Math 607 (Homological Algebra), Spring 2016
Exercises

There will be one exercise per class. I'll include readings by the exercise.

Week 1

Reading: Crawley-Boevey's notes, p3-6. Gabriel's theorem is on p19.

M. This exercise is about embedding the category of graded modules over a graded ring into ordinary modules over an ordinary ring.

Let $q \in \mathbb{C}$ be an invertible complex number which is not a root of unity. Let $R = \oplus_{i \in \mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, and $M = \oplus_{i \in \mathbb{Z}} M_i$ be a graded R-module. Define an invertible operator h on M, where $hm = q^i m$ for any $m \in M_i$. Note that h is diagonalizable, and its eigenspaces are the graded pieces M_i. Now M is a module over both R and $\mathbb{C}[h, h^{-1}]$. However, it is not a module over $R[h, h^{-1}]$ since the elements of R do not commute with h.

1. Compute hrh^{-1} for $r \in R$ homogeneous. Use this relation to define a new ring R_h, having the same size as $R[h, h^{-1}]$, and having both R and $\mathbb{C}[h, h^{-1}]$ as subrings, but where h acts by conjugation in the appropriate way.
2. Define a functor F from graded R-modules to R_h-modules, via the discussion above. Is this functor full? Is it faithful?
3. Is the functor F essentially surjective (that is, does every R_h-module come from a graded R-module)? If not, can you find a precise condition as to when an R_h-module comes from a graded R-module?

W. For each of the categories below, please find:

- All the simple modules.
- All the indecomposable modules.
- Which indecomposable modules are projective.
- All the non-split short exact sequences whose outer terms are indecomposable.

You need not give proof, just be sure its a complete list.

1. R-mod for the ring $R = \mathbb{C}[x]/(x^4)$.
2. Q-rep for the quiver $\bullet \to \bullet \to \bullet$.

F. Consider the following quiver Q. It has four vertices named $1, 2, 3$ and $*$. It has three arrows, one from i to $*$ for each $i \in \{1, 2, 3\}$. (This is a particular orientation on the D_4 quiver.)

1. Find all the simples and indecomposable projectives.
2. By the Gabriel theorem, there is one isomorphism class of indecomposable object M whose dimension at vertices $1, 2, 3$ is one, and whose dimension at vertex $*$ is 2. Find M, and show that any other indecomposable of this dimension is isomorphic to M.

3. Find a short exact sequence $0 \to P \to Q \to M \to 0$ where P and Q are projective.

Week 2

Reading: MacLane Chapter 8. Weibel 1.6 on the Freyd-Mitchell embedding theorem.

M. Consider the quiver Q_1 which is $\bullet \to \bullet \to \bullet$, and Q_2 which is $\bullet \leftarrow \bullet \to \bullet$.

1. Compute the split Grothendieck group $[Q_i]$ of projectives, with its induced bilinear form, for $i = 1, 2$.
2. Is there a $\mathbb{N}$-linear isometry between $[Q_1]$ and $[Q_2]$, i.e. an isomorphism which preserves the bilinear form? Are the categories of modules for Q_1 and Q_2 equivalent?
3. Find a $\mathbb{Z}$-linear isometry. (Hint: the projectives for vertices 2 and 3 are unchanged.)

We'll learn later that this isometry is the shadow of a reflection functor which changes the orientation around the first vertex, and which induces an equivalence of derived categories.

W. For a functor F between additive categories, prove that F preserves addition if and only if F preserves direct sums.

F. Two problems today! I have betrayed you.

1. Prove that in an abelian category (i.e. when every monic is a kernel, and every epic is a cokernel) that the canonical map from the coimage to the image is an isomorphism. (Addendum: Yikes! This problem is really hard. It is now just extra credit, replaced by the next problem instead.)
2. Prove that if g is monic (in an abelian category) then $g = \ker(\text{coker}(g))$.
3. Prove that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $gf = 0$ and for all members x of B with $g(x) = 0$, there exists a member y of A such that $f(y) = x$.

Week 3

Reading: Weibel Ch 1.1, 1.3, 1.4: basics on complexes, homology, and the long exact sequence. Weibel 2.1-2.5 on the basics of derived functors.

M. Find projective resolutions of:

1. All indecomposables in R-mod for the ring $R = \mathbb{C}[x]/(x^4)$.
2. All indecomposables for Q-rep for the quiver $\bullet \to \bullet \to \bullet$.
3. The module $\mathbb{C}[x,y]/(x^2, xy, y^2)$ over the ring $\mathbb{C}[x,y]$.

