Linear Algebra (with theory) Study Guide (Math 441/541)
Spring 2019

Definitions you should know:

- **Chapter 1:**
- Field*
- Vector space*
- * Note: For both of these definitions there is a long list of axioms, which you need not worry about reproducing. You should know their names when they are applied, e.g. if you are asked what commutativity of addition is you should know what that is, or if you are asked to justify $a + b = b + a$ you should know the name of this property. Most of these names were known to you before this class anyway.
- * Moreover, you need not call out these axioms in a proof, unless it is specifically asked for, or the proof is explicitly about the axioms, e.g. prove that there is a unique additive identity.
- Subspace (aside: the axioms here are more important since you actually have to check them in practice.)
- Sum of subspaces
- When a sum is direct
- **Chapter 2:**
- Span
- Linear independence
- Dimension, and infinite-dimensional (Aside: the book is careful, defining finite-dimensional before it defines dimension. Then it proves that any two bases of a finite-dimensional space have the same size, and uses that to define dimension. You don't have to worry so much about this, after the fact.)
- Basis
- Coordinates with respect to a basis
- **Chapter 3:**
- Linear transformation
- Kernel (nulspace)
- Image (range)
- Injective, surjective, bijective, invertible (note the difference between bijective and invertible in the definition, but not in practice)
- The matrix of a linear transformation with respect to chosen bases
- invariant subspaces, preserving a subspace
- Dual space
- Hom space
- Product space
- Quotient space
- Affine subspace, or subset parallel to a subspace
- **Chapter 4:**
- Polynomial, degree, root
- Polynomial applied to an operator

- **Chapter 5:**
- Upper triangular matrix
- Eigenvector, eigenvalue, eigenspace
- Diagonalizable operator, diagonalizable matrix
- **Chapter 8:**
- Generalized eigenvector, generalized eigenspace
- Block diagonal matrix
- Jordan block matrix
- Jordan normal form matrix
- Characteristic polynomial
- Minimal polynomial
- **Chapter 6:**
- Bilinear form, sesquilinear form
- symmetric, hermitian
- positive definite
- norm
- orthonormal set. orthonormal basis
- orthogonal complement. orthogonal projection.

Examples you should know well:

- Fields: Rational, real, and complex numbers.
- Vector spaces and their operators:
- $\mathbb{F}^n$ and matrix multiplication.
- Polynomials and those of bounded degree, operators like differentiation and multiplication by x.
- Functions on the real numbers, differentiable functions, etcetera. Operators like differentiation and multiplication by $\sin x$.
- Solutions to certain differential equations. Operators like differentiation.
- The vector space of sequences $\mathbb{R}^{\mathbb{N}}$ and the subspace $\mathbb{R}^{\oplus\infty}$ of sequences which are eventually zero.
- Examples involving matrices:
- Upper triangular matrices and how they act.
- Diagonal matrices and how they act.
- Jordan blocks and how they act.
- Generalized eigenvectors for the derivative: things like t^7e^{5t}.
- Inner products: the standard one on $\mathbb{F}^n$.
- The inner product on function spaces given by integrating the product of two functions.

Theorems/results/ideas/techniques whose proof is not hard if you have internalized the ideas, so you might be expected to reproduce it on an exam. (Don't focus on memorizing proofs, focus on internalizing the ideas!):

- **Chapter 1**

- Zeroes and additive inverses are unique, the cancellation law, and other basic properties.
- Sum of two subspaces is direct iff intersection is zero. Why this fails for more than two subspaces.
- Sum of arbitrary subspaces is direct iff 0 can be written in a unique way
- Subspaces don't have cancellation law
- **Chapter 2**
- The criterion for when you can remove a vector from a set without changing the span
- The criterion for when you can add a vector to a linearly independent set to get another linearly independent set.
- Can reduce a (finite) spanning set to a basis.
- Can extend a linearly independent set to a basis (in finite dimensional setting).
- If correct size, basis iff spans iff lin. indep.
- Implications for dimension (e.g. dimension of sum vs. dimension of intersection, a sum is direct if dimensions add up, etcetera)
- Complements exist.
- How you show something is infinite dimensional.
- **Chapter 3**
- Linear maps compose, add, rescale to be linear maps.
- Basics of linear maps (kernel is a subspace, etc)
- Injective iff kernel is zero, and related theorems (surjective iff...)
- Injective implies $\dim V \leq \dim W$, and other easy applications of rank-nullity
- Bijective and invertible are the same
- A linear map is determined by what it does to a basis
- A linear map can do whatever it wants to a basis
- Using these facts to construct linear maps satisfying desired properties.
- Every subspace is the image of a map. Every subspace is the kernel of a map.
- If an operator on V preserves U, it induces an operator on V/U.
- How changing the basis affects the coordinates of a vector, or the matrix of a linear map!!!!!
- ... and the implications of this for the interpretation of row reduction, column reduction.
- **Chapter 5**
- Eigenvectors with distinct eigenvalues are linearly independent.
- λ is eigenvalue iff $(L - \lambda I)$ is not invertible.
- Every operator on a complex f.d. v.s. has an eigenvector.
- Properties of upper-triangular matrices - possible eigenvalues, what eigenvectors look like, when they are invertible, etcetera.
- L is diagonalizable if and only if (various list of conditions)
- How commuting operators behave (each preserves the eigenspaces of the other, there is a simultaneous eigenspace over $\mathbb{C}$).
- Polynomials in an operator commute
- How a polynomial in an operator acts on a vector. Conclusion: if $p(L) = 0$ and λ is not a root of p then λ is not an eigenvalue.

