. \end{cases} \tag{0.6}$$

   Prove that M^0 is not a linear transformation when M is nonzero.

3. Suppose one has a linear transformation $M \colon U \to W$. Now choose a complement U' to U, i.e. a subspace $U' \subset V$ such that $U \oplus U' = V$. Show that there is a unique linear transformation $M' \colon V \to W$ such that

$$M'(v) = \begin{cases} M(v) & \text{if } v \in U, \\ 0 & \text{if } v \in U'. \end{cases} \tag{0.7}$$

   We call M' an *extension* of M to V.

4. Give an example where two different complements give rise to two different extensions of a linear transformation.

4F. Axler 3rd edition 3.D.9. Written out: If $S, T \colon V \to V$ are linear maps, then $S \circ T$ is invertible if and only if both S and T are invertible.

4W. Axler 3rd edition 3.B.4. Written out: show that

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) \mid \dim \operatorname{Ker}(T) > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

4M. Let $V = \mathcal{P}_{\leq 2}$. Recall that $\mathrm{ev}_\lambda \in V^*$ sends a polynomial to its evaluation at $\lambda \in \mathbb{R}$, that is, $\mathrm{ev}_\lambda(p) = p(\lambda)$. I claim that the set $\{\mathrm{ev}_{-1}, \mathrm{ev}_0, \mathrm{ev}_1, \mathrm{ev}_2\}$ is linearly dependent in V^*. Find a nontrivial linear combination for 0.

Week 3 Reading: Axler Chapter 2 (the rest) and start of Chapter 3. I have the idea that, even though there are more problems this week, they feel shorter to me. Let me know if I'm wrong!

3.1 Axler 3nd edition exercises 2.A.16, 2.B.4, 2.B.5, 2.C.6, 2.C.10, 2.C.14, 3.A.1.

3F. Two problems. Axler 3rd edition 2.C.4 and 2.C.13.

3W. Two problems. I assigned this late, so feel free to hand in on Monday instead of Friday. Axler 3rd edition 2.B.7 and 2.A.17.

3M. Prove that $\mathbb{R}^{\mathbb{N}}$ is infinite-dimensional.

Week 2 Reading: Axler Chapter 1 (sums and direct sums), Chapter 2 (spans, linear independence, bases)

2.1 Do Axler 2nd edition exercises 1.14, 2.1, 2.2, and Axler 3rd edition exercises 2.A.7, 2.A.9.

2.2 Let U be a subspace of V. For any vector $v \in V$, define U_{+v} to be the subset

$$U_{+v} = \{z \in V \mid z = u + v \text{ for some } u \in U\}. \tag{0.8}$$

(Note: Now that we've discussed quotient spaces, we call this set $v + U$.)

1. For which $v \in V$ is U_{+v} a subspace?
2. Let $v, v' \in V$. Prove that the sets U_{+v} and $U_{+v'}$ are either equal or disjoint. (Hint: first prove that if $U_{+v} \cap U_{+v'} \neq \emptyset$ then $v' \in U_{+v}$.)
3. Think about the connection between this exercise and the notion of parallel lines, or parallel planes in 3D. Say something meaningful and brief, both about the definition of U_{+v}, and about the result you just proved.

2F. Axler 2nd edition exercise 2.3 (same as Axler 3rd edition exercise 2.A.10).

2W. Axler 3rd edition exercise 1.C.24. (Hint: ONLY READ AFTER THINKING AND BEING STUMPED. Given an arbitrary function $f(x)$, consider the function $g(x) = \frac{f(x)+f(-x)}{2}$.)

2M. Do Axler (3rd edition) exercise 1.C.20 and 1.C.21. Here they are written out in case you only have 2nd ed. Both exercises give you a subspace U and ask for W such that $U \oplus W$ is the whole space.

- $U = \{(x, x, y, y)\} \subset \mathbb{F}^4$,

- $U = \{(x, y, x + y, x - y, 2x)\} \subset \mathbb{F}^5$.

Week 1 Reading: Axler Chapter 1

1.1 Do Axler exercises 1.2, 1.4, 1.5, 1.6, 1.9. You only need rigorous proofs for 1.4 and 1.9.

1.2 Consider $\mathbb{Q}(\sqrt{2})$ defined abstractly as $\{(a, b) \mid a, b \in \mathbb{Q}\}$ with addition and multiplication coming from the formulas in exercise 1W. For $z = (a, b)$ in $\mathbb{Q}(\sqrt{2})$ let us define $N(z) \in \mathbb{R}$ by the formula $N(z) = a^2 - 2b^2$. Confirm that $N(z \cdot w) = N(z) \cdot N(w)$ for any $z, w \in \mathbb{Q}(\sqrt{2})$.

Aside: For a complex number $z = a+bi$ we can define $N(z) = a^2+b^2 = |z|^2$. The exercise above is essentially the same computation one would use to show that $N(z\cdot w) = N(z)\cdot N(w)$ for the complex numbers, or that $|z \cdot w| = |z| \cdot |w|$.

1F. Do Axler exercise 1.8.

1W. Consider the set $X = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$. Equip X with an addition and multiplication structure just like we did for $\mathbb{Q}(\sqrt{2})$ in class, namely

- $(a, b) + (c, d) = (a + c, b + d)$, and
- $(a, b) \cdot (c, d) = (ac + 2bd, ad + bc)$.

Sadly, X is not a field, and we will NOT denote it by $\mathbb{R}(\sqrt{2})$.

- Which properties of a field hold for X? (You need not provide the proof.)
- Which properties of a field fail? (Give an example.)

1M. Prove that the set of all twice-differential functions $f \colon \mathbb{R} \to \mathbb{R}$ which satisfy

$$f'' - 26f' + 3f = 0$$

is closed under addition and rescaling.

Note: This is not a hard proof, but I want to see your style of proof-writing. Don't be too verbose or too sketchy please. Feel free to use basic facts from calculus, but you should call them out.

Proof-writing hint: When you want to avoid ambiguity and convoluted sentences, name things. For instance, I didn't name the set above, but when you write up the proof, an excellent first sentence is "Let Y denote the set of all twice-differentiable ..." It is a lot easier to refer to Y than "the set in question" or "it" or whatever else one might say. Similarly, the equation $f'' - 26f' + 3f = 0$ might itself be named something, like $(\star)$, which enables you to say "Suppose that f satisfies $(\star)$. Then ..."