W. Consider R-mod for the ring $R = \mathbb{C}[x]/(x^4)$. Let F be the (covariant) functor from R-mod to vector spaces that sends M to $\{m \in M \mid xm = 0\}$, and G be the functor sending M to $\{m \in M \mid x^2 m = 0\}$.

1. Why are F and G left exact?
2. Recall that R is injective as an R-module. Find injective resolutions of $M = R/(x)$ and $N = R/(x^2)$. (Just flip your sequences from Monday.)

3. Compute R^iF and R^iG of M and N.

4. There is an inclusion $M \hookrightarrow N$. Lift this to a map between your injective resolutions. Use this to compute the maps $R^iF(M) \to R^iF(N)$ and $R^iG(M) \to R^iG(N)$.

5. There is a surjection $N \twoheadrightarrow M$. Lift this to a map between your injective resolutions. Use this to compute the maps $R^iF(M) \to R^iF(N)$ and $R^iG(M) \to R^iG(N)$.

6. These maps fit into a non-split short exact sequence. Write down the corresponding long exact sequence, for F and for G. Major note: the computations you have done so far make it easy to guess the connecting maps δ, but not to compute them explicitly. Why not?

7. (Extra credit) Compute the connecting maps δ explicitly, using the Horseshoe lemma. This is a LOT of work.

F. This is a first introduction to cones and convolutions.

1. Let A and B be two complexes. Classify all complexes X where $X^i = A^i \oplus B^i$, and the inclusion map $A^i \to X^i$ induces a chain map $A \to X$, and the projection map $X^i \to B^i$ induces a chain map $X \to B$. What I really want you to do is write down the differential on X explicitly.

2. Show that these complexes X are in bijection with chain maps $B \to A[1]$. Here, $A[1]$ is the complex where $A[1]^i = A^{i+1}$, and the differential on $A[1]$ is minus the differential on A.

3. (Optional) When are two such complexes X isomorphic? When are they homotopy equivalent?

4. (Optional, but recommended) Now let A, B, C be complexes. Classify all complexes X where $X^i = A^i \oplus B^i \oplus C^i$, and where the various inclusions and projections give a filtration of X with quotient C, middle B, and sub A.

5. Read Weibel's proof of functoriality on p46-47 and really try to understand it!

Week 4

Reading: Weibel 2.5. I don't like his chapter 3, unfortunately. Will search for a good Frobenius algebra reference.

M. In class I described functors between representations of different quivers, which I will call the *source-push* and the *sink-pop*. This exercise hearkens back to week 2 monday: consider the quiver Q_1 which is $\bullet \to \bullet \to \bullet$, and Q_2 which is $\bullet \leftarrow \bullet \to \bullet$. The source-push gives a functor F from Q_1 modules to Q_2 modules, by inverting the orientation around the first vertex.

1. Compute how F acts on all indecomposables. Is F right exact?

2. Compute the higher derived functors of F.

3. Shifting tacks, suppose one has an exact bounded complex C in an abelian category. Prove that $(-1)^i[C^i]$ is zero in the Grothendieck group. (Hint: don't try to use Jordan-Holder series or anything, just use induction on the length of C.)

4. Prove that the map $[M] \mapsto [FM]-[L^1F(M)]$ is a well-defined map from the Grothendieck group to Q_1-modules to the Grothendieck group of Q_2-modules.

5. Compare this Grothendieck group map from the source-push to the isometry you computed in week 2.

W1. Two small vignettes. The first is: compute the Nakayama automorphism of $\Lambda^*(\mathbb{C}^3)$.

W2. For a Frobenius algebra, all projectives are injective. A consequence is that any free module appearing as a submodule is actually a summand! Using this fact can help to compute the (non-free) indecomposable objects, as they have no free submodules.

1. Let $H = \Lambda^*(\mathbb{C})$, i.e. $H = \mathbb{C}[d]/(d^2)$. Convince yourself that a finitely-generated graded module over H the same as a bounded complex of (finite-dimensional) vector spaces. What does a free rank 1 graded H-module look like as a complex (many different grading shifts are allowed)?

2. What does it mean (in practical terms) for a complex not to have any free submodules?

3. Find all the simple and indecomposable bounded chain complexes of vector spaces. (You've already done this problem, but in a different language).

W3? Ok, W2 was pretty easy, because it was a warmup. Here's an open-ended problem, I don't care when or if you turn it in, but it is worth thinking about.

1. Let $H = \Lambda^*(\mathbb{C}^2)$. This is a bigraded (i.e. $\mathbb{Z} \times \mathbb{Z}$ graded) algebra. Convince yourself that a finitely-generated bigraded module over H the same as a bounded bicomplex of (finite-dimensional) vector spaces. What does a free module look like?