- How to find an eigenvector by applying polynomials in an operator
- How to describe the inverse as a polynomial in an operator
- **Chapter 8**
- Given the long division algorithm, how to show that the minimal polynomial exists.
- Roots of minimal polynomial = roots of characteristic polynomial = set of eigenvalues.
- The Cayley-Hamilton theorem over $\mathbb{C}$: if A is an upper triangular matrix and $p(x) = \prod(x - \lambda_i)$ is the product over the diagonal entries, then $p(A) = 0$.
- Comparing the size of Jordan blocks to the minimal polynomial, and to the characteristic polynomial, and to $\dim Ker(L - \lambda)^k$.
- The degree of a generalized eigenvector is at most $\dim V$.
- **Chapter 6**
- Pythagorean theorem
- Parallelogram equality (you don't have to memorize this equality though)
- Using the fact that a vector is zero if its norm is zero (this was the solution to some exercises)
- Cauchy-Schwartz
- Coordinates of a vector with respect to an orthonormal basis
- Formula for projection of a vector to the line of another vector.
- Given Gram-Schmidt, various corollaries (existence of orthonormal bases, extension of orthonormal sets to orthonormal bases, etcetera).
- Using Gram-Schmidt to find orthonormal bases and orthogonal complements.
- **Meta-topic:** Interpreting properties of a linear map (e.g. preserving a subspace, how one acts on quotients, direct sums, etcetera) in terms of properties of the matrix with respect to a well-chosen basis!!!!!

Theorems/results/ideas whose proof is complicated, but where the big ideas in the proof are still important to know for true/false problems

- Two finite bases have the same size (which was implied by the next one)
- Spanning sets are bigger than linearly independent sets (the big idea in the proof is the exchange or replacement lemma)
- Rank nullity theorem (The big big idea in the proof is that a basis for the kernel can be extended to a basis by adding elements which are sent to a basis of the image.)
- The Euclidean/Long division algorithm for polynomials. (The proof is boring and constructive, but illustrates the idea of studying the leading terms in a polynomial.)
- Polynomial has a root λ if and only if $(x - \lambda)$ divides it. (The proof is boring and constructive, or is easy using long division.)
- Every operator on a complex f.d. v.s. has an upper-triangular basis. (That it has a one-dimensional invariant subspace U_1 is another important argument above. The inductive step is the big idea, applying this same result to V/U_1.)
- Generalized eigenvectors with distinct eigenvalues are linearly independent. (The big idea, like for the normal eigenvector statement, is to find a way to cancel one coefficient from the linear combination, and use some kind of induction.)

Theorems/results/ideas we talked about, where you should really know the result, but whose proof is not essential (i.e. annoying/uninteresting/not covered/ too hard/uses material from other classes/etc)

- The fact that ANY vector space (possibly infinite dimensional) has a basis
- The fundamental theorem of algebra - any polynomial over $\mathbb{C}$ has a root.
- Over $\mathbb{C}$, any polynomial factors uniquely (up to reordering) into products of linear terms $(x - \lambda_i)$ and a leading coefficient.
- Over $\mathbb{C}$, V is the direct sum of its generalized eigenspaces.
- Jordan normal form.
- Gram-Schmidt.
- Projecting to a subspace gives the closest vector.

Theorems/results/ideas you don't need to worry about, or that we didn't cover at all

- The actual definition of the real numbers, or its properties (completeness, ordered field)
- How polynomials factor over $\mathbb{R}$
- Real Jordan normal form, invariant subspaces over $\mathbb{R}$
- Adjoint maps
- ***DETERMINANTS!!! (and traces too)
- *** Note: if you want to be a serious mathematician and go further, it is highly recommended to spend some time reading the rest of the book, especially chapter 10, to get some exposure to this topic. Might as well do it while things are still fresh. Then you can go read another linear algebra textbook and see how they do things!