2. What does it mean for a bicomplex not to have any free submodules?

3. Find all the simple and indecomposable bounded bicomplexes of vector spaces.

F. In Weibel, Proposition 3.2.4 (no need to read the proof yet, it will be easier after next week) it is proven that an R-module B is flat if and only if $\mathrm{Tor}^1_R(I\backslash R, B) = 0$ for every right ideal $I \subset R$. (Remember, $\mathrm{Tor}^i_R(M,N)$ takes a right R-module M and a left R-module N to an abelian group, just like how $M \otimes_R N$ is just an abelian group.)

1. When $I = (r)$ is principal, and r is a nonzerodivisor, compute $\mathrm{Tor}^i(I\backslash R, B)$.

2. When R is a PID, prove that B is flat if and only if it is torsion-free.

3. Let $R = \mathbb{C}[x,y]$, which is not a PID. Let $B = (x,y)$ be the ideal of x and y. Compute $\mathrm{Tor}^i(B, \mathbb{C})$. Is B torsion-free? Is B flat?

Week 5

Reading: Weibel 4.1-4.5.

M. Let $R = \mathbb{Z}[x]/(x^2-1)$, i.e. R is the group algebra of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$. Let S and T denote the R-modules, both isomorphic to $\mathbb{Z}$ as $\mathbb{Z}$-modules, which correspond to the sign and trivial representations.

1. After base change from $\mathbb{Z}$ to a finite field $\mathbb{F}_p$, what is the homological dimension of $R \otimes \mathbb{F}_p$? (Just cite a theorem.) The rest of this problem explores the original ring over $\mathbb{Z}$, which is not an algebra.

2. Show that S (resp. T) is both a sub and a quotient of a free module, but is nonetheless not a summand.

3. Find free resolutions of S and T. What is $\mathrm{Ext}^i(S,S)$, $\mathrm{Ext}^i(S,T)$, $\mathrm{Ext}^i(T,S)$, and $\mathrm{Ext}^i(T,T)$ (don't show your work). Note: Since R is commutative, morphisms between R-modules themselves form an R-module, so these exts are all R-modules. Make sure to indicate the R-module structure.

4. What is the homological dimension of R?

5. (Extra credit:) What is the homological dimension of $\mathbb{Z}[G]$, for any finite group G?

W. Let $R = \oplus_{i\in\mathbb{Z}} R^i$ be a graded commutative $\Bbbk$-algebra (for a field $\Bbbk$), where $R^i = 0$ for $i < 0$, and $R^0 = \Bbbk$. Assume that R is finite dimensional overall; in particular, it is bounded above. Sketch an argument that R has infinite global dimension. (Hint: Use the fact that projective modules over a local ring are free. Resolve the one-dimensional representation.) For extra credit, make it rigorous.

F. Find the canonical resolutions of the following objects.

1. Some indecomposable quiver representations (give me two, make them at least slightly interesting).

2. The representation $\mathbb{C}[x,y]/(x^2, xy, y^2)$ of the ring $\mathbb{C}[x,y]$.

Week 6

Reading: Weibel 4.5, 3.4 (there are some errors in this chapter, or so I heard)

M. Let C be the cone of a chain map $f: A \to B$. Thus there is a short exact sequence of complexes

$$0 \to B \to C \to A[1] \to 0.$$

Prove that in the long exact sequence, the map $h^i(A[1]) \to h^{i+1}(B)$ agrees with the map $f_*: h^{i+1}(A) \to h^{i+1}(B)$.

M challenge. This is extra credit, for those interested in knot homology. Let $R = \mathbb{C}[x_1, x_2]$ be a polynomial ring, and $S = \mathbb{C}[y_1, y_2, z_1, z_2] \cong R \otimes_{\mathbb{C}} R$, with $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$. We are interested in taking the Hochschild homology HH_i of various (graded) R-bimodules; recall that these are the higher derived functors of $R \otimes_S (-)$, from S-modules to R-modules.

1. Compute HH_i of the R-bimodule R, for all i.

2. Compute HH_i of the R-bimodule B, for all i. Here, $B = S/(y_1 - z_1, y_2^2 - z_2^2)$.

3. Let $f: B \to R$ be the R-bimodule map with $f(y_1) = f(z_1) = x_1$ and $f(y_2) = f(z_2) = x_2$. Compute the map induced by f from $HH_i(B)$ to $HH_i(R)$ for all i.

4. For each i, one has a two-term complex $HH_i(B) \to HH_i(R)$. Compute its cohomology in each degree.

W. Let Q be the quiver with two vertices $\{1,2\}$, and five arrows from 1 to 2. Let S_i denote the simple concentrated at vertex i.

1. Compute $\mathrm{Ext}^1(S_1, S_2)$.
2. Explicitly match each element of $\mathrm{Ext}^1(S_1, S_2)$ with an extension in $\mathrm{Exten}^1(S_1, S_2)$.
3. Prove that for a general quiver, the number of arrows from vertex i to vertex j is the dimension of $\mathrm{Ext}^1(S_i, S_j)$. (Warning: make sure your proof works when there are edges both from i to j and from j to i.)

F. Prove that a k-extension $[X]$ corresponds to $0 \in \mathrm{Ext}^k(A, B)$ if and only if it is equivalent to a split k-extension.

F. challenge Let $R = \mathbb{C}[x]/x^k$. We have already computed that $\mathrm{Ext}^i(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ for each $i \geq 0$. Compute the ring structure on $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C})$. (This depends on k!) For each k, find a 2-extension representing a non-trivial element of $\mathrm{Ext}^2(\mathbb{C}, \mathbb{C})$.

Week 7

Reading: Weibel 1.4, 1.5, 10.1, 10.2

M. Prove Gaussian elimination.

W. Given a distinguished triangle $X \to Y \to Z \to X[1]$ with maps (u, v, w), prove that $w = 0$ if and only if $Y \cong X \oplus Z$, making u and v commute with the inclusion and projection maps. Moreover, prove that for any X and Z, the triangle $X \to (X \oplus Z) \to Z \to X[1]$ is distinguished. Optional: give a sketch why the direct sum of two distinguished triangles is distinguished.

F. Prove that the triangulated Grothendieck group $[\mathcal{K}^b(\mathrm{vect})]$ of the bounded homotopy category of vector spaces is $\mathbb{Z}$, where the symbol of a complex is given by its Euler characteristic. This statement is false when vect is replaced with an non-semisimple abelian category $\mathcal{A}$. Prove that there is an Euler characteristic map $[\mathcal{K}^b(\mathcal{A})] \to [\mathcal{A}]$. Prove that when $\mathcal{A}$ has finite homological dimension, then there is a section $[\mathcal{A}] \to [\mathcal{K}^b(\mathcal{A})]$. Hint: the section is NOT $[M] \mapsto [0 \to M \to 0]$. Why is this not well defined!? Find a counterexample.

Week 8

Reading: Weibel 10.3 - 10.5, Kiehl-Weissauer.

M. Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a triangulated functor, and Q be the class of morphisms which is sent to isomorphisms. Prove that

1. Given any $s\colon A \to B$ with $s \in Q$, and $f\colon Y \to B$ any morphism, one can extend this data to a commutative square $ft = sg$, with $t \in Q$.
2. Given $f\colon A \to B$, the existence of some $s \in Q$ with $sf = 0$ implies the existence of some $t \in Q$ with $ft = 0$.

Hint: You do not need the octahedron axiom. For both, take the cone of s, and apply the functor F. For the second problem, apply $\mathrm{Hom}(A, -)$ to the cone of s.

W. Consider modules over $\Lambda = \mathbb{C}[x]/x^2$. Let M be the complex $\Lambda \to \underline{\Lambda}$, where the differential is multiplication by x.

1. Compute the hom complex $\underline{\mathrm{Hom}}(\mathbb{C}, M)$ and its cohomology.

2. Compute the hom complex $\underline{\mathrm{Hom}}(P, M)$ and its cohomology, where P is the projective resolution of $\mathbb{C}$. (Hint: This complex should mostly consist of 4D vector spaces.)

3. Is there a reason why these two cohomologies agreed?

F. A long one. We return to the setup of Week 4 Monday: consider the quiver Q_1 which is $\bullet \to \bullet \to \bullet$, and Q_2 which is $\bullet \leftarrow \bullet \to \bullet$. The source-push gives a functor F from $Q_1 - \mathrm{Rep}$ to $Q_2 - \mathrm{Rep}$, by inverting the orientation around the first vertex, and the sink-pop functor G gives a functor back. Remember that F is right exact, while G is left exact.

1. You have already computed the higher derived functors of F on all simples, so look those up.

2. Compute the higher derived functors of G on all simples. (Hint: You need to find injective resolutions. But recall that for an algebra R, taking the dual vector space gives an equivalence between the finite-dimensional projective left modules for R and the finite-dimensional injective right modules for R. So you're effectively looking for a projective resolution of the right module T^* for each simple T.)

3. It is rarely true that a complex is quasi-isomorphic to its cohomology (i.e. it's cohomology, viewed as a complex with zero differential). However, show that this is always true when the cohomology is concentrated in a single degree, which is the first or the last degree in the support of the complex.

4. Compute (the total derived functors) $\mathbb{L}F(S)$ and $\mathbb{R}G(T)$ for each simple quiver representation S of Q_1 and T of Q_2. That is, find a nice complex which represents each quasi-isomorphism class in the derived category.

5. Deduce that $\mathbb{L}F(\mathbb{R}G(T)) \cong T$ and $\mathbb{R}G(\mathbb{L}F(S)) \cong S$.

6. Deduce that $\mathbb{L}F$ and $\mathbb{R}G$ are inverse functors between $D^b(Q_1 - \mathrm{Rep})$ and $D^b(Q_2 - \mathrm{Rep})$. (You need to prove this for all objects of the derived category, not just for objects in the abelian category! So use the 5-lemma.)

7. Identify the triangulated Grothendieck group of $D^b(Q_i - \mathrm{Rep})$ with the abelian Grothendieck group of $Q_i - \mathrm{Rep}$ in the usual way. Identify the Grothendieck groups of $Q_1 - \mathrm{Rep}$ and $Q_2 - \mathrm{Rep}$ by identifying their bases of simple objects in the natural way. How does the derived functor $\mathbb{L}F$ act on the Grothendieck group? Compare this to Week 4 Monday.

8. (If you know about Coxeter groups) Reflection representation! Discuss.

Week 9

M.

1. Does $\tau_{\leq 0}$ descend to a functor on $\mathcal{K}(\mathcal{A})$? Does $\tau'_{\leq 0}$?

2. Let $(D^{\leq 0}, D^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{D}$. Let M and M' be two objects, and consider distinguished triangles

$$X \to M \to Y \to X[1],$$

$$X' \to M' \to Y' \to X'[1],$$

with $X, X' \in \mathcal{D}^{\leq 0}$ and $Y, Y' \in \mathcal{D}^{\geq 1}$. Prove that any map $M \to M'$ induces a unique map $X \to X'$ and a unique map $Y \to Y'$. Prove that this assignment is functorial (i.e. for a composition $M \to M' \to M''$, the maps $X \to X' \to X''$ compose to be the map $X \to X''$ induced by the composition $M \to M''$).

W. Prove that any commutative square in the stable module category over a Frobenius algebra extends to a morphism between distinguished triangles. (Hint: You can use any injective module you want to define the triangles, so choose the injective module through which something is forced to factor...)

Week 10

M. This is an exercise which fills in some gaps from lecture. For the double complex attached to a chain map $f: A \to B$ of complexes, whose total complex is $\mathrm{Cone}(f)$, compute both spectral sequences (horizontal first, vertical first), and explicitly exhibit the filtration on $h^k(\mathrm{Cone}(f))$ coming from each one. This involves constructing a short exact sequence and a long exact sequence which was mentioned in class.

Extra Exercises

1. Consider $\mathcal{K}$ the homotopy category of p-complexes. Let $H_{(i)}: \mathcal{K} \to \mathrm{vect}_{\mathbb{Z}}$ be the homological functor to graded vector spaces which sends a p-complex to $\mathrm{Ker}\, d^i/\mathrm{Im}\, d^{p-i}$. Let $h^k_{(i)}: \mathcal{K} \to \mathrm{vect}$ be the homological functor to vector spaces which takes the degree k part of $H_{(i)}$. Prove that neither $H_{(i)}$ nor $h^k_{(i)}$ give rise to a naive t-structure, e.g.

$$D^{\leq 0} = \{X \mid H_{(i)}(X[\ell]) = 0 \text{ for all } \ell > 0\}$$

and a similar definition of $D^{\geq 0}$ does not give a t-structure.

2. Here is a double complex $X^{\bullet\bullet}$ of $\mathbb{Z}$-modules. The only nonzero entries are $X^{10} \cong X^{11} \cong X^{01} \cong X^{02} \cong \mathbb{Z}$. The horizontal differential is multiplication by 12 from X^{01} to X^{11}. The vertical differential is multiplication by 2 from X^{01} to X^{02}, and multiplicaiton by 8 from X^{10} to X^{11}.

1. Compute the total complex and its cohomology. (Hint: For abstract reasons, this will be an abelian group of order 16. Which abelian group is it? Does it have an element of order 16? Of order 8?)
2. Compute the spectral sequence where one takes vertical cohomology first. What filtration on the total complex do you get?
3. Compute the spectral sequence where one takes horizontal cohomology first. What filtration on the total complex do you get?